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Subspace profiles, q-Whittaker functions and Krylov methods

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Abstract. Bender, Coley, Robbins, and Rumsey posed the problem of counting the number of subspaces which have a given profile with respect to a linear endomorphism defined on a finite vector space. We settle this problem in full generality by giving an explicit counting formula in terms of a Hall scalar product involving dual q-Whittaker functions and another symmetric function that is determined by conjugacy class invariants of the endomorphism. As corollaries, we obtain new combinatorial interpretations for the coefficients in the q-Whittaker expansions of several symmetric functions. These include the power sum, complete homogeneous, products of modified Hall–Littlewood polynomials, and certain products of q-Whittaker functions. These results are used to derive a formula for the number of anti-invariant subspaces (as defined by Barría and Halmos) with respect to an arbitrary operator. We also give an application to an open problem in Krylov subspace theory.

Keywords: finite field, q-Whittaker function, Hall–Littlewood polynomial, invariant subspace lattice, Krylov subspace, anti-invariant subspace.

1 Introduction

Let $\mathbb{F}_q$ denote the finite field with q elements where q is a prime power. For each positive integer n, write $\mathrm{M}_n(\mathbb{F}_q)$ for the algebra of $n \times n$ matrices over $\mathbb{F}_q$.

Definition 1.1. *Given a matrix $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$, a subspace W of $\mathbb{F}_q^n$ has Δ-profile $\mu = (\mu_1, \mu_2, \ldots)$ if*

$$\dim(W + \Delta W + \cdots + \Delta^{j-1} W) = \mu_1 + \mu_2 + \cdots + \mu_j \text{ for } j \geq 1.$$

Let $\sigma(\mu, \Delta)$ denote the number of subspaces with Δ-profile μ. The Δ-profile of a subspace was referred to as 'dimension sequence' by Bender, Coley, Robbins and Rumsey [5, p. 2] who proposed the following combinatorial problem in 1992.

Problem 1.2. *Given μ and Δ, determine $\sigma(\mu, \Delta)$.*

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They showed, by a beautiful probabilistic argument involving Möbius inversion on the lattice of subspaces, that the $\sigma(\mu, \Delta)$ for μ varying satisfy a system of equations (see Theorem 5.1). They solved these equations in two special cases to obtain elegant product formulas. If Δ is regular nilpotent (nilpotent with one-dimensional null space), then $\sigma(\mu, \Delta) = \prod_{i\geq 2} q^{\mu_i^2} \begin{bmatrix} \mu_{i-1} \\ \mu_i \end{bmatrix}_q$. When Δ is simple (has irreducible characteristic polynomial),

$$\sigma(\mu, \Delta) = \frac{q^n - 1}{q^{\mu_1} - 1} \prod_{i\geq 2} q^{\mu_i^2 - \mu_i} \begin{bmatrix} \mu_{i-1} \\ \mu_i \end{bmatrix}_q . \tag{1.1}$$

They remarked that these formulas do not appear to have simple counting proofs. Several special instances of Problem 1.2 have been solved in the literature [9, 10, 2, 3, 7, 19, 20, 18, 22]. The consideration of subspace profiles when Δ is a regular diagonal operator has recently led to a new proof of the Touchard–Riordan formula concerned with crossings of chord diagrams [18].

In this paper we solve Problem 1.2 in full generality by giving an explicit formula for $\sigma(\mu, \Delta)$ for arbitrary μ and Δ. The fact that the theory of symmetric functions can be leveraged to answer the counting problem is a very recent development. We show that Problem 1.2 admits a compact solution involving q-Whittaker functions which occur as specializations of Macdonald polynomials. The following theorem (stated later as Theorem 5.2) is our main result.

Theorem 1.3. *For each partition μ,*

$$\sigma(\mu, \Delta) = (-1)^{\sum_{j\geq 2} \mu_j} q^{\sum_{j\geq 2} \binom{\mu_j}{2}} \langle F_\Delta(x), \widetilde{W}_\mu(x; q)\, h_{n-|\mu|} \rangle,$$

for each prime power q and each matrix $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$.

