

Relations between generalised Gelfand–Tsetlin and Kazhdan–Lusztig bases of S_n

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Abstract. We prove that the Kazhdan–Lusztig basis of Specht modules is upper triangular with respect to all generalized Gelfand–Tsetlin bases constructed from any multiplicity-free tower of standard parabolic subgroups.

Keywords: Specht modules, Kazhdan–Lusztig basis, generalised Gelfand–Tsetlin basis, standard Young tableaux, multiplicity free, parabolic subgroups

1 Introduction

In their classical paper “Relations between Young’s natural and the Kazhdan–Lusztig representations of S_n ” [3], Garsia and MacLarnan show that Young’s natural basis of a Specht module is unitriangular with respect to the Kazhdan–Lusztig (KL) basis. The aim of this note is to prove an analogue of this theorem for all generalised Gelfand–Tsetlin (GT) bases.

To explain more precisely, first note that Young’s natural basis is known to be upper-triangular with respect to either Young’s seminormal or orthogonal bases [1]. These bases are scalar multiples of each other, and are in fact examples of the standard GT basis. Recall that the latter is the basis obtained by restricting a Specht module along the tower of groups

$$1 < S_2 < S_3 < \cdots < S_n. \tag{1.1}$$

It is well-defined up to scalar multiplication. By combining the results of [3] and [1], one can show that the GT basis is upper triangular with respect to the KL basis. Of course, to make this statement precise one has to be careful about the orderings of the bases, but we ignore that for the moment.

Now observe that, on the one hand, the GT basis depends on a choice of tower (1.1). Indeed, there are many possible choices, and each one results in a different GT basis which we call “generalised GT bases” (Definition 3.1). On the other hand, the KL basis is *canonical* and so is independent of any choices. Therefore, it is reasonable to expect

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that the KL basis is upper triangular with respect to *any* tower. This is exactly what we prove (Theorem 4.3).

Our proof of Theorem 4.3 is independent of Garsia and MacLarnan's work. Moreover, the generalised GT bases are not upper triangular with respect to each other (cf. Example 4.6), so it is hopeless to try to bootstrap our result from the standard case to all generalised GT bases in the obvious manner.

Instead, our argument is based on two important inputs: the classical result about the action of the longest element on the KL basis (Theorem 2.1), and recent work of the second author with Gossow on the compatibility of the KL basis with restriction (Proposition 2.2). We also note that in order to formulate our theorem we introduce variations of the evacuation operator and dominance order on standard Young tableaux (Definition 4.1), which may be interesting in their own right.

2 Background

Throughout we work over the field of rational numbers \mathbb{Q} . Let $[a, b] = \{a, a+1, \dots, b\}$, for $1 \leq a < b$. Given a set X , $S_X = \text{Aut}(X)$ denotes the group of permutations of X and $S_n = S_{[1, n]}$. Let $w_0 \in S_n$ be the longest element, which interchanges i and $n-i+1$ for every i , and more generally for an interval $I \subseteq [1, n]$, let $w_I \in S_I$ be the longest element. Given a subgroup $H \leq S_n$ set $\bar{H} = w_0 H w_0$.

For groups $H < G$ and a representation V of G , let $\text{Res}_H^G(V)$ denote the restriction of V to H . Given an isomorphism class λ of an irreducible representation of G , let $\text{Iso}_\lambda(V)$ denote the λ -isotypic component of V . The irreducible Specht module [9, Definition 2.3.4] of S_n indexed by a partition $\lambda \vdash n$ is denoted V^λ .

Recall that $\dim(V^\lambda) = \#\text{SYT}(\lambda)$, where $\text{SYT}(\lambda)$ denotes the set of standard Young tableaux of shape λ . These are λ -shaped arrays of numbers which are increasing along rows from left to right, and down columns. For example, $\begin{smallmatrix} 1 & 2 & 4 \\ 3 & 5 \end{smallmatrix} \in \text{SYT}(3, 2)$. We number the rows from top to bottom, and let $\varepsilon_k(T)$ denote the row number containing k . For example, for the tableau above, $\varepsilon_3(T) = 2$ and $\varepsilon_4(T) = 1$. Let $\text{sh}(T)$ denote the shape of the tableau T .

