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The branching models of Kwon and Sundaram via flagged hives

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Abstract. We prove a bijection between the branching models of Kwon and Sundaram, conjectured previously by Lenart–Lecouvey. To do so, we use a symmetry of Littlewood–Richardson coefficients in terms of the hive model. Along the way, we introduce a new branching model in terms of *flagged hives*.

Keywords: branching rule, hives, Gelfand–Tsetlin patterns, tableaux

1 Introduction

In the representation theory of Lie algebras, branching problems study the restriction of a finite-dimensional irreducible highest-weight representation of a semisimple Lie algebra to nice subalgebras. Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{k}$ be a semisimple Lie subalgebra of $\mathfrak{g}$. Recall that the finite-dimensional irreducible highest-weight representations of $\mathfrak{g}$ are parametrized by dominant integral weights $\mathcal{P}^+(\mathfrak{g})$. Let $V(\nu)$ denote the irreducible highest weight representation indexed by $\nu \in \mathcal{P}^+(\mathfrak{g})$, and consider its restriction to $\mathfrak{k}$:

$$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{k}} V(\nu) = \bigoplus_{\mu \in \mathcal{P}^+(\mathfrak{k})} V(\mu)^{\oplus c^{\nu}_{\mu}}. \tag{1.1}$$

The coefficients c^{λ}_{μ} in (1.1) are called branching coefficients. A combinatorial rule for computing these coefficients is called a branching rule. By a combinatorial rule, we mean associating a combinatorial set (model) to each pair (ν, μ) whose cardinality is c^{ν}_{μ}.

Throughout this paper, we fix $\mathfrak{g}$ to be the special linear Lie algebra $\mathfrak{sl}(2n, \mathbb{C})$ and $\mathfrak{k}$ to be the symplectic Lie algebra $\mathfrak{sp}(2n, \mathbb{C})$ (thought of as the fixed point subalgebra for the non-trivial Dynkin diagram automorphism of $\mathfrak{sl}(2n, \mathbb{C})$). For the restriction problem in this case, various combinatorial models based on tableaux are known. There is the classical model by Littlewood, in terms of Littlewood–Richardson tableaux, for the stable

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case [11, 12]. An elegant extension of that rule beyond the stable case was found by Sundaram [16]. Later, a rule in terms of tableaux and Littelmann paths [14], which was conjectured by Naito–Sagaki, was proven via its relationship to Sundaram's rule. There is another, more recent extension of the Littlewood branching rule by Kwon [8, 9], which is formulated in a more general context, for all classical types, using a combinatorial model for classical crystals known as the spinor model.

In [10], Lenart–Lecouvey use the branching models of Kwon and Sundaram to obtain combinatorial descriptions of generalized exponents in type C_n. In this work, we prove a bijection conjectured by them in [10] between these models. The main tool in our proof is the hive model for the Littlewood–Richardson coefficients and Gelfand–Tsetlin patterns [1, 4]. In fact, we use the flagged hive models studied in [7, 6] where they use the same to study saturation property for some structure constants using their connections to crystals. Another important element in our proof is a symmetry (denoted by U here) of Littlewood–Richardson coefficients from [7].

Organization of the paper

In Section 2 we fix the basic notations to work with. In Section 3 we recall the branching models of Kwon (in fact, an equivalent formulation from [10]) and Sundaram in [8] and [16]. We also state the main theorem in this section. In Section 4 we recall and use the combinatorics of hives to spell out the proof. The arguments are sketched to the extent possible subject to the overall space restrictions. The detailed proofs are part of a forthcoming publication [13].

2 Notation

A *partition* is a non-increasing sequence of non-negative integers $\nu := \nu_1 \geq \nu_2 \geq \cdots$ such that $\nu_k = 0$ for some $k \geq 1$. The maximal j such that $\nu_j \neq 0$ is called the *number of parts* or *length* of ν and will be denoted by $\ell(\nu)$. We will abuse notation and denote a partition by $\nu = (\nu_1, ..., \nu_k)$ for $k \geq \ell(\nu)$. Given a partition ν we will often consider its Young diagram which is a left and top justified collection of boxes with ν_k many boxes in the k^{th} row for all $k \in \mathbb{Z}_{>0}$. We will denote the Young diagram of ν again by ν.