Here $\langle \cdot, \cdot \rangle$ denotes the Hall scalar product while h_λ and $\widetilde{W}_\lambda$ denote the complete homogeneous and dual (with respect to the Hall scalar product) q-Whittaker symmetric functions respectively. In addition, $F_\Delta(x)$ is a symmetric function depending on conjugacy class invariants of the matrix Δ that can be expressed in terms of plethystic substitutions involving modified Hall–Littlewood polynomials (see Proposition 2.10).

Several symmetric functions such as the power sum symmetric functions, the complete homogeneous symmetric functions, products of modified Hall–Littlewood polynomials, and certain products of q-Whittaker functions arise as F_Δ for suitably chosen Δ (see Section 2). In the case where μ is a partition of n, the theorem above entails new combinatorial interpretations of the coefficients in the q-Whittaker expansion of each of these symmetric functions (Corollary 4.2).

In Section 6, we use our results to derive an explicit formula for the number of anti-invariant subspaces with respect to a linear operator, a notion that goes back to Barría and Halmos [4]. In Section 7 we give an application of our results to computing a certain probability which is important for understanding and evaluating the efficiency of several algorithms called Krylov subspace methods. These algorithms have applications in many mathematical areas, including quadrature methods, the analytic theory of continued fractions, expansions of infinite series, orthogonalization algorithms, and the mathematical underpinnings of quantum mechanics (Liesen and Strakoš [16, p. 8]).

Detailed proofs of all results in this paper can be found in [21].

2 Generating function for flags of invariant subspaces

We begin with a brief overview of symmetric functions; our references are Macdonald [17] and Stanley [24, Chapter 7]. A weak composition of an integer n is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots)$ of nonnegative integers with sum n. A partition of n is a weak composition of n in which the sequence of integers is weakly decreasing. If λ is a partition of n, we write $\lambda \vdash n$. Nonzero terms in the sequence λ are called parts of λ and the number of parts of λ is denoted $\ell(\lambda)$. We ignore trailing zeroes in weak compositions and partitions; thus, the weak compositions $(3,1,2,0,1,0,0,\ldots)$, $(3,1,2,0,1,0)$ and $(3,1,2,0,1)$ are all considered equivalent. For a weak composition α, write $|\alpha|$ for the sum $\sum_{i\geq 1} \alpha_i$.

Let $\mathbb{Q}(t)$ denote the field of rational functions in an indeterminate t. Denote by $\Lambda_{\mathbb{Q}(t)}$ the algebra of formal symmetric functions in infinitely many variables $x = (x_1, x_2, \ldots)$ with coefficients in $\mathbb{Q}(t)$. The algebra $\Lambda_{\mathbb{Q}(t)}$ admits several natural bases indexed by integer partitions: monomial m_λ, elementary e_λ, power sum p_λ, complete homogeneous h_λ and Schur s_λ. The ring of symmetric functions is also equipped with an involutory automorphism ω which satisfies

$$\omega e_\lambda = h_\lambda; \quad \omega h_\lambda = e_\lambda; \quad \omega s_\lambda = s_{\lambda'}.$$

Here λ' denotes the partition conjugate to λ. The q-Whittaker functions $W_\lambda(x;t)$ and Hall–Littlewood polynomials $P_\lambda(x;t)$ are two more bases of $\Lambda_{\mathbb{Q}(t)}$. Both occur as specializations of a more general class of two-parameter symmetric functions, the Macdonald polynomials. The ring of symmetric functions is endowed with the Hall scalar product $\langle \cdot, \cdot \rangle$ with respect to which the bases m_λ and h_λ are dual. The dual basis of the Hall–Littlewood polynomials $P_\lambda(x;t)$ with respect to the Hall scalar product consists of the transformed Hall–Littlewood polynomials $H_\lambda(x;t)$. They are related to the q-Whittaker functions by $\omega H_{\lambda'}(x;t) = W_\lambda(x;t)$.