Recall Schützenberger's evacuation operator $\text{ev} : \text{SYT}(\lambda) \rightarrow \text{SYT}(\lambda)$ is an involution on the set of standard Young tableaux [10, Definition A1.2.8]. For convenience, we also write $\bar{T} := \text{ev}(T)$. Most importantly for us, evacuation encodes the action of w_0 on the Kazhdan–Lusztig basis (cf. Theorem 2.1 below). For $\lambda \vdash n$, set:

$$\begin{aligned} \text{SYT}_k(\lambda) &= \{T \in \text{SYT}(\lambda) \mid \varepsilon_n(T) = k\}, \\ \text{SYT}_{\bar{k}}(\lambda) &= \{T \in \text{SYT}(\lambda) \mid \varepsilon_n(\bar{T}) = k\}. \end{aligned}$$

Let $d(T)$ denote the tableau obtained from $T \in \text{SYT}(\lambda)$ by removing the n -box. Conversely, suppose $\mu \vdash n-1$ is obtained by removing a box from λ . Then for $S \in \text{SYT}(\mu)$,

$S^+ \in \text{SYT}(\lambda)$ is the tableau obtained by adding an n -box in the correct row. The notation S^+ is ambiguous, as it depends on λ , but this will always be clear from context.

Of central interest for us is the Kazhdan–Lusztig (KL) basis of V^λ , denoted $\{c_T \mid T \in \text{SYT}(\lambda)\}$. This is a canonical basis obtained from the (left) KL cellular representation of the Hecke algebra of S_n [6, 3]. The KL basis has close connections to the geometry of Schubert varieties and the Kazhdan–Lusztig conjectures. It is difficult to compute in general (it is precisely as difficult to compute as the Kazhdan–Lusztig polynomials), and in particular the action of a generic permutation on basis elements c_T is hard to express. Nevertheless, we have the following remarkable classical result about the action of the longest element, which is due separately to Berenstein–Zelevinsky, Mathas and Stembridge:

Theorem 2.1 ([2, 8, 11]). *Let $\lambda \vdash n$ and let $T \in \text{SYT}(\lambda)$. Then $w_0 \cdot c_T = \pm c_{\bar{T}}$, and the sign depends only on λ .*

In recent work of the second author with Gossow, we generalised the above theorem to the class of separable permutations [4, 5]. For present purposes, we recall a result from these works that will play a central role.

Let $\lambda \vdash n$ and suppose λ has r removable boxes, which appear in rows $1 \leq a_1 < a_2 < \dots < a_r$. Let $\mu_k \vdash n-1$ be the partition obtained by removing the box in row a_k , and let $V_a^\lambda := \text{span}\{c_T \mid \varepsilon_n(T) \leq a\}$. This defines a filtration:

$$0 \subset V_{a_1}^\lambda \subset V_{a_2}^\lambda \subset \dots \subset V_{a_r}^\lambda = V^\lambda.$$

Proposition 2.2 ([4, Theorem 1.2]). *The subspaces $V_{a_k}^\lambda$ are S_{n-1} invariant, and the map*

$$f_k : V_{a_k}^\lambda / V_{a_{k-1}}^\lambda \longrightarrow V^{\mu_k}, \quad c_T + V_{a_{k-1}}^\lambda \longmapsto c_{d(T)}$$

is an isomorphism of S_{n-1} -modules.

Below we use binary sequences to label chains of subgroups. Let \mathbb{B}_n be the set of binary sequences of length n . Given $b = b_1 \dots b_n \in \mathbb{B}_n$, set

$$\begin{aligned} \bar{b} &= \bar{b}_1 \dots \bar{b}_n, \\ b_+ &= b_2 \dots b_n, \end{aligned}$$

where $\bar{0} = 1$ and vice-versa.