Let ν, μ be partitions with $\mu \subset \nu$, (that is, the Young diagram of μ is a subset of the Young diagram of ν, or equivalently, $\mu_j \leq \nu_j$ for all $j \in \mathbb{Z}_{>0}$). A semi-standard tableau of skew shape ν/μ is a function assigning a natural number to each box of ν such that it is weakly increasing along rows and strictly increasing along columns, and such that it is constant and equal to 0 precisely on the boxes corresponding to μ. For aesthetic purposes, when displaying a tableau, we will leave the boxes with a zero filling simply blank. The image of a box will be simply referred to as the entry in the box. Usually, a positive integer k will be fixed and $[0,k] := \{0,1,...,k\}$ will be used as co-domain for the filling function. In this case, we will denote the set of semi-standard tableaux of skew shape ν/μ by $\mathrm{SSYT}_k(\nu/\mu)$. Whenever $\mu = (0)$, the set of semi-standard tableaux of skew shape ν/μ are typically known as semi-standard Young tableaux of (straight) shape ν, while semi-standard Young tableaux of skew shape $(m)^d/\nu$ are typically referred to as contretableux of shape ν', where ν' is the complement of ν in $(m)^d$. A semistandard Young tableau can be equivalently defined via a sequence of k partitions

$$\nu^{(0)} = \mu \subset \nu^{(1)} \subset \cdots \subset \nu^{(k)} = \nu,$$

where $\nu^{(i)}$ is defined to be the sub-shape of ν/μ which is the pre-image of $[0,i]$.

The *north western row word*, a.k.a. the *reverse row word* of a semi-standard Young tableau T is obtained by reading the entries of its rows from top to bottom and right to left. We will denote this word by $w(T)$. The *content* of a semi-standard tableau T is the content of its reverse row word. The content of a word w is $\alpha = (\alpha_1, \ldots, \alpha_n)$ such that α_i equals the number of times i appear in w. A *Yamanouchi word* is a word $w = w_1 \cdots w_l$ such that for each $1 \leq k \leq l$, the content of the subword $w^k := w_1 \cdots w_k$ is a partition.

We now briefly recall the Schützenberger involution on the set of semi-standard Young tableaux. For $T \in \mathrm{SSYT}_k(\lambda)$, if $w(T) = w_1 w_2 \ldots w_m$ then the Schützenberger involution $S(T)$ is the unique tableau in the plactic class of the word $S(w(T)) := w'_m \ldots w'_2 w'_1$ where, w'_t denotes $k+1-w_t$. It is a well-known fact that the tableau $S(T)$ has shape equal to λ and the map S is an involution on $\mathrm{Tab}_k(\lambda)$. We remark here that the word $S(w(T))$ is the reading word of a contretableau of shape λ. Thus $S(T)$ can be defined equivalently as the rectification of this contretableau.

3 The branching models of Kwon and Sundaram

Let ν, μ, λ be partitions, with $\lambda, \mu \subset \nu$. From now on we will fix a positive integer n, and assume, unless otherwise stated, that $\ell(\nu) \leq 2n-1$. A semi-standard tableau of skew shape ν/μ and content λ is said to be a Littlewood–Richardson (LR) tableau if its reverse row reading word is a Yamanouchi word. We denote the set of LR tableaux of shape ν/μ and content λ by $\mathrm{LR}(\nu/\mu, \lambda)$. The numbers $c^{\nu}_{\mu,\lambda} := |\,\mathrm{LR}(\nu/\mu, \lambda)|$ are called the *LR coefficients*. Let T_μ be the unique semi-standard Young tableau of shape and content μ. We say that a semi-standard Young tableau T is *μ-dominant* if it satisfies the following condition:

$$w(T_\mu) * w(T) \text{ is a Yamanouchi word.} \tag{3.1}$$

We denote the set of all μ-dominant semi-standard Young tableau of shape λ and content $\nu - \mu$ by $\mathrm{LR}^{\nu}_{\mu,\lambda}$.

Example 3.1. Let $\nu = (5,3,1), \mu = (3,1), \lambda = (3,1,1)$, with $n \geq 3$. In this case, there is a unique Littlewood–Richardson tableau of shape ν/μ and content λ:

$$\begin{array}{|c|c|c|c|c|}\hline \ & \ & \ & 1 & 1 \\ \hline \ & 1 & 2 \\ \cline{1-3} 3 \\ \cline{1-1}\end{array}\,.$$

Now, from a Littlewood–Richardson tableau T of shape ν/μ and content λ one can easily obtain its *companion tableau* $c(T)$ by placing in the k-th row of the Young diagram of λ the indices of the rows of T containing an entry k.

Example 3.2. For T as in Example 3.1, we have

$$c(T) = \begin{array}{|c|c|c|}\hline 1 & 1 & 2 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array} \text{ and also } T_\mu = \begin{array}{|c|c|c|}\hline 1 & 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array}, w(T_\mu) = 1112, w(c(T)) = 21123.$$

Note that $w(T_\mu) * w(T) = 111221123$ is indeed a Yamanouchi word.

