Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #118, 12 pp.

Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

On e-positivity of trees and connected partitions

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Abstract. We prove that a tree with a vertex of degree at least five must be missing a connected partition of some type and therefore its chromatic symmetric function cannot be e-positive. We prove that this also holds for a tree with a vertex of degree four as long as it is not adjacent to any leaf. This brings us very close to the conjecture by Dahlberg, She, and van Willigenburg of non-e-positivity for all trees with a vertex of degree at least four. We also prove that spiders with four legs cannot have an e-positive chromatic symmetric function.

Keywords: chromatic symmetric function, connected partition, cut vertex, e-positive, proper colouring, spider graph, Stanley–Stembridge conjecture, tree isomorphism conjecture

1 Introduction

Stanley [20] defined the *chromatic symmetric function* of a graph $G = (V, E)$ to be

$$X_G = \sum_{\substack{\kappa: V \to \{1,2,3,\dots\} \\ \text{if } ij \in E, \text{ then } \kappa(i) \neq \kappa(j)}} \prod_{v \in V} x_{\kappa(v)}. \tag{1.1}$$

There are two major problems in the study of chromatic symmetric functions. The first is the Stanley–Stembridge conjecture [21], which asserts that unit interval graphs G are *e-positive*, meaning that X_G is a positive sum of elementary symmetric functions. This problem is connected to positivity of immanants of Jacobi–Trudi matrices [21], cohomology of Hessenberg varieties [1, 5, 10, 11, 18], and characters of Hecke algebras [6, 19]. Hikita [13] announced a proof of the Stanley–Stembridge conjecture by finding a positive e-expansion in terms of probabilities associated to standard Young tableaux. However, a q-analogue of this problem proposed by Shareshian and Wachs [17] is still open.

The second is the tree isomorphism conjecture, which asserts that nonisomorphic trees T and T' must have $X_T \neq X_{T'}$. This is intriguing because all trees with n vertices have the same chromatic polynomial. Many authors proved that X_T determines certain invariants of a tree and therefore it distinguishes particular subclasses of trees [4, 2, 3, 9, 12, 14, 15, 16]. For example, *spiders* $S(\lambda)$, which consist of paths of lengths $\lambda_1, \dots, \lambda_\ell$ joined at a single vertex, are distinguished by their chromatic symmetric function [15].

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Dahlberg, She, and van Willigenburg [7] considered a hybrid of these two questions by asking which trees T are e-positive. Many authors proved e-positivity or non-e-positivity for particular subclasses of trees [8, 7, 23, 25, 27]. Dahlberg, She, and van Willigenburg checked the following conjecture for all trees with at most 12 vertices.

Conjecture 1.1 ([7, Conjecture 6.1])**.** *If T is a tree with a vertex of degree $d \geq 4$, then it is not e-positive.*

Wolfgang found a powerful necessary condition for e-positivity. Let $G = (V, E)$ be an n-vertex graph and let $\lambda \vdash n$ be an integer partition of size n. A *connected partition of G of type λ* is a set partition $\mathcal{S} = \{S_1, \ldots, S_\ell\}$ of V such that

- the induced subgraph $G[S_i]$ is connected for every $1 \leq i \leq \ell$, and
- $|S_i| = \lambda_i$ for every $1 \leq i \leq \ell$.

Lemma 1.2 ([26, Proposition 1.3.3])**.** *A connected n-vertex graph G must have a connected partition of type λ for every $\lambda \vdash n$ in order to be e-positive.*

Example 1.3. *Figure 1 shows some spiders and their chromatic symmetric functions. In particular, Wolfgang's necessary condition for e-positivity is not sufficient.*

Dahlberg, She, and van Willigenburg proved the following result.

Theorem 1.4 ([7, Theorem 4.1])**.** *If G is an n-vertex connected graph with a cut vertex whose deletion produces a graph with $d \geq \log_2 n + 1$ connected components, then G is missing a connected partition of some type and therefore is not e-positive.*

In particular, if T is an n-vertex tree with a vertex of degree $d \geq \log_2 n + 1$, then T is not e-positive. Zheng strengthened this result considerably.