3 Generalised Gelfand–Tsetlin bases

A **multiplicity-free chain** of S_n is a sequence of groups $(G_1, G_2, \dots, G_{n-1})$ such that

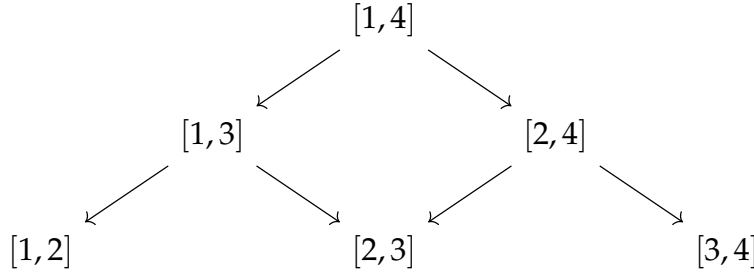
$$S_n = G_1 \supset G_2 \supset \dots \supset G_{n-1},$$

and G_m is a standard parabolic subgroup of S_n isomorphic to S_{n-m+1} . The **standard chain** is

$$(S_n, S_{n-1}, \dots, S_2).$$

Writing $G_m = S_{I_m}$ for I_m an interval, I_{m+1} is obtained from I_m by either removing the smallest or largest element. We can thus index the multiplicity-free chains by binary sequences $b = b_1 \cdots b_{n-2}$, where $b_{m+1} = 0$ if I_{m+1} is obtained from I_m by removing the largest element, and is 1 otherwise. The standard chain corresponds to the zero sequence.

For example, if $n = 4$ we have four multiplicity-free chains, corresponding to the four two-step paths in the directed graph below:



From left to right, these chains are labelled by 00, 01, 10 and 11. In general, S_n has $N := 2^{n-2}$ multiplicity-free chains. Note that our multiplicity-free chains stop at rank S_2 since there is no “choice” of S_1 at the next step.

Let (G_1, \dots, G_{n-1}) be a multiplicity-free chain labelled by b , and let V be an irreducible representation of S_n .

Definition 3.1. A nonzero vector $v \in V$ is a *b -Gelfand–Tsetlin (GT) vector* if for $1 \leq m < n$, there exist $\mu^m \vdash n - m + 1$ such that

$$v \in \text{Iso}_{\mu^m} \text{Res}_{G_m}^{S_n}(V).$$

In this case, we say v is of **type** T , where $T \in \text{SYT}(\lambda)$ corresponds to the sequence of partitions $(\mu^1, \mu^2, \dots, \mu^{n-1}, (1))$. A basis $\{v_T^b \mid T \in \text{SYT}(\lambda)\}$ is a *b -GT basis* if for all T , v_T^b is a b -GT vector of type T . We call these **generalised GT bases**.

The following proposition is an immediate consequence of the classical fact that the restriction of any irreducible representation of G_m to G_{m+1} is multiplicity-free:

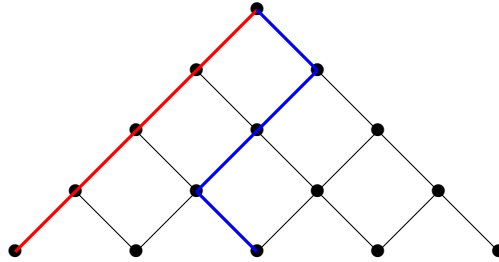
Proposition 3.2. Let $\lambda \vdash n$ and let $b \in \mathbb{B}_{n-2}$. Then there exists a b -GT basis $\{v_T^b \mid T \in \text{SYT}(\lambda)\}$ of V^λ which is unique up to scalar factors.