The companion $c(T)$ of T is a μ-dominant semi-standard Young tableau of shape λ and weight $\nu - \mu$. In fact, it is well known that the *companion* map induces a bijection

$$c : \mathrm{LR}(\nu/\mu, \lambda) \xrightarrow{\sim} \mathrm{LR}^{\nu}_{\mu,\lambda}\,.$$

The set of dominant integral weights for the special linear Lie algebra $\mathfrak{sl}(2n,\mathbb{C})$ is in bijection with the set of integer partitions ν for which $\ell(\nu) \leq 2n-1$. Therefore, the irreducible finite-dimensional highest weight representations of $\mathfrak{sl}(2n,\mathbb{C})$ are indexed by partitions ν for which $\ell(\nu) \leq 2n-1$. Let $V(\nu)$ be the finite-dimensional irreducible $\mathfrak{sl}(2n,\mathbb{C})$ module indexed by ν. For partitions λ and μ whose length is at most $2n-1$, the Littlewood–Richardson coefficients are the tensor product multiplicities:

$$V(\lambda) \otimes V(\mu) = \bigoplus_{\nu} V(\nu)^{\oplus c^{\nu}_{\mu,\lambda}}.$$

By the symmetry of tensor products, it is then clear that $c^{\nu}_{\mu,\lambda} = c^{\nu}_{\lambda,\mu}$; this property is called the *symmetry of Littlewood–Richardson coefficients*. In this work, we will recurr to a bijection $\mathrm{LR}^{\nu}_{\mu,\lambda} \overset{U}{\longleftrightarrow} \mathrm{LR}^{\nu}_{\lambda,\mu}$ via the *hive model* (cf. Section 4) from [7].

The set of irreducible finite-dimensional highest weight representations for the symplectic Lie algebra $\mathfrak{sp}(2n,\mathbb{C})$ are indexed by partitions μ for which $\ell(\mu) \leq n$. Let $V^{\sigma}(\mu)$ be the simple $\mathfrak{sp}(2n,\mathbb{C})$ module of highest weight μ. Consider the branching of $V(\nu)$ after restriction to $\mathfrak{sp}(2n,\mathbb{C})$:

$$\mathrm{res}^{\mathfrak{sl}(2n,\mathbb{C})}_{\mathfrak{sp}(2n,\mathbb{C})} V(\nu) = \bigoplus_{\mu} V^{\sigma}(\mu)^{\oplus c^{\nu}_{\mu}}.$$

We say that a LR tableau of shape ν/μ and content λ satisfies the *Sundaram* property, if, for each $i = 0, ..., \frac{1}{2}\ell(\lambda)$, the entry $2i+1$ appears in row $n+i$ or above in the Young diagram of ν. We denote the set of $T \in \mathrm{LR}(\nu/\mu, \lambda)$ satisfying the Sundaram property by $\mathrm{LRS}(\nu/\mu, \lambda)$.

Example 3.3. Let $n = 3$. The tableau in Example 3.1 satisfies the Sundaram condition, but $\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline 1 \\ \hline 2 \\ \hline \end{array}$ does not.

A partition λ is *even* if $\lambda_{2i-1} = \lambda_{2i}$ for each $i \in \mathbb{Z}_{\geq 0}$. The following theorem is due to Sundaram.

Theorem 3.4 ([16, Theorem 12.1])**.** *The branching coefficient c^{ν}_{μ} equals the cardinality of the set*

$$\mathrm{LRS}(\nu, \mu) := \bigcup \mathrm{LRS}(\nu/\mu, \lambda),$$

where the union is taken over all even partitions λ.

A tableau of shape λ with $\ell(\lambda) \leq n$ is said to satisfy the *Kwon property* if the entries in row i are at least $2i-1$, for $i = 1, ..., n$. Denote the subset of $\mathrm{LR}^{\nu}_{\lambda,\mu}$ consisting of tableaux T such that their evacuation $S(T)$ satisfies the Kwon property by $LRK^{\nu}_{\lambda,\mu}$. The following is the reformulation of Kwon's branching rule by Lecouvey–Lenart [10, Lemma 6.11].

Theorem 3.5 ([8])**.** *The branching coefficient c^{ν}_{μ} equals the cardinality of the set*

$$\mathrm{LRK}(\nu, \mu) := \bigcup LRK^{\nu}_{\lambda,\mu}$$

where, the union is taken over even partitions $\lambda \subset \nu$.

We state our main theorem below.

Theorem 3.6. *The composition*

$$\mathrm{LR}(\nu/\mu, \lambda) \xrightarrow{c} \mathrm{LR}^{\nu}_{\mu,\lambda} \xleftrightarrow{U} \mathrm{LR}^{\nu}_{\lambda,\mu} \xrightarrow{S} S(\mathrm{LR}^{\nu}_{\lambda,\mu})$$

restricts to a bijection

$$\mathrm{LRS}(\nu/\mu, \lambda) \longleftrightarrow \mathrm{LRK}^{\nu}_{\lambda,\mu}\,.$$

We prove Theorem 3.6 in Section 4. As a byproduct, we obtain a new branching model in terms of the *flagged hive model*.