The modified Hall–Littlewood polynomials are indexed by integer partitions and are defined by $\tilde{H}_\lambda(x;t) = t^{n(\lambda)} H_\lambda(x;t^{-1})$, where $n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i$. The modified Hall–Littlewood polynomial also satisfies $\tilde{H}_\lambda(x;t) = \sum_\mu \tilde{K}_{\mu\lambda}(t) s_\mu$, where s_μ denotes a Schur function and $\tilde{K}_{\mu\lambda}(t)$ is a modified Kostka–Foulkes polynomial.

For each nonnegative integer n, define the q-analogs $[n]_q := 1 + q + \cdots + q^{n-1}$ and $[n]_q! := [1]_q[2]_q \cdots [n]_q$. For a weak composition α of n, the q-multinomial coefficient is defined by

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_q := \frac{[n]_q!}{\prod_{i\geq 1}[\alpha_i]_q!}.$$

Definition 2.1. *Given a matrix $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$ and a weak composition $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ of n, let $X_\alpha(\Delta)$ denote the number of flags $(0) = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_\ell = \mathbb{F}_q^n$ of Δ-invariant subspaces satisfying $\dim W_i/W_{i-1} = \alpha_i$ for $1 \leq i \leq r$.*

Example 2.2. *If $\Delta = cI$ where I denotes the $n \times n$ identity matrix and $c \in \mathbb{F}_q$, then $X_\alpha(\Delta) = \begin{bmatrix} n \\ \alpha \end{bmatrix}_q$, a q-multinomial coefficient.*

Definition 2.3. *For $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$ the* invariant flag generating function *$F_\Delta(x)$ is defined by*

$$F_\Delta(x) := \sum_\alpha X_\alpha(\Delta) x^\alpha,$$

where the sum is taken over all weak compositions α of n and x^α denotes the product $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$.

Proposition 2.4. *For each matrix $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$, we have $F_\Delta(x) = \sum_\lambda X_\lambda(\Delta) m_\lambda$, where the sum is taken over all partitions λ of n and m_λ denotes the monomial symmetric function.*

Example 2.5. *If $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$ is* simple *(it has irreducible characteristic polynomial), then the only Δ-invariant subspaces are the zero subspace and $\mathbb{F}_q^n$. Therefore $F_\Delta(x) = \sum_{i\geq 1} x_i^n = p_n$, the power sum symmetric function.*

Example 2.6. *If $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$ is regular nilpotent (nilpotent with one-dimensional null space), then there is precisely one Δ-invariant subspace of dimension k for each $0 \leq k \leq n$. In this case $F_\Delta(x) = \sum_\alpha x^\alpha = h_n$, the complete homogeneous symmetric function.*

Our objective now is to determine the invariant flag generating function for an arbitrary matrix $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$. Recall that the action of Δ on $\mathbb{F}_q^n$ defines an $\mathbb{F}_q[t]$-module structure on $\mathbb{F}_q^n$. By the structure theorem for finitely generated modules over a principal ideal domain, this module is isomorphic to a direct sum

$$\bigoplus_{i=1}^{k} \bigoplus_{j=1}^{\ell_i} \frac{\mathbb{F}_q[t]}{(g_i^{\lambda_{i,j}})}, \tag{2.1}$$

where $g_i(t) \in \mathbb{F}_q[t]$ are distinct monic irreducible polynomials and the sequence $\lambda^i = (\lambda_{i,1}, \lambda_{i,2}, \ldots, \lambda_{i,\ell_i})$ is an integer partition for each $1 \leq i \leq k$. Let d_i denote the degree of g_i for $1 \leq i \leq k$.

Definition 2.7. *With d_i and λ_i as above, the* similarity class type *of the matrix Δ is the multiset $\tau = \{(d_1, \lambda^1), (d_2, \lambda^2), \ldots, (d_k, \lambda^k)\}$. The* size *of τ is the integer $\sum_{i=1}^k d_i |\lambda^i|$.*

The notion of similarity class type can be traced back to the work of Green [11] who studied the characters of the finite general linear groups. Considering similarity class types allows for a q-independent classification of conjugacy classes in these groups.