The b -GT basis is related to the standard GT basis by the action of S_n , as we now explain. First note that a permutation $w \in S_n$ acts on a multiplicity-free chain by conjugation:

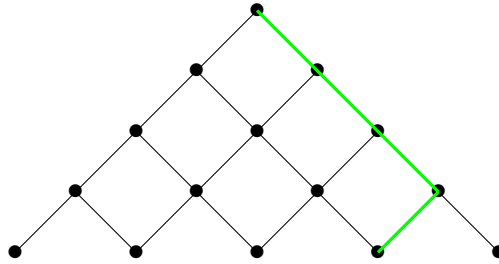
$$w \cdot (G_1, \dots, G_{n-1}) = (wG_1w^{-1}, \dots, wG_{n-1}w^{-1}).$$

This may not be a multiplicity-free chain since the conjugate of a standard parabolic subgroup is not necessarily a standard parabolic. But the multiplicity-free chains are always related in this way. This is best explained by example.

Consider the two chains in S_6 in the figure below. To simplify the picture we replaced the intervals with nodes, so here the red path is the standard chain and the blue path is the chain $([1, 6], [2, 6], [2, 5], [2, 4], [3, 4])$. We'll find a permutation $u \in S_6$ that conjugates the standard chain to the blue one.



The key point is that given a multiplicity-free chain, suppose the j -th group is the symmetric group on an interval I . Then the action of the long element w_I reflects the chain about the vertical ray pointed downwards from the node labelled by I . It doesn't affect the part of the chain above that node. For example, the action of the long element $w_{[2,6]}$ flips the blue chain above to:



Now, if we begin with the red path and reflect at the first node, then at the second node and then at the fourth node we obtain the blue path. These nodes correspond to the intervals $[1, 6]$, $[2, 6]$ and $[2, 4]$ respectively. Therefore the desired permutation is given by $u = w_{[2,4]}w_{[2,6]}w_{[1,6]} \in S_6$, and is uniquely determined by the chain we started with. We note that u is always a *separable* permutation, which is by definition a permutation that avoids the patterns 2413 and 3142 [7].

Lemma 3.3. *Let (G_1, \dots, G_{n-1}) be a multiplicity-free chain labelled by b and corresponding to the permutation u . Then for every $T \in \text{SYT}(\lambda)$, $u \cdot v_T^0 = v_T^b$ (up to scalar factors). Consequently, $w_0 \cdot v_T^b = v_T^{\bar{b}}$.*

Proof. Let (μ^1, \dots, μ^n) be the sequence of partitions corresponding to T . The vector v_T^0 is

determined (up to scalar factors) by the containments $v_T^0 \in \text{Iso}_{\mu^m}(\text{Res}_{S_{n-m+1}}^{S_n}(V^\lambda))$. Since $uS_{n-m+1}u^{-1} = G_m$, $u \cdot v_T^0 \in \text{Iso}_{\mu^m}(\text{Res}_{G_m}^{S_n}(V^\lambda))$, and so $u \cdot v_T^0 = v_T^b$ (up to scalar factors).

Now, note that \bar{b} labels the multiplicity-free chain $(\bar{G}_1, \dots, \bar{G}_{n-1})$, and the corresponding permutation for this chain is w_0u . Therefore $w_0 \cdot v_T^b = w_0u \cdot v_T^0 = v_T^{\bar{b}}$. \square

4 The main theorem

To state our main result we introduce a family of partial orders and bijections on $\text{SYT}(\lambda)$ labelled by binary sequences.

Definition 4.1. Let $\lambda \vdash n$ and $b \in \mathbb{B}_{n-2}$.

1. Define a bijection $\varphi_b = \varphi_{\lambda,b} \in \text{Aut}(\text{SYT}(\lambda))$ recursively:

$$\varphi_{\lambda,b}(T) = \begin{cases} \varphi_{\mu,b_+}(d(T))^+ & \text{if } b_1 = 0, \\ \varphi_{\mu,\bar{b}_+}(d(\bar{T}))^+ & \text{if } b_1 = 1, \end{cases}$$

where $\mu = \text{sh}(d(T))$ or $\text{sh}(d(\bar{T}))$.