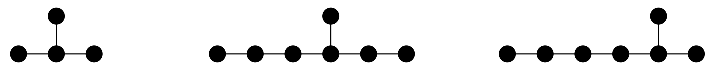
Theorem 1.5 ([27, Theorem 3.13])**.** *If G is a connected graph with a cut vertex whose deletion produces a graph with at least 6 connected components, then G is missing a connected partition of some type.*

Our main result lowers this bound further.

Theorem 1.6 ([24, Theorem 1.5])**.** *Let G be a connected graph with a cut vertex whose deletion produces a graph with either at least 5 connected components, or 4 connected components that each have at least two vertices. Then G is missing a connected partition of some type.*

Corollary 1.7. *If T is a tree with a vertex of degree at least 5, or with a vertex of degree 4 that is not adjacent to any leaf, then T is missing a connected partition of some type and therefore is not e-positive.*

Figure 1: $S(1,1,1)$: missing a connected partition of type $(2,2)$, not e-positive. $S(3,2,1)$: has a connected partition of every type $\lambda \vdash 7$ and is e-positive. $S(4,1,1)$: has a connected partition of every type $\lambda \vdash 7$ but is not e-positive.



$$
\begin{aligned}
X_{S(1,1,1)} &= e_{211} - 2e_{22} + 5e_{31} + 4e_4 \\
X_{S(3,2,1)} &= e_{2221} + 2e_{3211} + 5e_{322} + 4e_{331} + 12e_{421} + 5e_{43} + 4e_{511} + 13e_{52} + 11e_{61} + 7e_7 \\
X_{S(4,1,1)} &= e_{2221} + 4e_{3211} - 3e_{322} + 10e_{331} + 10e_{421} + 17e_{43} + 4e_{511} + 3e_{52} + 11e_{61} + 7e_7
\end{aligned}
$$

It turns out that there exists a tree T with a vertex of degree 4 that is adjacent to a leaf, and which has a connected partition of every type. Therefore, in this sense, Theorem 1.6 cannot be improved. Specifically, for every $m \geq 1$ the spider graph $S(6m, 6m-2, 1, 1)$ has a connected partition of every type [24, Proposition 3.19], so there are infinitely many examples. This means that to prove non-e-positivity of trees with a vertex of degree 4 in general, we need to know more about the tree to calculate particular coefficients. In the specific case of spiders, we will prove that Conjecture 1.1 holds.

Theorem 1.8 ([24, Theorem 1.6])**.** *If S is a spider with four legs, then it is not e-positive.*

In the next section, we discuss how Theorem 1.6 and Theorem 1.8 can be proven by cleverly choosing the parts of our integer partitions.

2 Proof of main result

We first fix notation that we will use for the remainder of this extended abstract. Let G be an n-vertex connected graph with a cut vertex v whose deletion produces a graph with at least 3 connected components $A, B, C_1, \ldots, C_k$, which have sizes

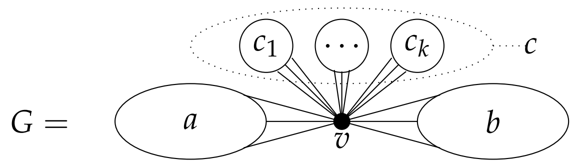
$$a \geq b \geq c_1 \geq \cdots \geq c_k \geq 1, \text{ where } k \geq 1.$$

This situation is illustrated in Figure 2. Let $c = c_1 + \cdots + c_k$. We will always assume that $c \geq 2$. We cannot say much when $c = 1$ because there are infinite families of e-positive spiders of the form $S(a, b, 1)$ [8, 23, 25]. Note that

$$n = a + b + c + 1 \geq 2b + c + 1.$$

We now state our main result using this notation.

Figure 2: A graph G with a cut vertex v joining at least 3 connected components.



Theorem 2.1. *Suppose that $c \geq 2$. If any of the following hold, then G is missing a connected partition of some type $\lambda \vdash n$.*

1. *We have $b \leq 2c - 2$.*
2. *We have $b = 2c - 1$ and $c \geq c_1 + 1$.*
3. *We have $2c \leq b \leq \frac{c^2}{2}$.*
4. *We have $b \geq \frac{c^2}{2}$ and $c \geq c_1 + 2$.*

In particular, if $c \geq c_1 + 2$, then G is missing a connected partition of some type $\lambda \vdash n$.