Example 2.8. *Consider $n \times n$ matrices over $\mathbb{F}_q$. A simple matrix has type $\{(n,(1))\}$ while a scalar multiple of the identity has type $\{(1,(1^n))\}$. A regular nilpotent matrix has type $\{(1,(n))\}$.*

Remark 2.9. *For arbitrary τ and q, there may be no matrix of similarity class type τ over $\mathbb{F}_q$. For a fixed τ, a matrix of type τ over $\mathbb{F}_q$ always exists for sufficiently large prime powers q.*

It is not difficult to see that $F_\Delta(x)$ is multiplicative over the primary components of Δ. In general we have the following result.

Proposition 2.10. *If $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$ is a matrix of similarity class type $\tau = \{(d_i, \lambda^i)\}_{1 \le i \le k}$, then*

$$F_\Delta(x) = \prod_{i=1}^{k} \tilde{H}_{\lambda^i}(x_1^{d_i}, x_2^{d_i}, \ldots; q^{d_i}) = \prod_{i=1}^{k} p_{d_i}[\tilde{H}_{\lambda^i}(x;t)]_{|t=q}, \tag{2.2}$$

where $\tilde{H}_\lambda$ denotes a modified Hall–Littlewood polynomial. Here, the plethystic substitution $p_{d_i}[\tilde{H}_{\lambda^i}(x;t)]$ is performed before evaluating at $t = q$.

The expression for F_Δ in Proposition 2.10 is a product of symmetric functions where the parameter t is specialized to a prime power q. Rather than specialize the parameter, one can work directly with the parametric versions $\tilde{H}_\lambda(x;t)$ by considering similarity class types instead of matrices. For a similarity class type $\tau = \{(d_i, \lambda^i)\}_{1 \le i \le k}$, define

$$F_\tau(x;t) := \prod_{i=1}^{k} p_{d_i}[\tilde{H}_{\lambda^i}(x;t)]. \tag{2.3}$$

Given a prime power q and a matrix $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$ with similarity class type τ, it is clear that $F_\Delta(x) = F_\tau(x;q)$. We now give examples of symmetric functions which arise as F_Δ for some Δ. In Section 4 we will see that a combinatorial interpretation can be given for the coefficients in the q-Whittaker expansions of each of these functions.

Example 2.11. *A matrix is* regular semisimple *if its characteristic polynomial is a product of distinct irreducible polynomials $g_i(t)(1 \le i \le k)$ over $\mathbb{F}_q$. Let $\lambda = (\lambda_1, \ldots, \lambda_k)$ denote the integer partition obtained by arranging the degrees of the g_i in weakly decreasing order. The invariant flag generating function is given by $\prod_{i=1}^{k} p_{\lambda_i} = p_\lambda$.*

Example 2.12. *A matrix is* regular split *if its minimal polynomial is equal to its characteristic polynomial and each is a product of linear factors: $\prod_{i=1}^{k}(x - a_i)^{\lambda_i}$. In this case the invariant flag generating function is $\prod_{i=1}^{k} h_{\lambda_i} = h_\lambda$.*

The following is a simultaneous generalization of Examples 2.6 and 2.12.

Example 2.13. *A matrix is* triangulable *if it is similar to an upper triangular matrix. For such a matrix $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$, we have $d_i = 1$ in the decomposition (2.1) above for $1 \leq i \leq k$. It follows from Proposition 2.10 that $F_\Delta = \prod_{i\geq 1} \tilde{H}_{\lambda^i}(x;q)$.*

Example 2.14. *Let $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$ be a diagonalizable matrix with characteristic polynomial $\prod_{i=1}^{k}(x-a_i)^{\nu_i}$ for some partition ν of n and distinct elements $a_i \in \mathbb{F}_q (1 \leq i \leq k)$. Since Δ acts by a scalar multiple of the identity on each eigenspace, it follows from Example 2.2, that $F_\Delta = \prod_{i\geq 1} F_{\nu_i}$ where, for each positive integer m, we have $F_m := \sum_{\lambda\vdash m} \binom{m}{\lambda}_q m_\lambda = W_{(m)}(x;q)$, a q-Whittaker function. Therefore $F_\Delta = \prod_{i\geq 1} W_{(\nu_i)}(x;q)$.*