2. Define a linear order $\leq_b = \leq_{\lambda,b}$ recursively: $S \leq_{\lambda,b} T$ if and only if:

$$\begin{cases} \varepsilon_n(S) < \varepsilon_n(T) \text{ or } \varepsilon_n(S) = \varepsilon_n(T) \text{ and } d(S) \leq_{\mu,b_+} d(T) \text{ if } b_1 = 0, \\ \varepsilon_n(\bar{S}) < \varepsilon_n(\bar{T}) \text{ or } \varepsilon_n(\bar{S}) = \varepsilon_n(\bar{T}) \text{ and } d(\bar{S}) \leq_{\mu,\bar{b}_+} d(\bar{T}) \text{ if } b_1 = 1, \end{cases}$$

where $\mu = \text{sh}(d(T))$ or $\text{sh}(d(\bar{T}))$.

In the table below we describe the bijection and linear order in some examples. The order is described by associating a sequence of numbers $\text{sig}_b(T)$ to a tableau T , and ordering the sequences *reverse lexicographically*.

b	$\varphi_b(\mathbf{T})$	$\text{sig}_b(\mathbf{T})$
00...0	id	$(\varepsilon_1(T), \dots, \varepsilon_n(T))$
11...1	\bar{T}	$(\varepsilon_1(\bar{T}), \dots, \varepsilon_n(\bar{T}))$
01...1	$(\overline{d(T)})^+$	$(\varepsilon_1(\bar{T}), \dots, \varepsilon_{n-1}(\bar{T}), \varepsilon_n(T))$
10...0	$(\overline{d(\bar{T})})^+$	$(\varepsilon_1(\overline{d(\bar{T})}), \dots, \varepsilon_{n-1}(\overline{d(\bar{T})}), \varepsilon_n(\bar{T}))$

Remark 4.2. As pointed out to us by Fern Gossow, there is an alternate way to describe the linear order \leq_b . For example, suppose b corresponds to the chain $([1,5], [1,4], [2,4], [3,4], [3,3])$. Rewrite this as a sequence $(3,4,2,1,5)$, describing which element gets added moving up the chain, and proceed as follows:

1. Begin with a tableau of a single box labelled 3. Make this the root of an ordered tree.
2. Next insert 4. There are two rows to put this box. Make two branches of the tree for these two choices, where the small branch is the higher row.
3. Next insert 2. Again we order the outcomes by the inserted rows. Since 2 is small, we jeu-de-taquin slide the 2 into each tableaux.
4. Continue in this way. For large values, just place by rows. For small values, place by rows and then jeu-de-taquin slide them in. Don't place a box in rows that make the tableaux bigger than your desired shape.

In the end we get the ordering \leq_b on standard Young tableaux.

We can now state our main result, and make a few remarks immediately following:

Theorem 4.3. *Let $\lambda \vdash n$ and $b \in \mathbb{B}_{n-2}$. Then for any $T \in \text{SYT}(\lambda)$,*

$$c_T = a_T v_{\varphi_b(T)}^b + \sum_{S \leq_b T} a_S v_{\varphi_b(S)}^b, \quad (4.1)$$

where $a_T \neq 0$. Moreover, the b -GT basis can be normalised so that the coefficients $a_T, a_S \in \mathbb{Z}$.

Remark 4.4. The coefficients a_T, a_S that appear in (4.1) depend on b . This is perhaps not clear from our notation, which we chose to declutter the formula. Note, the set of b -GT bases can be normalised so that the coefficients a_T, a_S are minimally integral in the following sense: $\{a_S \mid S \leq_b T\}$ is integral and at least one pair of elements are coprime. In this case, the leading term coefficient a_T is independent (up to sign) of $b \in \mathbb{B}_{n-2}$. We are not aware of an interesting (geometric) interpretation of these coefficients.

Remark 4.5. If $b = 0 \cdots 0$, i.e. we consider the standard GT basis, then this result can be deduced from [3, 1]. Indeed, Garsia and MacLarnan prove that the KL basis is integral and unitriangular with respect to Young's natural basis. The latter is not the standard GT basis, but Armon and Halverson prove that it is upper triangular with respect to the seminormal basis, which is the standard GT basis. Moreover, the relevant orderings are compatible, from which the desired result follows. From this argument we lose integral unitriangularity, and indeed in general we cannot impose that the leading coefficient $a_T = 1$ and the other coefficients are integral in (4.1).