Remark 2.2. *The condition that $c \geq c_1 + 2$ is precisely that the deletion of v produces either at least 5 connected components, or 4 connected components that each have at least two vertices, so Theorem 2.1 implies Theorem 1.6. We cannot remove this hypothesis because for every $m \geq 1$ the spider graph $S(6m, 6m-2, 1, 1)$ has a connected partition of every type $\lambda \vdash 12m + 3$.*

Remark 2.3. *The condition that $c \geq c_1 + 1$ is precisely that the deletion of v produces at least 4 connected components. We cannot remove this hypothesis because the spider graph $S(5, 3, 2)$ has a connected partition of every type $\lambda \vdash 11$ and is in fact e-positive.*

Remark 2.4. *We cannot remove the hypothesis that $c \geq 2$ because for every $m \geq 1$ the spider graph $S(m+1, m, 1)$ has a connected partition of every type $\lambda \vdash 2m + 3$ and is in fact e-positive.*

The rest of this abstract is dedicated to proving Theorem 2.1. It turns out that Part 4 is the easiest to prove because the total number of vertices n is large compared to c.

Proposition 2.5. *If $c \geq c_1 + 2$ and $b \geq \frac{c^2}{2}$, then G is missing a connected partition of some type $\lambda \vdash n$.*

Proof. The idea is to take a partition $\lambda \vdash n$ with parts equal to $(c-1)$ or c. We first check that this is possible. The well-known Frobenius coin problem [22] tells us that given integers x and y with $\gcd(x, y) = 1$, every integer $n \geq (x-1)(y-1)$ can be written in the form $n = a_1 x + a_2 y$ for some integers $a_1, a_2 \geq 0$. Because $b \geq \frac{c^2}{2}$, we have

$$n \geq 2b + c + 1 \geq c^2 + c + 1 \geq (c-1)(c-2), \tag{2.1}$$

so we can write $n = a_1 c + a_2(c-1)$ and take λ consisting of a_1 c's and a_2 $(c-1)$'s. We now show that G is missing a connected partition of type λ.

Suppose that $\mathcal{S}$ is a connected partition of G of type λ and let $S \in \mathcal{S}$ be the subset with $v \in S$. If there is any vertex $u \in C_i$ with $u \notin S$, then the subset $U \in \mathcal{S}$ with $u \in U$ would have $U \subseteq C_i$ because v is a cut vertex of G, and therefore

$$|U| \leq |C_i| \leq c_i \leq c_1 \leq c-2, \tag{2.2}$$

which is impossible because every subset in $\mathcal{S}$ must have size $(c-1)$ or c. Therefore every $C_i \subseteq S$. However, this is also impossible because $v \in S$ and therefore

$$|S| \geq |C_1| + \cdots + |C_k| + 1 = c+1, \tag{2.3}$$

and again every subset in $\mathcal{S}$ must have size $(c-1)$ or c. So G is missing a connected partition of type λ. □

We now describe another way to choose the parts of λ so that G is missing such a connected partition.

Lemma 2.6. *Let q be a positive integer and let $J = \{x, x+1, \ldots, y-1, y\}$, where*

$$x = \left\lceil \frac{b+1}{q} \right\rceil \text{ and } y = \left\lfloor \frac{b+c}{q} \right\rfloor. \tag{2.4}$$

Also suppose that $x \geq c+1$. If $\lambda \vdash n$ is a partition with all parts in the interval J, then G is missing a connected partition of type λ.