3 Diagonal operators and q-Whittaker functions

As noted earlier, the q-Whittaker functions $W_\lambda(x;t)$ form a basis for $\Lambda_{\mathbb{Q}(t)}$. They occur as joint eigenfunctions of q-deformed Toda chain Hamiltonians with support in the positive Weyl chamber (see Etingof [8] or Ruijsenaars [23]). They interpolate between the Schur and elementary symmetric functions: $W_\lambda(x;0) = s_\lambda(x)$, and $W_\lambda(x;1) = e_{\lambda'}(x)$. They also expand positively in the Schur basis $W_\mu = \sum_\lambda K_{\lambda'\mu'}(t)s_\lambda$ where $K_{\lambda\mu}(t)$ denotes a Kostka–Foulkes polynomial. In a representation theoretic context, the q-Whittaker functions arise in the setting of the graded Frobenius characteristic of the cohomology ring of Springer fibers. Diagonalizable operators play a central role in the proof of the main theorem. In this section we relate them to monomial coefficients of q-Whittaker functions.

Definition 3.1. *The type of a diagonalizable matrix is the integer partition obtained by sorting the dimensions of its eigenspaces in weakly decreasing order.*

Note that a diagonalizable matrix of type ν over $\mathbb{F}_q$ exists if and only if the number of parts of ν is at most q. The following result was proved in [22, Theorem 4.21].

Theorem 3.2. *For each pair μ, ν of integer partitions of n, there exist polynomials $b_{\mu\nu}(t) \in \mathbb{Z}[t]$ such that*

$$\sigma(\mu,\Delta) = (q-1)^{\sum_{j\geq 2}\mu_j} q^{\sum_{j\geq 2}\binom{\mu_j}{2}} b_{\mu\nu}(q),$$

for each prime power q and each diagonal matrix $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$ of type ν.

The following result was derived from a decomposition of the open Schubert cells of the Grassmanian according to subspace profiles with respect to a diagonal operator.

Proposition 3.3 ([22, Theorem 5.3])**.** *For partitions μ and ν,*

$$b_{\mu\nu}(t) = \frac{\prod_{i\geq 1}[\nu_i]_t!}{\prod_{i\geq 1}[\mu_i - \mu_{i+1}]_t!}\langle W_\mu, h_\nu\rangle.$$

The polynomials $b_{\mu\nu}(t)$ can also be obtained by summing a statistic on a suitably defined class of set partitions. In fact, the expressions for these polynomials in terms of both set partition statistics and semistandard tableaux has led to an elementary correspondence between these two classical combinatorial objects. This correspondence yields a way to associate a set partition statistic to each Mahonian statistic on multiset permutations. The polynomials $b_{\mu\nu}(t)$ also have several interesting specializations. In particular, when $\mu = (m, m)$ and $\nu = (1^{2m})$ the polynomial $b_{\mu\nu}(t)$ coincides with the Touchard–Riordan generating polynomial for chord diagrams by their number of crossings. For more on this topic and some connections with q-rook theory, the reader is referred to [22].

4 Full profiles

In this section we obtain an explicit formula for $\sigma(\mu, \Delta)$ when μ is a partition of n. Let $a_{\mu\lambda}(t)$ and $\tilde{a}_{\mu\lambda}(t)$ denote the transition coefficients between the Hall–Littlewood functions and the elementary symmetric functions:

$$e_\mu = \sum_\lambda a_{\mu\lambda}(t) P_\lambda(x;t) \text{ and } P_\mu(x;t) = \sum_\lambda \tilde{a}_{\mu\lambda}(t) e_\lambda.$$

