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Chow polynomials of posets

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Abstract. Three decades ago, Stanley and Brenti initiated the study of the Kazhdan–Lusztig–Stanley (KLS) functions, putting on common ground several polynomials appearing in algebraic combinatorics, discrete geometry, and representation theory. In the present paper we develop a theory that parallels the KLS theory. To each kernel in a given poset, we associate a polynomial function that we call the *Chow function*. The Chow function often exhibits remarkable properties, and sometimes encodes the graded dimensions of a cohomology or Chow ring. The framework of Chow functions provides natural polynomial analogs of graded module decompositions that appear in algebraic geometry, but that work for arbitrary posets, even when no graded module decomposition is known to exist. In this general framework, we prove a number of unimodality and positivity results without relying on versions of the Hard Lefschetz theorem. Our framework shows that there is an unexpected relation between positivity and real-rootedness conjectures about chains on face lattices of polytopes by Brenti and Welker, Hilbert–Poincaré series of matroid Chow rings by Ferroni and Schröter, and flag enumerations on Bruhat intervals of Coxeter groups by Billera and Brenti.

Keywords: partially ordered sets, Kazhdan–Lusztig theory, matroids, face lattices of polytopes, Coxeter groups, real-rooted polynomials, gamma-positivity, unimodality

1 Introduction

In the foundational paper [37], Stanley developed a notable framework to study polynomials arising from partially ordered sets. This puts on common ground and unifies several—a priori unrelated—theories that are of fundamental importance in mathematics. Three prominent examples are i) the enumeration of points, lines, planes, etc. in a matroid, ii) the enumeration of faces in convex polytopes, and iii) the combinatorics and representation theory associated to Coxeter groups. In accordance with another

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influential paper by Brenti [11], we call this the Kazhdan–Lusztig–Stanley (KLS) theory for posets. We also suggest a recent self-contained survey on KLS theory and its algebro-geometric consequences by Proudfoot [32].

Assume that P is a locally finite, weakly ranked, partially ordered set, and let $\mathrm{Int}(P)$ be the set of all closed intervals of P. We denote by $\rho\colon \mathrm{Int}(P) \to \mathbb{Z}$ the weak rank function of P. Consider the incidence algebra $\mathcal{I}(P)$ of P over the univariate polynomial ring $\mathbb{Z}[x]$. The weak rank function ρ gives rise to the subalgebra $\mathcal{I}_\rho(P) \subseteq \mathcal{I}(P)$ consisting of the elements $f \in \mathcal{I}(P)$ such that $\deg f_{st} \leq \rho_{st}$ for each closed interval $[s,t]$. Stanley realized the importance of special elements $\kappa \in \mathcal{I}_\rho(P)$ which are called (P,ρ)-kernels or, when ρ is understood from context, just P-kernels. To each such kernel κ one associates two important elements $f, g \in \mathcal{I}_\rho(P)$. The element f (resp. g) is often called the right (resp. left) Kazhdan–Lusztig–Stanley (KLS) function associated to the (P,ρ)-kernel κ.

In the three examples from the first paragraph, the posets and the kernels are, respectively, i) the lattice of flats of a matroid with the characteristic function as kernel, ii) the face lattice of a convex polytope with the kernel $[s,t] \mapsto (x-1)^{\dim t - \dim s}$, and iii) the strong Bruhat order poset of a Coxeter group with the R-polynomials as kernel. In these three cases the posets are graded and bounded, and the assignment $[s,t] \mapsto \rho_{st}$ is given by the length of an arbitrary saturated chain starting at s and ending at t. Correspondingly, the KLS functions that arise in each of these cases are i) the Kazhdan–Lusztig polynomial of the matroid defined by Elias, Proudfoot, and Wakefield in [17], ii) the toric g-polynomial of the polytope introduced by Stanley [35], and iii) the Kazhdan–Lusztig polynomial(s) of the Coxeter group discovered by Kazhdan and Lusztig [28].