Example 4.6. Consider the first non-trivial case: $\lambda = (2, 1) \vdash 3$. We realise V^λ as the subspace in \mathbb{C}^3 of vectors whose entries sum to zero. Set $S = \begin{smallmatrix} 1 & 3 \\ 2 \end{smallmatrix}$ and $T = \begin{smallmatrix} 1 & 2 \\ 3 \end{smallmatrix}$. Then $c_S = (1, -1, 0)$ and $c_T = (0, 1, -1)$.

If $b = 0$ then $\varphi_0 = id$ and $S <_0 T$. The 0-GT basis is $v_S^0 = \frac{1}{2}c_S$ and $v_T^0 = \frac{1}{2}(1, 1, -2)$ (we choose the normalisation so that the change of basis matrix is integral). The upper-triangularity result is simply

$$\begin{aligned} c_S &= 2v_S^0, \\ c_T &= v_T^0 - v_S^0. \end{aligned}$$

On the other hand, if $b = 1$ then φ_1 interchanges S and T and $T <_1 S$. The 1-GT basis is $v_S^1 = \frac{1}{2}(0, 1, -1)$ and $v_T^1 = \frac{1}{2}(2, -1, -1)$. The upper-triangularity in this case is

$$\begin{aligned} c_T &= 2v_{\varphi_1(T)}^1, \\ c_S &= v_{\varphi_1(S)}^1 - v_{\varphi_1(T)}^1. \end{aligned}$$

5 Proof of Theorem 4.3

The remainder of the paper is devoted to the proof of Theorem 4.3. Let $b = b_1 \cdots b_{n-2} \in \mathbb{B}_{n-2}$ label the multiplicity-free chain (G_1, \dots, G_{n-1}) . First note that the second part of the theorem is a trivial consequence of the first: if two bases are related by a rational matrix, then one can rationally rescale one of the bases so that the matrix is integral. We now proceed to prove Theorem 4.3(1) by induction on $n \geq 2$.

If $n = 2$ the result is trivial, and we normalise the GT basis in this case to equal the KL basis. Suppose now that $n > 2$ and the result holds for S_{n-1} . Let $\lambda \vdash n$ and suppose λ has r removable boxes, which appear in rows $1 \leq a_1 < a_2 < \cdots < a_r$. Let $\mu_k \vdash n-1$ be the partition obtained by removing the box in row a_k . Then, for any $1 \leq k \leq r$ and $T \in \text{SYT}(\mu_k)$, by induction we have an equality in V^{μ_k} :

$$c_{d(T)} = a_{d(T)} v_{\varphi_{\mu_k, b_+}(d(T))}^{b_+} + \sum_{\substack{S \in \text{SYT}(\mu_k) \\ S <_{\mu_k, b_+} d(T)}} a_S v_{\varphi_{\mu_k, b_+}(S)}^{b_+}, \quad (5.1)$$

with rational coefficients.

Recall from Proposition 2.2 we have a filtration

$$0 \subset V_{a_1}^\lambda \subset \cdots \subset V_{a_r}^\lambda = V^\lambda,$$

and an isomorphism of S_{n-1} -modules $f_k : V_{a_k}^\lambda / V_{a_{k-1}}^\lambda \rightarrow V^{\mu_k}$ given by $c_T + V_{a_{k-1}}^\lambda \mapsto c_{d(T)}$. Therefore, in the unique decomposition $V^\lambda = V_1 \oplus \cdots \oplus V_r$, where $V_r \cong V^{\mu_k}$ as S_{n-1} -modules, $V_{a_k}^\lambda = V_1 \oplus \cdots \oplus V_k$. The induction now breaks into two cases depending on the value of b_1 .

5.1 Case 1

We first consider the case $b_1 = 0$.