4 The flagged hive model

In this section, we recollect the notions of Gelfand–Tsetlin patterns, hives and their connections with tableaux.

Fix a positive integer m. A Gelfand–Tsetlin (GT) pattern P is a triangular array of numbers $(p_{i,j})_{\substack{1\le i\le m,\\ 1\le j\le i}}$ such that

$$p_{i+1,j} \geq p_{ij} \geq p_{i+1,j+1} \tag{4.1}$$

for all appropriate values of i and j.

$$\begin{array}{ccccccc} & & & p_{1,1} & & & \\ & & p_{2,1} & & p_{2,2} & & \\ & p_{3,1} & & p_{3,2} & & p_{3,3} & \\ p_{4,1} & & p_{4,2} & & p_{4,3} & & p_{4,4} \end{array} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & 3 & & 1 & & \\ & 4 & & 3 & & 0 & \\ 6 & & 3 & & 2 & & 0 \end{array}$$

Figure 1: Gelfand–Tsetlin patterns

The inequalities in (4.1) imply that $P_k := (p_{k,1}, p_{k,2}, \ldots, p_{k,k}, 0, 0, \ldots, 0)$ is weakly decreasing ($1 \leq k \leq m$) and hence an integral GT pattern P (i.e., $p_{i,j} \in \mathbb{Z}_{\geq 0}$) is in fact a sequence of partitions $(0) \subset P_1 \subset P_2 \subset \ldots \subset P_m$ such that the length of P_k is at most k and the successive quotients P_{k+1}/P_k are horizontal strips ($1 \leq k \leq m-1$). Recall that this defines a semi-standard Young tableau of shape P_m which we denote by $T(P)$. Also, this map defines a bijection between GT patterns and semi-standard Young tableaux. Given a semi-standard Young tableau R, we denote the associated GT pattern by $GT(R)$.

Given an integral GT pattern P, one could also define a contretableau $C(P)$ as follows: First, let k denote the largest part of P_m (i.e., $k = p_{m,1}$). Then the sequence of partitions $\mathbf{k} - \mathrm{rev}(P_m) \subset \mathbf{k} - \mathrm{rev}(P_{m-1}) \subset \cdots \subset \mathbf{k} - \mathrm{rev}(P_1) \subset \mathbf{k}$ defines a contretableau, where $\mathrm{rev}(\gamma)$ denotes $(\gamma_m, \ldots, \gamma_1)$, the reverse of $\gamma = (\gamma_1, \ldots, \gamma_m)$ and $\mathbf{k}$ denotes the partition $(k, \ldots, k)$ with m parts. This is because, if λ and μ are partitions with $\mu \subset \lambda$ then and if $k \geq \lambda_1$ then $\mathbf{k} - \mathrm{rev}(\lambda) \subset \mathbf{k} - \mathrm{rev}(\mu)$; moreover, it is easy to see with help of Young diagrams that if λ/μ is a horizontal strip, then so is $\mathbf{k} - \mathrm{rev}(\mu)/\mathbf{k} - \mathrm{rev}(\lambda)$.

Example 4.1. Let P be the GT pattern appearing in Figure 1. We have $m = 4$, and $P_4 = (6,3,2,0), P_3 = (4,3,0,0), P_2 = (3,1,0,0), P_1 = (1,0,0,0)$. Then

$$T(P) = \begin{array}{|c|c|c|c|c|c|} \hline 1 & 2 & 2 & 3 & 4 & 4 \\ \hline 2 & 3 & 3 & \multicolumn{3}{c}{} \\ \cline{1-3} 4 & 4 & \multicolumn{4}{c}{} \\ \cline{1-2} \end{array} \quad \text{and } C(P) = \begin{array}{|c|c|c|c|c|c|} \hline & & & & & \\ \hline & & & & 1 & 1 \\ \hline & & & 2 & 2 & 3 \\ \hline 1 & 1 & 2 & 3 & 3 & 4 \\ \hline \end{array}.$$

Recall the Schützenberger involution on semi-standard Young tableaux defined in Section 2. It is well-known that this operation coincides with the Lusztig–Schützenberger involution in the context of crystals [15, 3, 1]. Then if $\mathrm{rect}(T)$ denotes the rectification of a skew semi-standard Young tableau, we have:

Proposition 4.2. *Given a GT pattern P, the tableau and contretableau associated with it are swapped by the Schützenberger involution, that is, $S(T(P)) = \mathrm{rect}(C(P))$.*