The central contribution of the present work is the introduction of a new class of functions, that we call *Chow functions*, associated to any (P,ρ)-kernel κ. As opposed to the case of the KLS functions where a convention of left versus right constitutes an essential part of the definition, in our case there is a single distinguished element $\mathrm{H} \in \mathcal{I}_\rho(P)$ called the κ-Chow function associated to (P,ρ). Notably, the KLS functions are required to satisfy a very restrictive degree bound: $\deg f_{st} < \frac{1}{2}\rho_{st}$ and $\deg g_{st} < \frac{1}{2}\rho_{st}$ for each $s < t$. In our case, the Chow function H satisfies a weaker degree bound: $\deg \mathrm{H}_{st} < \rho_{st}$ for each $s < t$ but, in order to compensate the additional degrees of freedom, one imposes that the polynomials H_{st} are *palindromic*.

As we will demonstrate in this paper, Chow functions and KLS functions are tightly related to each other. Often, properties of one have an impact on the other. The most significant example of this phenomenon in the our paper is the following result.

Theorem 1.1. *Let κ be a (P,ρ)-kernel. If the right KLS function f or the left KLS function g is non-negative, then the Chow function* H *is non-negative and unimodal.*

Whenever we say that an element $a \in \mathcal{I}(P)$ is non-negative (resp. unimodal, symmetric, γ-positive, etc.) we mean that each of the polynomials $a_{st}(x)$ is non-negative (resp. unimodal, symmetric, γ-positive, etc.)

We prove Theorem 1.1 motivated by a module decomposition called the *canonical decomposition* of the matroid Chow ring in [10]. Furthermore, our proof is entirely combinatorial, in the sense that we do not deal with any algebraic structures but only with polynomials. Theorem 1.1 yields unimodality results in the three aforementioned main examples. This is because the KLS functions were proved to be non-negative in groundbreaking papers: i) by Braden, Huh, Matherne, Proudfoot, and Wang [10] via the introduction of the matroid intersection cohomology[1]; ii) by Karu in [25] building upon earlier work of McMullen [29], Barthel, Brasselet, Fieseler, and Kaup [7], and Bressler and Lunts [13]; and iii) by Elias and Williamson [18] via the machinery of Soergel bimodules [34], and relying on techniques by De Cataldo and Migliorini [14, 15].

The main inspiration for our definition of Chow functions, and the reason for choosing the name, stems from the first on-going example concerning matroids. The Chow function encodes the Hilbert series of the Chow rings of all minors of a matroid. These Chow rings were introduced by Feichtner and Yuzvinsky in [21] and played a primary role in the resolution of long-standing conjectures in combinatorics [2, 3, 10]; for amenable surveys we refer to [30, 24, 4, 19]. The case of Chow functions arising from matroids was studied in a previous paper together with Matthew Stevens [22].

A further motivation to develop the theory in the present paper was to understand to what extent one can hope to derive other versions of some crucial module decompositions concerning matroid intersection cohomologies, by Braden, Huh, Matherne, Proudfoot, and Wang in [9, 10]. We came to realize that a number of the module decompositions that constitute the intricate induction appearing in [10] can be shadowed step by step, but working instead with *polynomials* rather than *graded modules*. There are some advantages in this approach.

- Our framework does not require us to work with matroids nor posets with characteristic polynomials displaying any specific sign pattern in their coefficients. More so, we can apply these constructions to the examples of face lattices of polytopes or Bruhat intervals, by considering the corresponding kernel.

- We are able to state results that would not be possible to obtain by taking graded dimensions of any module or ring. A priori, our identities may involve polynomials that cannot possibly be Hilbert series or Poincaré polynomials, e.g., when one of the coefficients is negative.

- We are able to provide combinatorial proofs of statements that were known to be valid via the application of difficult results from algebraic geometry, and we achieve so for more general classes of posets.

- This framework is amenable to build upon intuition from one setting (say, polytopes) and use it in another one (say, Coxeter groups). For example, the use of

[1] We note, however, that the *left* KLS function is trivial in this case.

the **cd**-index in the case of polytopes led us to consider the complete **cd**-index of Bruhat intervals introduced by Billera–Brenti [8].

In addition to the key object $\mathrm{H} \in \mathcal{I}_\rho(P)$ introduced in this paper, we also study two related functions: $F \in \mathcal{I}_\rho(P)$ the *right augmented Chow function*, and $G \in \mathcal{I}_\rho(P)$ the *left augmented Chow function*. These are obtained by convolving H with the right and left KLS functions respectively, and they also exhibit remarkable properties. The element G plays a key role in the singular Hodge theory of matroids [10], where it encodes the Hilbert–Poincaré series of augmented Chow rings (hence the name), but F is more subtle. Due to space reasons, we do not address them in this extended abstract.