One can view the polynomials $a_{\mu\lambda}(t)$ as the generating polynomial for a suitably defined combinatorial statistic on $(0, 1)$-matrices with row sums μ and colums sums λ (Macdonald [17, p. 211]). On the other hand, the polynomials $\tilde{a}_{\mu\lambda}(t)$ do not have nonnegative coefficients in general and do not appear to have a nice combinatorial description. A very intricate explicit formula for $\tilde{a}_{\mu\lambda}(t)$ was found by Lassalle and Schlosser [14, Theorem 7.5] who gave an expression for the more general two-parameter coefficients in the elementary expansion of Macdonald polynomials. For each partition λ, let $\epsilon_\lambda = (-1)^{|\lambda| - \ell(\lambda)}$, where $\ell(\lambda)$ denotes the number of parts of λ. The symmetric functions $\widetilde{W}_\lambda$ defined by $\widetilde{W}_\lambda = \omega P_{\lambda'}$ are dual to the q-Whittaker functions with respect to the Hall scalar product. The following theorem is the main result of this section.

Theorem 4.1. *For each partition μ of n,*

$$\sigma(\mu, \Delta) = \epsilon_{\mu'}\, q^{\sum_{j \geq 2} \binom{\mu_j}{2}} \langle F_\Delta, \widetilde{W}_\mu(x;q) \rangle,$$

for every prime power q and every matrix $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$.

Theorem 4.1 gives the following combinatorial interpretation for the coefficients in the q-Whittaker expansion of the invariant flag generating function $F_\tau(x;t)$ (see Equation (2.3)) associated to a similarity class type τ.

Corollary 4.2. *For each similarity class type $\tau = \{(d_1, \lambda^1), (d_2, \lambda^2), \ldots, (d_k, \lambda^k)\}$ of size n,*

$$\prod_{i=1}^{k} p_{d_i}[\tilde{H}_{\lambda^i}(x;t)] = \sum_{\mu \vdash n} \epsilon_{\mu'} t^{-\sum_{j\geq 2}\binom{\mu_j}{2}} \sigma(\mu,\tau) W_\mu(x;t),$$

where $\sigma(\mu,\tau)$ denotes the number of subspaces which have profile μ with respect to a matrix of similarity class type τ over the finite field $\mathbb{F}_t$ for sufficiently large prime powers t.

In view of the examples in Section 2, Corollary 4.2 gives a combinatorial finite-field interpretation for the coefficients in the q-Whittaker expansion of several symmetric functions which arise as invariant flag generating functions.

Example 4.3. *Specializing Corollary 4.2 to the regular nilpotent and simple similarity class types, we obtain the following expansions for homogeneous and power sum symmetric functions:*

$$h_n = \sum_{\mu \vdash n} (-1)^{n-\mu_1} \left(\prod_{i \geq 2} t^{\binom{\mu_i+1}{2}} \begin{bmatrix} \mu_{i-1} \\ \mu_i \end{bmatrix}_t \right) W_\mu(x;t),$$
$$p_n = \sum_{\mu \vdash n} (-1)^{n-\mu_1} \frac{t^n - 1}{t^{\mu_1} - 1} \left(\prod_{i \geq 2} t^{\binom{\mu_i}{2}} \begin{bmatrix} \mu_{i-1} \\ \mu_i \end{bmatrix}_t \right) W_\mu(x;t).$$

The q-Whittaker functions have recently arisen in the work of Karp and Thomas [13] in the context of counting certain partial flags compatible with a nilpotent endomorphism over a finite field. They also obtain an elegant probabilistic bijection between nonnegative integer matrices and pairs of semistandard tableaux. It would be interesting to relate their work to the counting problem considered in this paper.

5 Arbitrary profiles

An explicit formula for $\sigma(\mu,\Delta)$ when μ is a partition of the ambient vector space dimension was obtained in Theorem 4.1. In this section we extend the result to arbitrary partitions μ. The following result of Bender, Coley, Robbins and Rumsey shows that $\sigma(\mu,\Delta)$ satisfies a remarkably simple system of equations.