Proof. Let P be a GT pattern with fixed m. It follows from the definitions of $T(P)$ and $C(P)$ that the reverse reading word of $T(P)$, respectively the reading word of $C(P)$, are obtained from the NE diagonals of P in the following way. The i-th NE diagonal of P, namely, $p_{m,i}, \dots, p_{i,i}$ determines the entries in the i-th row of $T(P)$ read from right to left, respectively the entries in the $m-i$-th row of $C(P)$ read from left to right. This is done very simply: the number of $j's$ in row i, respectively the number of $m-j's$ in row $m-i$, is given by $p_{j,i} - p_{j-1,i}$, working with the convention that $p_{i,j} = 0$ whenever $j < i$. □

Fix a positive integer m. A $m+1$-triangular grid as in Figure 2 is a m-hive triangle.

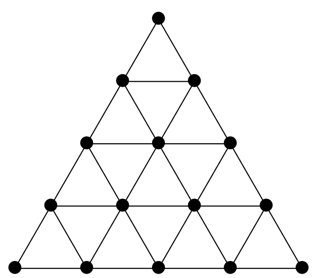
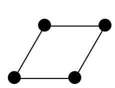
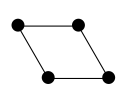
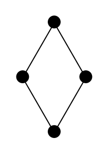


Figure 2: The 4-hive triangle

Observe that the unit rhombi in a m-hive triangle are of three kinds based on their orientation:

Northeast (NE):  Southeast (SE):  Vertical: 

A m-hive is a labelling of the $m+1$-hive triangle such that the content of each small rhombus is positive. Here, the *content* of a small rhombus is the sum of the labels on its obtuse-angled nodes minus the sum of the labels on its acute-angled nodes (in Figure 5, the content of the displayed NE rhombus would be $h_{i,j} + h_{i+1,j+1} - h_{i+1,j} - h_{i,j+1}$). Given partitions λ, μ and ν with at most m non-zero parts, the hive polytope $\mathrm{Hive}(\lambda, \mu, \nu)$ is

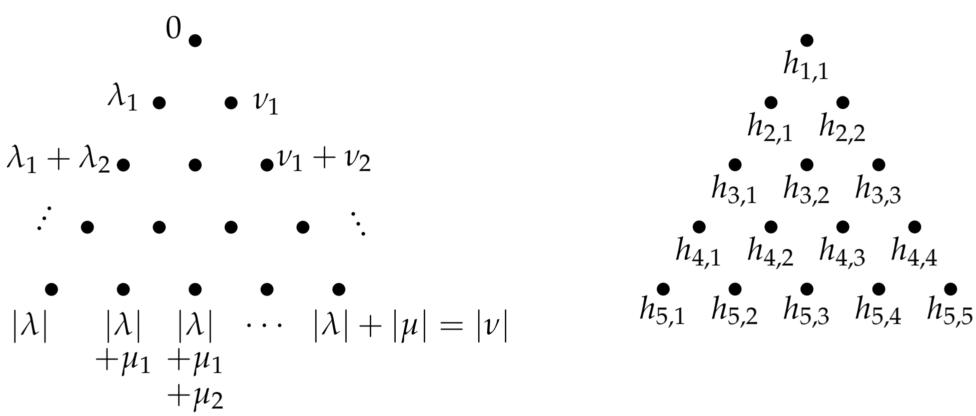


Figure 3

the set of all labellings of the m-hive triangle with the boundaries labelled as in Figure 3.

Theorem 4.3 ([2, 5]). *The LR coefficient $c^{\nu}_{\lambda,\mu}$ is given by the number of integral points in the hive polytope* $\text{Hive}(\lambda, \mu, \nu)$.

We present here the bijection $\varphi : \text{LR}^{\nu}_{\lambda,\mu} \longrightarrow \text{Hive}(\lambda, \mu, \nu)$ for the comfort of the reader. For $R \in \text{LR}^{\nu}_{\lambda,\mu}$ we compute its GT pattern $GT(R)$. Next, we obtain a new triangular arrangement of numbers $GT(R)^p$ defined by taking partial sums along the rows of $GT(R)$. More precisely, this arrangement is defined by $a_{i,i} = 0$ and $a_{i,j} := \sum_{k=1}^{j} p_{i,k}$, where $(p_{i,j})$ are the entries of the GT pattern $GT(R)$. Now let $\lambda^p = (\lambda^p_1, ..., \lambda^p_n)$ denote the n vector consisting of partial sums for λ, that is $\lambda^p_j = \sum_{i=1}^{j} \lambda^p_i$. The hive $\varphi(R)$ is obtained from $GT(R)^p$ by adding λ^p_j to each entry of $GT(R)^p$ that has the form $p'_{i,j}$.