2 The definition and the three main examples

A *weak rank function* on P is a map $\rho\colon \operatorname{Int}(P) \to \mathbb{Z}_{\geq 0}$ satisfying the following properties:

1. If $s < t$, then $\rho_{st} > 0$.
2. If $s \leq w \leq t$, then $\rho_{st} = \rho_{sw} + \rho_{wt}$.

Observe that the second condition guarantees that $\rho_{ss} = 0$ for every $s \in P$. By definition, a *weakly ranked poset* is a pair (P, ρ) consisting of a partially ordered set P and a weak rank function ρ on P. We note explicitly that it is *not* required that ρ be combinatorially invariant. If P has a minimum element $\widehat{0}$, we will often write $\rho(w) := \rho_{\widehat{0},w}$ for any $w \in P$.

A weak rank function ρ on a locally finite poset P induces a special subalgebra $\mathcal{I}_\rho(P) \subseteq \mathcal{I}(P)$, defined by

$$\mathcal{I}_\rho(P) = \{a \in \mathcal{I}(P) : \deg a_{st}(x) \leq \rho_{st} \text{ for all } s \leq t \text{ in } P\}. \tag{2.1}$$

This subalgebra admits an involution $a \mapsto a^{\mathrm{rev}}$ defined via the following identity:

$$\left(a^{\mathrm{rev}}\right)_{st}(x) = x^{\rho_{st}}\, a_{st}(x^{-1}). \tag{2.2}$$

The name "rev" stems from the fact that this involution reverses (with respect to the weak rank function) the coefficients of the polynomials associated to each interval.[2] It is immediate from the definition that this involution respects products, that is, $(ab)^{\mathrm{rev}} = a^{\mathrm{rev}} \cdot b^{\mathrm{rev}}$. Similarly, whenever $a \in \mathcal{I}_\rho(P)$ is invertible, our involution commutes with taking inverses $(a^{-1})^{\mathrm{rev}} = (a^{\mathrm{rev}})^{-1}$.

A key object in the subalgebra $\mathcal{I}_\rho(P)$ is the *characteristic function*, denoted by χ. It is defined by

$$\chi = \mu \cdot \zeta^{\mathrm{rev}} = \zeta^{-1} \cdot \zeta^{\mathrm{rev}}. \tag{2.3}$$

[2] We warn the reader that in other sources this involution is denoted by $a \mapsto \overline{a}$; however, in the present work we will reserve that notation for a different operation.

More explicitly, to each interval $[s,t]$ of P we associate the polynomial

$$\chi_{st}(x) = \sum_{s\leq w\leq t} \mu_{sw}\, x^{\rho_{wt}}.$$

Whenever P is bounded, the polynomial $\chi_P(x) := \chi_{\hat{0}\hat{1}}(x)$ will often be called the *characteristic polynomial of P*.

From the basic properties of the involution rev, we have that the characteristic function enjoys an important property:

$$\chi^{\mathrm{rev}} = \left(\zeta^{-1}\cdot\zeta^{\mathrm{rev}}\right)^{\mathrm{rev}} = (\zeta^{\mathrm{rev}})^{-1}\cdot\zeta = (\zeta^{\mathrm{rev}})^{-1}\cdot\left(\zeta^{-1}\right)^{-1} = \left(\zeta^{-1}\cdot\zeta^{\mathrm{rev}}\right)^{-1} = \chi^{-1}. \tag{2.4}$$

In other words, inverting χ reverses its coefficients. This motivates a key definition.

Definition 2.1. *Let (P,ρ) be a weakly ranked poset. An element $\kappa\in\mathcal{I}_\rho(P)$ is said to be a* (P,ρ)-kernel *if $\kappa_{ss}(x)=1$ for all $s\in P$ and*

$$\kappa^{-1} = \kappa^{\mathrm{rev}}.$$

The corresponding reduced (P,ρ)-kernel *is the element $\overline{\kappa}\in\mathcal{I}_\rho(P)$ given by*

$$\overline{\kappa}_{st}(x) = \begin{cases} \frac{1}{x-1}\,\kappa_{st}(x) & \text{if } s<t\\ -1 & \text{if } s=t.\end{cases}$$

This definition implicitly carries the fact that for a (P,ρ)-kernel, the polynomials $\kappa_{st}(x)$ are divisible by $x-1$ whenever $s<t$. Now we are ready to define Chow functions.