Theorem 5.1 ([5, Equation 4])**.** *Let n be a positive integer and suppose ν is a partition with $|\nu| < n$. For each matrix $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$,*

$$\sum_{\mu : |\mu| \leq n} (-1)^{\mu_1} q^{-\mu\cdot\nu + \binom{\mu_1}{2}} \sigma(\mu,\Delta) = 0,$$

where the sum is taken over all partitions μ of size at most n (including the empty partition) and $\mu \cdot \nu := \sum_{j\geq 1} \mu_j \nu_j$.

Bender, Coley, Robbins and Rumsey used Theorem 5.1 to derive an explicit formula for $\sigma(\mu, \Delta)$ in the cases where Δ is simple or regular nilpotent. The following theorem is our main result.

Theorem 5.2. *For each partition μ,*

$$\sigma(\mu, \Delta) = \epsilon_{\mu'}\, q^{\sum_{j\geq 2}\binom{\mu_j}{2}} \langle F_\Delta, \widetilde{W}_\mu(x;q)\, h_{n-|\mu|}\rangle,$$

for each prime power q and each matrix $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$.

6 Partial profiles and anti-invariant subspaces

Definition 6.1. *Given a matrix $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$, a subspace $W \subseteq \mathbb{F}_q^n$ has partial Δ-profile $\rho = (\rho_1, \rho_2, \ldots, \rho_r)$ if*

$$\dim(W + \Delta W + \cdots + \Delta^{j-1}W) = \rho_1 + \rho_2 + \cdots + \rho_j \text{ for } 1 \leq j \leq r.$$

Let $\pi(\rho, \Delta)$ denote the number of subspaces with partial Δ-profile ρ.

Example 6.2. *We have $\pi((m), \Delta) = \begin{bmatrix} n \\ m \end{bmatrix}_q$ for each $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$ and $m \geq 0$.*

Note that $\pi((m, 0), \Delta)$ is the number of m dimensional Δ-invariant subspaces which is distinct from $\pi((m), \Delta)$ in general. Therefore, we think of ρ as a tuple rather than as an integer partition. It is evident that the number of subspaces with Δ-profile $\mu = (\mu_1, \ldots, \mu_k)$ equals the number of subspaces with partial Δ-profile $(\mu_1, \ldots, \mu_k, 0)$.

Definition 6.3. *Given $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$ and a positive integer t, a subspace W of $\mathbb{F}_q^n$ is said to be t-fold Δ-anti-invariant if*

$$\dim(W + \Delta W + \cdots + \Delta^t W) = (t+1)\dim W.$$

Thus an m-dimensional Δ-anti-invariant subspace is precisely one with partial Δ-profile (m^{t+1}). Anti-invariant subspaces were originally defined (for $t = 1$) by Barría and Halmos [4], motivated by earlier work of Hadwin, Nordgren, Radjavi and Rosenthal [12] on the weak density of certain sets of operators on Banach spaces. Barría and Halmos determined the maximum possible dimension of an anti-invariant subspace.

Theorem 6.4. *Given $\rho = (\rho_1, \ldots, \rho_r)$ with $\rho_r \neq 0$ and $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$, the number of subspaces with partial Δ-profile ρ is given by $\pi(\rho, \Delta) = \langle \omega F_\Delta, G_\rho\rangle$, where*

$$G_\rho = (-1)^{\sum_{j\geq 2}\rho_j} q^{\sum_{j\geq 2}\binom{\rho_j}{2}} \sum_{\substack{\eta \vdash n \\ \ell(\eta) = r}} P_{\eta'}(x;q) \prod_{i\geq 1} \begin{bmatrix} \eta_i - \eta_{i+1} \\ \eta_i - \rho_i \end{bmatrix}_q.$$

Corollary 6.5. *Given $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$ and a positive integer t, the number of t-fold Δ-anti-invariant subspaces of dimension m is given by*

$$(-1)^{mt} q^{t\binom{m}{2}} \langle \omega F_\Delta, P_{((t+1)^m, 1^{n-m(t+1)})}(x;q)\rangle.$$