Proof. Suppose that $\mathcal{S}$ is a connected partition of G of type λ and let $S \in \mathcal{S}$ be the subset with $v \in S$. As before, because $|S| \geq x \geq c+1 \geq c_1 + 1$, we must have every $C_i \subseteq S$. Thus $\mathcal{S}$ is of the form

$$\mathcal{S} = \{A_1, \ldots, A_i, S, B_1, \ldots, B_j\},$$

where every $A_{i'} \subseteq A$ and every $B_{j'} \subseteq B$. But now, if $j \geq q$, we would have

$$|B| \geq |B_1| + \cdots + |B_j| \geq jx \geq q \left\lceil \frac{b+1}{q} \right\rceil \geq b+1,$$

which is a contradiction. Similarly, if $j \leq q-1$, we would have

$$|B| + |C_1| + \cdots + |C_k| + 1 \leq |S| + |B_1| + \cdots + |B_j| \leq (j+1)y \leq q \left\lfloor \frac{b+c}{q} \right\rfloor \leq b+c,$$

again a contradiction. So G is missing a connected partition of type λ. □

To apply Lemma 2.6, it will be useful to generalize the result of the Frobenius coin problem by seeing which numbers can be written as a sum of numbers in an interval J.

Lemma 2.7. *Let x and y be positive integers with $x < y$. Then there exists a partition $\lambda \vdash n$ with all parts in the interval $J = \{x, \dots, y\}$ as long as*

$$n \geq \left\lceil \frac{x-1}{y-x} \right\rceil x. \tag{2.5}$$

Proof. For a positive integer t, let $tJ = \{tx, tx+1, \dots, ty-1, ty\}$. First note that every integer $n \in tJ$ has a partition $\lambda \vdash n$ consisting of t integers in J. Indeed, for $n = ty$ we can take λ to be t y's, and for $tx \leq n \leq ty - 1$ we can write $n = tq + r$ for some integers $x \leq q \leq y - 1$ and $0 \leq r \leq t - 1$, and take λ consisting of r $(q+1)$'s and $(t-r)$ q's.

We now see that as t increases, the intervals tJ become closer to each other. Specifically, for $t \geq \left\lceil \frac{x-1}{y-x} \right\rceil \geq \frac{x-1}{y-x}$, we have $ty + 1 \geq (t+1)x$, which means that as soon as we exit the interval tJ, we are already in the interval $(t+1)J$. This means that every integer n satisfying (2.5) is in some interval tJ and has a partition $\lambda \vdash n$ with all parts in J. □

In fact, this argument shows that we can find an *equitable* partition $\lambda \vdash n$, which means that all of the parts of λ are i or $(i+1)$ for some i.

Example 2.8. *Let $x = 6$ and $y = 9$, so that $J = \{6, 7, 8, 9\}$ and we can write the numbers*

$$\{\underbrace{6,7,8,9}_{J}\} \cup \{\underbrace{12,13,14,15,16,17,18}_{2J}\} \cup \{\underbrace{18,19,20,21,22,23,24,25,26,27}_{3J}\} \cup \cdots$$

as a sum of numbers in J. We see that once $t \geq \left\lceil \frac{x-1}{y-x} \right\rceil = 2$, we no longer have a gap between the intervals tJ and $(t+1)J$, so we are able to express any integer $n \geq 2x = 12$.

By considering the special case of $q = 1$ in Lemma 2.6, we can now prove Part 1 of Theorem 2.1. The proof of Part 2 is similar, instead using $q = 2$, and is omitted.

Proposition 2.9. *If $c \geq 2$ and $b \leq 2c - 2$, then G is missing a connected partition of some type $\lambda \vdash n$.*

Proof. Because $b \leq 2c - 2$, we have

$$\left\lceil \frac{b}{c-1} \right\rceil (b+1) \leq 2(b+1) \leq 2b + c + 1 \leq n,$$

so Lemma 2.7 tells us that there exists a partition $\lambda \vdash n$ with all parts in the interval $J = \{b+1, \dots, b+c\}$. By Lemma 2.6 with $q = 1$ and noting that $x = b + 1 \geq c + 1$, G is missing a connected partition of type λ. □

Now it remains to prove Part 3, so we assume that $2c \leq b \leq \frac{c^2}{2}$. Our goal is to find a positive integer q such that $x \geq c + 1$ and there exists a partition $\lambda \vdash n$ with all parts in the interval J, so that we can apply Lemma 2.6. By Lemma 2.7, this will happen if

$$\left\lceil \frac{x-1}{y-x} \right\rceil x \leq 2b + c + 1. \tag{2.6}$$

Example 2.10. *Let $b = 75$ and $c = 14$, so that $n \geq 2b + c + 1 = 165$. We will try to find a value of q for which we can apply Lemma 2.6.*