Definition 2.2. *Let κ be a (P,ρ)-kernel. We define the* Chow function *associated to κ, or* κ-Chow function, *as the element $\mathrm{H}\in\mathcal{I}_\rho(P)$ defined by*

$$\mathrm{H} = -\left(\overline{\kappa}\right)^{-1}.$$

*If the poset P is bounded, the polynomial $\mathrm{H}_P(x) = \mathrm{H}_{\hat{0}\hat{1}}(x)$ will be customarily called the κ-*Chow polynomial *of the poset.*

As a consequence of having defined $\overline{\kappa}_{ss}(x)$ as -1 for every $s\in P$, the minus sign appearing in the above definition guarantees that $\mathrm{H}_{ss}(x)=1$ for every $s\in P$. In the subsequent sections of this article we will focus our attention on a number of interesting examples of Chow functions. Notice that our definition of Chow functions as $-(\overline{\kappa})^{-1}$ is equivalent to either of the following properties:

$$\mathrm{H}_{st}(x) = \sum_{s<w\leq t}\overline{\kappa}_{sw}(x)\,\mathrm{H}_{wt}(x) \qquad \text{or, dually,} \tag{2.5}$$

$$\mathrm{H}_{st}(x) = \sum_{s\leq w<t}\mathrm{H}_{sw}(x)\,\overline{\kappa}_{wt}(x), \qquad \text{for all } s<t \text{ in } P. \tag{2.6}$$

The following is the basic toolkit of properties that general Chow functions satisfy regarding degree and symmetry.

Proposition 2.3. *Let κ be a (P,ρ)-kernel, and let $\mathrm{H} \in \mathcal{I}_\rho(P)$ be the corresponding Chow function. Then, the following properties hold true:*

1. *For every $s < t$, we have that*

$$[x^{\rho_{st}-1}]\mathrm{H}_{st}(x) = [x^{\rho_{st}}]\kappa_{st}(x).$$

   In particular, if $\deg \kappa_{st} = \rho_{st}$ for every $s \leq t$, we have that $\deg \mathrm{H}_{st} = \rho_{st} - 1$ for every $s < t$.

2. *The Chow function is symmetric, i.e.,*

$$\mathrm{H}_{st}(x) = x^{\rho_{st}-1}\,\mathrm{H}_{st}(x^{-1}) \qquad \textit{for every } s < t.$$

2.1 Characteristic Chow functions

As is pointed out in [32], the characteristic function $\chi \in \mathcal{I}_\rho(P)$ is a P-kernel in any weakly ranked locally finite poset P, and lattices of flats of matroids are just a special case. In particular, there is no formal obstruction to consider the KLS functions f and g and the Chow function H arising from this setup. For the sake of clarity, we will refer to this Chow function H as the *characteristic Chow function* or, for brevity, the *χ-Chow function* of (P,ρ). In the matroid setting, one has the following result.

Theorem 2.4. *Let M be a loopless matroid and let $P = \mathcal{L}(\mathsf{M})$ be its lattice of flats. Then the characteristic Chow polynomial of P coincides with the Hilbert–Poincaré series of the Chow ring of M. In particular, it is unimodal.*

The first part of the above statement is proved in our prequel [22], whereas the second follows from the Hard Lefschetz theorem, proved by Adiprasito, Huh, and Katz [2].

In [22] we proved a strengthening of unimodality in the above statement: the Hilbert–Poincaré series of a matroid Chow ring is in fact γ-positive [22, Theorem 1.8]. The main tool to prove that was a key result of Braden, Huh, Matherne, Proudfoot, and Wang [9], who established a semi-small decomposition for the Chow ring of a matroid.

In the present paper we deal with much more general posets, for which the Chow ring is not even defined. By applying our numerical analog of the canonical decomposition of matroid Chow rings from [10] we have the following result.

Theorem 2.5. *Let P be any graded bounded poset. The χ-Chow polynomial of P is unimodal.*

Notice that this can be viewed as a corollary of Theorem 1.1, because the left KLS function is identically 1. The latter fact is just equivalent to the inclusion-exclusion principle. Most of the previous proofs of the above unimodality result (for geometric lattices only) relied on versions of the Hard Lefschetz theorem.