Lemma 5.1. *For any $T \in \text{SYT}(\lambda)$, $f_k(v_T^b + V_{a_{k-1}}^\lambda) = v_{d(T)}^{b_+}$ (up to nonzero scalar factors).*

Proof. Suppose $T \in \text{SYT}_{a_k}(\lambda)$. Then, by definition, $v_T^b \in V_k$, and for $1 < m < n - 1$ there exist $\lambda^m \vdash n - m + 1$ such that

$$v_T^b \in \text{Iso}_{\lambda^m} \text{Res}_{G_m}^{S_{n-1}}(V_k).$$

This shows that $\{v_T^b \mid T \in \text{SYT}_{a_k}(\lambda)\}$ is a b_+ -GT basis of V_k , and moreover v_T^b is of type $d(T)$. Since f_k is an isomorphism of S_{n-1} modules, the result follows. \square

We normalize the b -GT basis of V^λ so that [Lemma 5.1](#) is an equality on the nose. Now we induct on $1 \leq k \leq r$. Let $k = 1$. Applying f_1^{-1} to [\(5.1\)](#) we obtain

$$c_T = a_{d(T)} v_{\varphi_{\mu_1, b_+}(d(T))^+}^b + \sum_{\substack{S \in \text{SYT}(\mu_1) \\ S <_{\mu_1, b_+} d(T)}} a_S v_{\varphi_{\mu_1, b_+}(S)^+}^b.$$

By [Definition 4.1](#), and setting $a_P := a_{d(P)}$ for all $P \in \text{SYT}_{a_1}(\lambda)$, the right hand side can be rewritten as

$$a_T v_{\varphi_{\lambda, b}(T)}^b + \sum_{\substack{Q \in \text{SYT}_{a_1}(\lambda) \\ Q <_{\lambda, b} T}} a_Q v_{\varphi_{\lambda, b}(Q)}^b.$$

This completes the base case of the induction on k .

Now let $k > 1$. Applying f_k^{-1} to [\(5.1\)](#), and using [Definition 4.1](#) as in the base case, we obtain a congruence:

$$c_T \equiv a_T v_{\varphi_{\lambda, b}(T)}^b + \sum_{\substack{Q \in \text{SYT}_{a_k}(\lambda) \\ Q <_{\lambda, b} T}} a_Q v_{\varphi_{\lambda, b}(Q)}^b \pmod{V_{a_{k-1}}^\lambda}.$$

Therefore, there exists some $x \in V_{a_{k-1}}^\lambda$ such that

$$c_T = a_T v_{\varphi_{\lambda, b}(T)}^b + \sum_{\substack{Q \in \text{SYT}_{a_k}(\lambda) \\ Q <_{\lambda, b} T}} a_Q v_{\varphi_{\lambda, b}(Q)}^b + x.$$

Now, $x = \sum_R a_R c_R$, the sum ranging over $R \in \text{SYT}(\lambda)$ such that $\varepsilon_n(R) \leq a_{k-1}$. By induction (on k), [Theorem 4.3](#) holds for c_R . Since $R <_{\lambda, b} T$ this completes the induction in this case.

5.2 Case 2

We now consider the case $b_1 = 1$. The proof here follows a similar strategy, but requires some subtle changes. For any $k \in \mathbb{N}$ define,

$$V_{\bar{k}}^\lambda = \text{span}\{c_T \mid T \in \text{SYT}(\lambda), \varepsilon_n(\bar{T}) \leq k\},$$

$$f_{\bar{k}} : V_{\bar{a}_k}^\lambda / V_{\bar{a}_{k-1}}^\lambda \rightarrow V^{\mu_k}, \quad c_T + V_{\bar{a}_{k-1}}^\lambda \mapsto \pm c_{d(\bar{T})}.$$

where the sign in the definition of $f_{\bar{k}}$ agrees with the one appearing in $w_0 \cdot c_T = \pm c_{\bar{T}}$ (cf. [Theorem 2.1](#)), and in particular it only depends on λ . First note:

Lemma 5.2. *For any a , $w_0(V_a^\lambda) = V_{\bar{a}}^\lambda$, and hence $V_{\bar{a}}^\lambda$ is $\overline{S_{n-1}}$ -invariant.*

Proof. The first statement is an immediate consequence of that fact that evacuation is an involution and [Theorem 2.1](#). The second now follows since V_a^λ is S_{n-1} -invariant. \square

By this lemma we can regard $V_{\bar{a}}^\lambda$ as an S_{n-1} -module by twisting the action with w_0 , i.e. by transferring the structure via the isomorphism $w_0 : V_a^\lambda \rightarrow V_{\bar{a}}^\lambda$.

Lemma 5.3. *The map $f_{\bar{k}}$ is an isomorphism of S_{n-1} -modules.*

Proof. Consider the diagram

$$\begin{array}{ccc} V_{\bar{a}_k}^\lambda / V_{\bar{a}_{k-1}}^\lambda & \xrightarrow{w_0} & V_{a_k}^\lambda / V_{a_{k-1}}^\lambda \\ & \searrow f_{\bar{k}} & \downarrow f_k \\ & & V^{\mu_k} \end{array}$$

We claim this commutes. Indeed, following the diagram right and down we get:

$$\begin{aligned} c_T + V_{\bar{a}_{k-1}}^\lambda &\mapsto w_0(c_T) + V_{a_{k-1}}^\lambda \\ &= \pm c_{\bar{T}} + V_{a_{k-1}}^\lambda \\ &\mapsto \pm c_{d(\bar{T})}, \end{aligned}$$

agreeing with the value of $c_T + V_{\bar{a}_{k-1}}^\lambda$ under $f_{\bar{k}}$. Therefore $f_{\bar{k}}$ is a composition of S_{n-1} -module isomorphisms. \square

Lemma 5.4. *For any $T \in \text{SYT}(\lambda)$, $f_{\bar{k}}(v_T^b + V_{\bar{a}_{k-1}}^\lambda) = \pm v_{d(T)}^{\bar{b}_+}$ (up to nonzero scalar factors).*

Proof. By [Lemma 3.3](#), $v_T^b = w_0 \cdot v_T^{\bar{b}}$. Hence

$$\begin{aligned} f_{\bar{k}}(v_T^b + V_{\bar{a}_{k-1}}^\lambda) &= f_{\bar{k}}(w_0 \cdot v_T^{\bar{b}} + V_{\bar{a}_{k-1}}^\lambda) \\ &= f_k(v_T^{\bar{b}} + V_{a_{k-1}}^\lambda) \\ &= v_{d(T)}^{\bar{b}_+} \end{aligned}$$

Note the second equality uses the commutative diagram in [Lemma 5.3](#) and the third equality uses [Lemma 5.1](#). \square

To continue the proof, now we induct on $1 \leq k \leq r$ in the filtration:

$$0 \subset V_{\bar{a}_1}^\lambda \subset \cdots \subset V_{\bar{a}_r}^\lambda = V^\lambda.$$

Let $k = 1$. Consider (5.1) with T replaced by \bar{T} , and b by \bar{b} . Applying f_1^{-1} to this, we obtain

$$c_T = \pm a_{d(T)} v_{\varphi_{\mu_1, \bar{b}_+}(d(\bar{T}))}^b + \sum_{\substack{S \in \text{SYT}(\mu_1) \\ S <_{\mu_1, \bar{b}_+} d(\bar{T})}} a_S v_{\varphi_{\mu_1, \bar{b}_+}(S)}^b.$$

As in the previous case, the right hand side can be rewritten as

$$a_T v_{\varphi_{\lambda, b}(T)}^b + \sum_{\substack{Q \in \text{SYT}_{\bar{a}_1}(\lambda) \\ Q <_{\lambda, b} T}} a_Q v_{\varphi_{\lambda, b}(Q)}^b,$$

proving the base case of the induction on k . One can similarly complete the inductive step as in the previous case, and thus the proof of part (1) of [Theorem 4.3](#) is done.

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