- *For $q \geq 6$, we have $x \leq \lceil \frac{76}{6} \rceil = 13 \leq c$, so this will not work.*
- *For $q = 5$, we have $x = \lceil \frac{76}{5} \rceil = 16$ and $y = \lfloor \frac{89}{5} \rfloor = 17$. We have $x \geq c + 1$, but*
$$\left\lceil \frac{x-1}{y-x} \right\rceil x = 240 > 165,$$
so this will not work. For example, we cannot write $n = 239$ as a sum of 16's and 17's.
- *For $q = 4$, we have $x = \lceil \frac{76}{4} \rceil = 19$ and $y = \lfloor \frac{89}{4} \rfloor = 22$. We have $x \geq c + 1$ and*
$$\left\lceil \frac{x-1}{y-x} \right\rceil x = 114 \leq 165,$$
so this will work. Lemma 2.7 tells us that there exists a partition $\lambda \vdash n$ with all parts in the interval $J = \{19, 20, 21, 22\}$ and Lemma 2.6 tells us that G will be missing a connected partition of type λ.

The next example will show that there may be no single value of q for which the inequality (2.6) holds. However, for every value of $n \geq 2b + c + 1$, there will exist some q such that there exists a partition $\lambda \vdash n$ with all parts in the corresponding interval J.

Example 2.11. *Let $b = 24$ and $c = 7$, so that $n \geq 2b + c + 1 = 56$.*

- *For $q \geq 4$, we have $x \leq 7 \leq c$, so this will not work.*
- *For $q = 3$, we have $x = 9$, $y = 10$, and $\left\lceil \frac{x-1}{y-x} \right\rceil x = 72 > 56$, so this will not work.*
- *For $q = 2$, we have $x = 13$, $y = 15$, and $\left\lceil \frac{x-1}{y-x} \right\rceil x = 78 > 56$, so this will not work.*
- *For $q = 1$, we have $x = 25$, $y = 31$, and $\left\lceil \frac{x-1}{y-x} \right\rceil x = 100 > 56$, so this will not work.*

We find that no single value of q satisfies (2.6). For $n \geq 72$, we can take $q = 3$, but this will not work for every $n \geq 56$. Nevertheless, for each $56 \leq n \leq 71$, there is some value of q such that we can find a partition $\lambda \vdash n$ with all parts in the corresponding interval J.

$$q = 3 : \{\underbrace{54, 55, 56, 57, 58, 59, 60}_{6J}\} \cup \{\underbrace{63, 64, 65, 66, 67, 68, 69, 70}_{7J}\} \cup \{\underbrace{72, 73, 74, \ldots}_{8J}\} \cup \cdots$$
$$q = 2 : \{\underbrace{52, 53, 54, 55, 56, 57, 58, 59, 60}_{4J}\} \cup \{\underbrace{65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75}_{5J}\} \cup \cdots$$
$$q = 1 : \{\underbrace{50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62}_{2J}\} \cup \{\underbrace{75, 76, 77, \ldots}_{3J}\} \cup \cdots$$

It turns out that there exist exactly 65 (b,c) pairs with $2 \le c \le 40$ and $2c \le b \le \frac{c^2}{2}$ for which we cannot find a single value of q such that $x \ge c+1$ and the inequality (2.6) holds. Nevertheless, we checked by computer that in each case, for every $n \ge 2b+c+1$ there is some value of q such that we can find a partition $\lambda \vdash n$ with all parts in the corresponding interval J.

Proposition 2.12 ([24, Proposition 3.13]). *If $2 \le c \le 40$ and $2c \le b \le \frac{c^2}{2}$, then G is missing a connected partition of some type $\lambda \vdash n$.*

When $c \ge 41$, we can always find a single value of q that works. We make precise estimates, the details of which are omitted. The argument is technical but elementary.