7 Application to Krylov subspace methods

Let $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$ and consider a subset $S = \{v_1, \ldots, v_k\}$ of column vectors in $\mathbb{F}_q^n$. The *truncated Krylov subspace* of order ℓ generated by S is defined by

$$\mathrm{Kry}(\Delta, S, \ell) := \left\{ \sum_{i=1}^{k} f_i(\Delta) v_i : f_i(x) \in \mathbb{F}_q[x] \text{ and } \deg f_i < \ell \right\}.$$

Let $\psi_{k,\ell}(\Delta)$ denote the probability of selecting a k-tuple of vectors uniformly at random from $\mathbb{F}_q^n$ such that the truncated Krylov subspace of order ℓ spanned by them is all of $\mathbb{F}_q^n$. Thus

$$\psi_{k,\ell}(\Delta) := \frac{1}{q^{nk}} |\{(v_1, \ldots, v_k) \in (\mathbb{F}_q^n)^k : \mathrm{Kry}(\Delta, \{v_1, \ldots, v_k\}; \ell) = \mathbb{F}_q^n\}|. \tag{7.1}$$

Computing $\psi_{k,\ell}(\Delta)$ is crucial for analyzing a class of algorithms that solve large, sparse linear systems over finite fields, commonly encountered in number theory and computer algebra (Watkins [25]). These algorithms, collectively referred to as Krylov subspace methods, have origins that can be traced back to contributions by Lagrange, Euler, Gauss, Hilbert and von Neumann, among others (Liesen and Strakoš [16, p. 8]). For instance, the linear algebra step in the Number Field Sieve, a well-known algorithm for large integer factorization, depends on Krylov subspace methods (Lenstra, Lenstra, Manasse and Pollard [15]). Another example is Wiedemann's algorithm, which is employed to determine the minimal polynomials of large matrices over finite fields (Liesen and Strakoš [16, p. 19]). The quantity $\psi_{k,\ell}(\Delta)$ plays a key role in evaluating the effectiveness of these algorithms, and determining bounds on this probability represents a challenging and crucial task in the field (Brent, Gao and Lauder [6, p. 277]). We give an explicit formula for this probability.

Theorem 7.1. *For each matrix $\Delta \in \mathrm{M}_n(\mathbb{F}_q)$, we have $\psi_{k,\ell}(\Delta) = \langle F_\Delta, G(n,k,\ell) \rangle$, where*

$$G(n,k,\ell) = q^{-nk} \sum_{\substack{\mu \vdash n \\ \ell(\mu) \leq \ell}} (-1)^{n-\mu_1} (q-1)^{\mu_1} q^{\sum_{j \geq 1} \binom{\mu_j}{2}} \begin{bmatrix} k \\ \mu_1 \end{bmatrix}_q [\mu_1]_q! \widetilde{W}_\mu(x;q).$$

8 Recent developments

Denote by $[n]$ the set of the first n positive integers. A Hessenberg function is a weakly increasing function $\mathsf{m} : [n] \to [n]$ satisfying $\mathsf{m}(i) \geq i$ for each $i \in [n]$. For a linear operator Δ on $\mathbb{F}_q^n$, the Hessenberg variety is defined by

$$\mathscr{H}(\mathsf{m}, \Delta) := \{\text{complete flags } V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = \mathbb{F}_q^n : \Delta V_i \subseteq V_{\mathsf{m}(i)} \text{ for } i \in [n]\}.$$

The following theorem was proved in joint work with Abreu and Nigro [1].

Theorem 8.1. *For each operator Δ, the number of $\mathbb{F}_q$-rational points on the Hessenberg variety $\mathscr{H}(\mathsf{m}, \Delta)$ is given by*

$$|\mathscr{H}(\mathsf{m}, \Delta)| = \langle F_\Delta, \omega X_{G(\mathsf{m})}(x; q) \rangle,$$

where $X_{G(\mathsf{m})}(x; t)$ denotes the chromatic quasisymmetric function of the unit interval graph $G(\mathsf{m})$ associated to m.

This result entails an expression for the Poincaré polynomials of complex Hessenberg varieties involving modified Hall–Littlewood polynomials.

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