Example 2.6. *If $P = C_n$ is a chain on $n \geq 2$ elements, the χ-Chow polynomial of P is given by*

$$\mathrm{H}_P(x) = (x+1)^{n-2}.$$

The above identity can be easily proved by induction. On the other hand, if $P = B_n$ is a Boolean lattice on $n \geq 1$ atoms, the Chow polynomial of P is

$$\mathrm{H}_P(x) = A_n(x),$$

the n-th Eulerian polynomial*, which has i-th coefficient the number of permutations $\sigma \in \mathfrak{S}_n$ having exactly i descents. This can be proved directly by induction, or by using the fact that this is a geometric lattice corresponding to Boolean matroids, whose Chow ring is isomorphic to the cohomology of the permutohedral variety. Note that the example of Boolean lattices shows that characteristic Chow polynomials behave erratically under Cartesian products, because B_n is the n-fold Cartesian product of B_1 with itself.*

Besides unimodality, one may consider the stronger property of being γ-positive. For geometric lattices this property is known to hold true thanks to [22, Theorem 1.8]. We go far beyond geometric lattices and prove the following.

Theorem 2.7. *Let P be any Cohen–Macaulay poset. The χ-Chow polynomial of P is γ-positive.*

For a general Cohen–Macaulay poset there is no obvious way of defining the Chow ring and therefore no clear analogue of the semi-small decomposition of [9]. The proof of the last statement in fact leads to the generalization of a result by Stump [38, Theorem 1.1], which was a key motivation for our strategy.

Example 2.8. *Consider the poset P whose Hasse diagram is depicted in Figure 1. The χ-Chow polynomial equals:*

$$\mathrm{H}_P(x) = x^4 + 7x^3 + 11x^2 + 7x + 1.$$

This polynomial is not γ-positive, because $\gamma_P(x) = -x^2 + 3x + 1$. Of course, one expects that P is not Cohen–Macaulay, which can be seen from the fact that $\beta_P(\{2,4\}) = -1$.

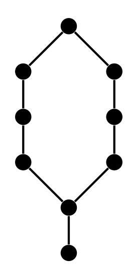

Figure 1: The poset of Example 2.8

In [23, Conjecture 8.18], Ferroni and Schröter conjectured that whenever $P = \mathcal{L}(\mathsf{M})$ is the lattice of flats of a matroid M, then the Hilbert–Poincaré series of the Chow ring of M is a real-rooted polynomial. We formulate the stronger conjecture that this property also holds true for the χ-Chow polynomials of all Cohen–Macaulay posets. Proving our conjecture would also imply another conjecture by Huh on the real-rootedness of Hilbert series of augmented Chow rings of matroids.

Conjecture 2.9. *Let P be any Cohen–Macaulay poset. The χ-Chow polynomial of P is real-rooted.*

2.2 Eulerian Chow functions

Whenever P is an Eulerian poset, the element $\varepsilon \in \mathcal{I}_\rho(P)$ given by $\varepsilon_{st} = (x-1)^{\rho_{st}}$ is a P-kernel. The resulting Chow function will be customarily called the *Eulerian Chow function*, or *ε-Chow function* for brevity, associated to P. We prove the following result.

Theorem 2.10. *The Eulerian Chow polynomial of P equals the h-polynomial of the barycentric subdivision of P.*

By barycentric subdivision of a poset P we mean the simplicial complex whose faces are the flags of elements of P. We do not know whether Eulerian Chow polynomials are always non-negative. Moreover, we explain why we expect this question to be very subtle. By the positivity of the KLS functions proved in certain special cases (e.g., for face posets of simplicial polytopes [36], of general polytopes [25], or of simplicial spheres [1, 31]), Theorem 1.1 guarantees that the ε-Chow function is non-negative and unimodal. However, another deep result by Karu [26] about the **cd**-index of Gorenstein* posets (that is, posets that are both Eulerian and Cohen–Macaulay) can be used to obtain the following stronger property.

Theorem 2.11. *Let P be a Gorenstein* poset. The ε-Chow function of P is γ-positive.*

It is natural to ask whether the above property can be upgraded to real-rootedness. That is equivalent to a long-standing folklore conjecture, posed as an open question by Brenti and Welker [12], when P is the face poset of a polytope.