A flag $\phi = (\phi_1, \ldots, \phi_m)$ is a weakly increasing m-tuple of positive integers such that $i \leq \phi_i \leq m$ for all $1 \leq i \leq m$. The *flagged hive polytope* corresponding to a flag ϕ is the set of all hives in $\text{Hive}(\lambda, \mu, \nu)$ for which given any k, the first $m - \phi_k$ northeast rhombi in the k^{th} (slanted) column are *flat*, i.e., their contents are 0. We will denote flagged hive polytopes by $\text{Hive}(\lambda, \mu, \nu, \phi)$. See Figure 4 for an illustration of the set of all northeast rhombi determined by the flag $(2, 3, 3, 4)$ whose contents are all required to be 0.

Given a partition ν and a flag ϕ, we define the set of *flagged tableaux* $\text{SSYT}(\nu, \phi)$ to be the set of all semi-standard tableaux of shape ν such that each entry in the k^{th} row is at most ϕ_k for all k.

Proposition 4.4 ([7]). *The set of λ-dominant flagged tableaux of shape μ with flag ϕ and weight $\nu - \lambda$ is enumerated by the number of integral points in* $\text{Hive}(\lambda, \mu, \nu, \phi)$.

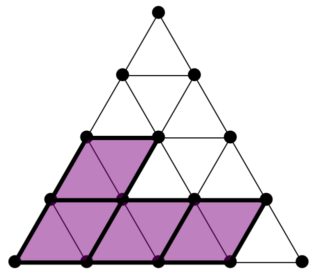


Figure 4: The region corresponding to the flag $\phi = (2, 3, 3, 4)$ and $n = 4$.

Given a hive $h = (h_{i,j})$, one can obtain a GT pattern $P(h) := p = (p_{i,j})$ by taking the successive differences along the rows (i.e., $p_{i,j} := h_{i+1,j+1} - h_{i+1,j}$). The assumption that the contents of the northeast rhombi of h are non-negative translates exactly to the GT inequality in (4.1). One could also take the northeast differences and get a GT pattern $\hat{P}(h)$ out of it in a similar fashion (i.e., $\hat{p}_{i,j} := h_{m-i+j,m-i} - h_{m-i+j-1,m-i}$).

Proposition 4.5 ([7, Proposition 4], [2, Appendix A]). *Let $h \in \mathrm{Hive}(\lambda, \mu, \nu)$ be an integral hive. Then,*

1. *$T(P(h))$ is a λ-dominant tableau of shape μ and weight $\nu - \lambda$.*
2. *$C(\hat{P}(h))$ is a μ-dominant contretableau of shape λ (i.e., a skew tableau of shape $m - \mathrm{rev}(\lambda)$).*

In fact, the map $T \circ P$ (resp. $C \circ \hat{P}$) is a bijection from $\mathrm{Hive}(\lambda, \mu, \nu)$ onto $LR^{\nu}_{\lambda,\mu}$ (resp. the set of μ-dominant contretableaux of shape λ and weight $\nu - \mu$).

Example 4.6. Let $n = 3, m = 6, \nu = (5, 4, 3, 3), \lambda = (2, 1, 1)$ and $\mu = (4, 3, 2, 2)$, and let

$$T = \begin{array}{|c|c|c|c|c|} \hline \ & \ & 1 & 1 & 1 \\ \hline \ & 2 & 2 & 2 \\ \cline{1-4} \ & 3 & 3 \\ \cline{1-3} 1 & 4 & 4 \\ \cline{1-3} \end{array} \in \mathrm{LR}(\nu/\lambda, \mu), \text{ whose companion tableau is } c(T) = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 4 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 & 3 \\ \cline{1-2} 4 & 4 \\ \cline{1-2} \end{array}.$$

The first triangular grid is the GT pattern corresponding to $c(T)$. The partial sums vector for λ is $(0, 2, 3, 4, 4)$. Therefore, to construct the hive $\phi(c(T))$ we first obtain the partial sums pattern, which is the second triangular grid.

We then proceed to add the entries of the partial sums vector to the partial sums pattern to get the hive $\varphi(c(T))$, that is, we add 0 to the first row, 2 to the second row, 3 to the third row, and 4 to the 4-th to 7-th rows.

$$\begin{array}{c} 3 \\ 3\quad 3 \\ 3\quad 3\quad 2 \\ 4\quad 3\quad 2\quad 2 \\ 4\quad 3\quad 2\quad 2\quad 0 \\ 4\quad 3\quad 2\quad 2\quad 0\quad 0 \end{array} \qquad \begin{array}{c} 0 \\ 0\quad 3 \\ 0\quad 3\quad 6 \\ 0\quad 3\quad 6\quad 8 \\ 0\quad 4\quad 7\quad 9\quad 11. \\ 0\quad 4\quad 7\quad 9\quad 11\quad 11 \\ 0\quad 4\quad 7\quad 9\quad 11\quad 11\quad 11 \end{array} \qquad \begin{array}{c} 0 \\ 2\quad 5 \\ 3\quad 6\quad 9 \\ 4\quad 7\quad 10\quad 12 \\ 4\quad 8\quad 11\quad 13\quad 15 \\ 4\quad 8\quad 11\quad 13\quad 15\quad 15 \\ 4\quad 8\quad 11\quad 13\quad 15\quad 15\quad 15 \end{array}$$