Conjecture 2.12 (see [12, Question 1])**.** *Let P be the face poset of a polytope (or even just a Gorenstein* poset). Then the ε-Chow polynomial of P is real-rooted.*

The question for Gorenstein* posets is strongly related to questions of Athanasiadis and Tzanaki [6, Question 7.4] and of Athanasiadis and Kalampogia-Evangelinou [5, Question 5.2].

2.3 Coxeter Chow functions

The chief example of KLS functions are precisely the Kazhdan–Lusztig polynomials of Bruhat intervals, defined by Kazhdan and Lusztig in [28]. The kernels in this case are the so-called R-polynomials. A powerful result by Dyer [16] allows for the computation of the R-polynomials via a computation on Bruhat graphs. We use this to prove the following interpretation for the Chow function.

Theorem 2.13. *Let W be a Coxeter group with a reflection order $<$ and two elements $u, v \in W$. Then,*

$$\mathrm{H}_{uv}(x) = \sum_{\Delta \in B(u,v)} x^{\frac{\rho_{uv} - \ell(\Delta)}{2} + \operatorname{asc}(\Delta)} = \sum_{\Delta \in B(u,v)} x^{\frac{\rho_{uv} - \ell(\Delta)}{2} + \operatorname{des}(\Delta)}.$$

In the above statement $B(u, v)$ stands for all the paths in the Bruhat graph of W that go from u to v, $\ell(\Delta)$ stands for the length of the path Δ, des stands for the number of descents of the path, whereas asc stands for the number of ascents. In particular, the Chow function is enumerating these paths according to a descent-like statistic. We show that the combinatorial invariance conjecture for Chow functions is equivalent to the combinatorial invariance conjecture for Kazhdan–Lusztig or R-polynomials.

Example 2.14. *By considering the whole Bruhat poset on $\mathfrak{S}_n$, and denoting the corresponding R-Chow polynomial by $\mathrm{H}_{\mathfrak{S}_n}(x)$, we obtain the following first few values:*

$$\mathrm{H}_{\mathfrak{S}_n}(x) = \begin{cases} 1 & n = 1, \\ 1 & n = 2, \\ x^2 + 3\,x + 1 & n = 3, \\ x^5 + 20\,x^4 + 84\,x^3 + 84\,x^2 + 20\,x + 1 & n = 4, \\ x^9 + 115x^8 + 2856x^7 + 21429x^6 + 56840x^5 + 56840x^4 + 21429x^3 + 2856x^2 + 115x + 1 & n = 5. \end{cases}$$

The sequences of coefficients of these polynomials do not appear in the OEIS [33]. In the authors' opinion, providing a closed formula for these polynomials or, at least, an efficient way of computing them would be very interesting.

Thanks to the breakthrough of Elias and Williamson [18], and as a consequence of Theorem 1.1, we obtain that the above enumeration of paths yields a unimodal polynomial. By shadowing the discussion of the two previous examples, we are led to consider γ-positivity and real-rootedness. In the case of polytopes (or Gorenstein* posets), the key tool to prove γ-positivity is the result on the **cd**-index proved by Karu [26]. In this case, we need to rely on a more complicated non-commutative polynomial called the *complete* **cd**-*index*, introduced by Billera and Brenti [8]. We prove the following.

Theorem 2.15. *Let W be a Coxeter group and let $u < v$ in W. The γ-polynomial of the Coxeter Chow polynomial H_{uv} is a positive specialization of the complete* **cd**-*index of the interval $[u, v]$.*

Billera and Brenti conjecture the non-negativity of the coefficients of the complete **cd**-index for any interval in a Coxeter group [8, Conjecture 6.1]. Some special cases of that conjecture are known to be true (see, e.g., [27], [20]), but it remains open in general. The preceding theorem implies that if Billera and Brenti's conjecture is true, then the Coxeter Chow functions of a Coxeter group are γ-positive; this leads to the next conjecture.

Conjecture 2.16. *Coxeter Chow polynomials of Bruhat intervals of Coxeter groups are γ-positive.*

Emboldened by Conjecture 2.9 and Conjecture 2.12, and numerous experiments on Bruhat intervals of rank ≤ 7, we also pose the following conjecture.

Conjecture 2.17. *Coxeter Chow polynomials of Bruhat intervals of Coxeter groups are real-rooted.*

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