Proposition 4.7. *The image of $LRS(\nu/\mu, \lambda)$ under the companion map c is the set of μ-dominant tableaux in* $\mathrm{SSYT}(\lambda, \phi)$ *of weight $\nu - \mu$ where $\phi = (\phi_1, \dots, \phi_{2n})$ is a flag for which $\phi_k = n + \lfloor k/2 \rfloor$.*

Proof. The image is a μ-dominant tableau of shape λ and weight $\nu - \mu$ is known already (see Section 3). The description of the companion map c implies that $\phi_k = n + \lfloor k/2 \rfloor$ for all odd positive integers k. The remaining bounds follow from the fact that $c(T)$ is μ-dominant and μ has at most n parts. □

By Proposition 4.4, we see that $\mathrm{LRS}(\nu/\mu, \lambda)$ is in one-to-one correspondence with $\mathrm{Hive}(\mu, \lambda, \nu, \phi)$ where ϕ is as in Proposition 4.7.

Proposition 4.8. *Given any $h \in \mathrm{Hive}(\mu, \lambda, \nu, \phi)$, we have $\mathrm{rect}(C(\hat{P}(h))) \in \mathrm{LRK}^{\nu}_{\lambda,\mu}$.*

Proof. Combine Propositions 4.2 and 4.5 while keeping in mind that Knuth equivalence preserves μ-dominance. It is enough to prove that the image satisfies the Kwon property. Observe that a flat NE rhombus like the one in Figure 5 gives rise to the equality

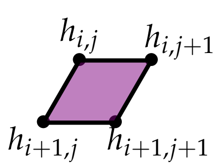
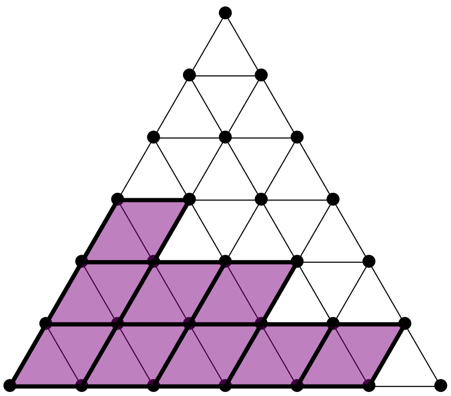


Figure 5

$\hat{p}_{n-j,i-j} = \hat{p}_{n-j+1,i-j+1}$ because the content of the NE rhombus is $h_{i,j} - h_{i+1,j} - (h_{i,j+1} - h_{i+1,j+1}) = \hat{p}_{n-j,i-j} - \hat{p}_{n-j+1,i-j+1} = 0$). These precisely translate to the Kwon condition. □

Theorem 4.9. *The following map is a bijection:*

$$\mathrm{rect} \circ C \circ \hat{P} \circ h \circ c : \ \mathrm{LRS}(\nu/\mu, \lambda) \longrightarrow \mathrm{LRK}(\nu/\lambda, \mu). \tag{4.2}$$



$$\begin{array}{ccccccccccccc}
 & & & & & \hat{p}_{1,1} & & & & & & & \\
 & & & & \hat{p}_{2,1} & & \hat{p}_{2,2} & & & & & & \\
 & & & \hat{p}_{3,1} & & \hat{p}_{3,2} & & \hat{p}_{3,3} & & & & & \\
 & & \hat{p}_{4,1} & & \hat{p}_{4,2} & & \hat{p}_{4,3} & & \hat{p}_{4,4} & & & & \\
 & \hat{p}_{5,1} & & \hat{p}_{5,2} & & \hat{p}_{5,3} & & \hat{p}_{5,4} & & \hat{p}_{5,5} & & & \\
\hat{p}_{6,1} & & \hat{p}_{6,2} & & \hat{p}_{6,3} & & \hat{p}_{6,4} & & \hat{p}_{6,5} & & \hat{p}_{6,6} & & \\
 & & & & & & & 0 & & 0 & & 0 &
\end{array}$$

Figure 6: A flagged hive h and the corresponding GT pattern $\hat{P}(h)$

Proof. By Proposition 4.7 the companion map c induces a bijection between $\mathrm{LRS}(\nu/\mu, \lambda)$ and $\mathrm{SSYT}(\nu, \phi)$. By Proposition 4.4, the map φ induces a bijection between $\mathrm{SSYT}(\nu, \phi)$ and $\mathrm{Hive}(\lambda, \mu, \nu, \phi)$. Finally, apply Proposition 4.8. □

The map is Theorem 4.9 is the LR symmetry map U described in [7]. Theorem 3.6 follows from Theorem 4.9 combined with Proposition 4.2.

Example 4.10. Let $n = 3, m = 6$ and

$$T = \begin{array}{|c|c|c|c|c|}\hline & & 1 & 1 & 1 \\ \hline & 1 & 2 & 2 \\ \cline{1-4} & 2 & 3 \\ \cline{1-3} 2 & 3 & 4 \\ \cline{1-3}\end{array} \in \mathrm{LRS}(\nu/\mu, \lambda)$$

where $\nu = (5, 4, 3, 3), \mu = (2, 1, 1)$, and $\lambda = (4, 4, 2, 1)$. Let $h := \varphi(c(T))$ be the corresponding hive. Then

$$c(T) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & 4 \\ \hline 3 & 4 \\ \cline{1-2} 4 \\ \cline{1-1}\end{array}, T(\hat{P}(h)) = \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 4 \\ \cline{1-1} 6 \\ \cline{1-1}\end{array}, C(\hat{P}(h)) = \begin{array}{|c|c|}\hline & 1 \\ \hline & 3 \\ \hline 4 & 4 \\ \hline\end{array} \text{ and } \mathrm{rect}(C(\hat{P}(h))) = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}.$$

References

[1] A. Berenstein and A. Zelevinsky. "Canonical bases for the quantum group of type A_r and piecewise-linear combinatorics". *Duke Math. J.* **82**.3 (1996), pp. 473–502. DOI.

[2] A. S. Buch. "The saturation conjecture (after A. Knutson and T. Tao)". *Enseign. Math. (2)* **46**.1-2 (2000). With an appendix by William Fulton, pp. 43–60.

[3] E. R. Gansner. "On the equality of two plane partition correspondences". *Discrete Math.* **30**.2 (1980), pp. 121–132. DOI.

[4] A. Henriques and J. Kamnitzer. "The octahedron recurrence and $\mathfrak{gl}_n$ crystals". *Adv. Math.* **206**.1 (2006), pp. 211–249. DOI.

[5] A. Knutson and T. Tao. "The honeycomb model of $\mathrm{GL}_n(\mathbf{C})$ tensor products. I. Proof of the saturation conjecture". *J. Amer. Math. Soc.* **12**.4 (1999), pp. 1055–1090. DOI.

[6] S. Kundu, K. N. Raghavan, V. Sathish Kumar, and S. Viswanath. "Saturation for flagged skew Littlewood-Richardson coefficients". *Algebr. Comb.* **7**.3 (2024), pp. 659–678. DOI.

[7] M. S. Kushwaha, K. N. Raghavan, and S. Viswanath. "The saturation problem for refined Littlewood-Richardson coefficients". *Sém. Lothar. Combin.* **85B** (2021), Art. 52, 12 pp.

[8] J.-H. Kwon. "Combinatorial extension of stable branching rules for classical groups". *Trans. Amer. Math. Soc.* **370**.9 (2018), pp. 6125–6152. DOI.

[9] J.-H. Kwon. "Lusztig data of Kashiwara-Nakashima tableaux in types B and C". *J. Algebra* **503** (2018), pp. 222–264. DOI.

[10] C. Lecouvey and C. Lenart. "Combinatorics of generalized exponents". *Int. Math. Res. Not. IMRN* 16 (2020), pp. 4942–4992. DOI.

[11] D. E. Littlewood. "On invariant theory under restricted groups". *Philos. Trans. Roy. Soc. London Ser. A* **239** (1944), pp. 387–417. DOI.

[12] D. E. Littlewood. *The theory of group characters and matrix representations of groups*. AMS Chelsea Publishing, Providence, RI, 2006, pp. viii+314. DOI.

[13] V. Sathish Kumar and J. Torres. "The branching models of Kwon and Sundaram via flagged hives". 2024. arXiv:2412.19721.

[14] B. Schumann and J. Torres. "A non-Levi branching rule in terms of Littelmann paths". *Proc. Lond. Math. Soc. (3)* **117**.5 (2018), pp. 1077–1100. DOI.

[15] M. P. Schützenberger. "Promotion des morphismes d'ensembles ordonnés". *Discrete Math.* **2** (1972), pp. 73–94. DOI.

[16] S. Sundaram. "On the combinatorics of representations of Sp(2n,C)". PhD thesis. Massachusetts Institute of Technology, 1986. Link.