Lemma 2.13 ([24, Lemma 3.17]). *Fix positive integers b and c with $c \ge 41$ and $2c \le b \le \frac{c^2}{2}$. There exists a positive integer q such that, letting x and y be as in (2.4), we have $x \ge c+1$ and the inequality (2.6) holds.*

Remark 2.14. *We have a rough approximation*

$$\left\lceil \frac{x-1}{y-x} \right\rceil x \approx \frac{b/q}{c/q}\frac{b}{q} = \frac{b^2}{cq}.$$

To minimize this, we want q to be large, but to have $x \ge c+1$, we must have $q \le \frac{b+1}{c} \approx \frac{b}{c}$. For $q \approx \frac{b}{c}$, we have $\left\lceil \frac{x-1}{y-x} \right\rceil x \approx b$, so this is promising. However, the floors and ceilings mean that the denominator $(y-x)$ may in fact be closer to $(\frac{c}{q}-2)$, and this (-2) term can really hurt us if q is close to $\frac{c}{2}$. Therefore, when b is close to $\frac{c^2}{2}$, we will need to make a clever choice of q and perform careful estimates.

Proof sketch. There are finitely many cases where $41 \le c \le 499$, which we checked by computer, so we may now assume that $c \ge 500$. If $2c \le b \le \frac{c^2}{4.2}$, we can take $q = \left\lfloor \frac{b}{c} \right\rfloor$ and we will have $x \ge c+1$ and

$$\left\lfloor \frac{x-1}{y-x} \right\rfloor x \le 1.955b + c \le 2b+c+1.$$

If $\frac{c^2}{4.2} \le b \le \frac{c^2}{3.4}$, we can take $q = \lfloor 0.46\sqrt{b} \rfloor$ and we will have $x \ge c+1$ and

$$\left\lfloor \frac{x-1}{y-x} \right\rfloor x \le 1.997b \le 2b+c+1.$$

If $\frac{c^2}{3.4} \le b \le \frac{c^2}{2}$, then one of $q = \left\lfloor \frac{\sqrt{b}}{\sqrt{3.5}} \right\rfloor$ or $q = \left\lfloor \frac{\sqrt{b}}{\sqrt{3.5}} \right\rfloor - 1$ will have $x \ge c+1$ and

$$\left\lfloor \frac{x-1}{y-x} \right\rfloor x \le 1.92b \le 2b+c+1.$$

□

Now the remaining Part 3 of Theorem 2.1 follows from Proposition 2.12, Lemma 2.13, Lemma 2.6, and Lemma 2.7.

Proposition 2.15. *If $c \geq 2$ and $2c \leq b \leq \frac{c^2}{2}$, then G is missing a connected partition of some type $\lambda \vdash n$.*

We can also prove that spiders with four legs are not e-positive.

Proof of Theorem 1.8. Because S has four legs, we have $c \geq c_1 + 1 \geq 2$. If $b \leq \frac{c^2}{2}$, the result follows from Parts 1, 2, and 3 of Theorem 2.1. If $b \geq \frac{c^2}{2}$, then

$$n \geq 2b + c + 1 \geq c^2 + c + 1,$$

and Zheng showed [27, Corollary 4.6] that such a spider S is not e-positive. □

Because a spider with two legs is a path and is well-known to be e-positive [20, Proposition 5.3], the problem of classifying e-positive spiders is now reduced to classifying e-positivity of spiders $S(a, b, c)$ with exactly three legs.

For $c = 1$, there are infinite families of e-positive and non-e-positive spiders of the form $S(a, b, 1)$ [8, 23, 25]. Although some necessary and sufficient conditions are known in terms of divisibility properties of a and b, Figure 3 suggests that a full classification would be difficult.

For $c \geq 2$, we can use Theorem 2.1 to rule out many cases and by calculating specific coefficients, we can check many cases by computer. We conjecture that there are in fact only finitely many e-positive spiders of the form $S(a, b, c)$ where $c \geq 2$.

Conjecture 2.16. *Let $S = S(a, b, c)$ be a spider with three legs and suppose that $c \geq 2$. Then S is e-positive if and only if*

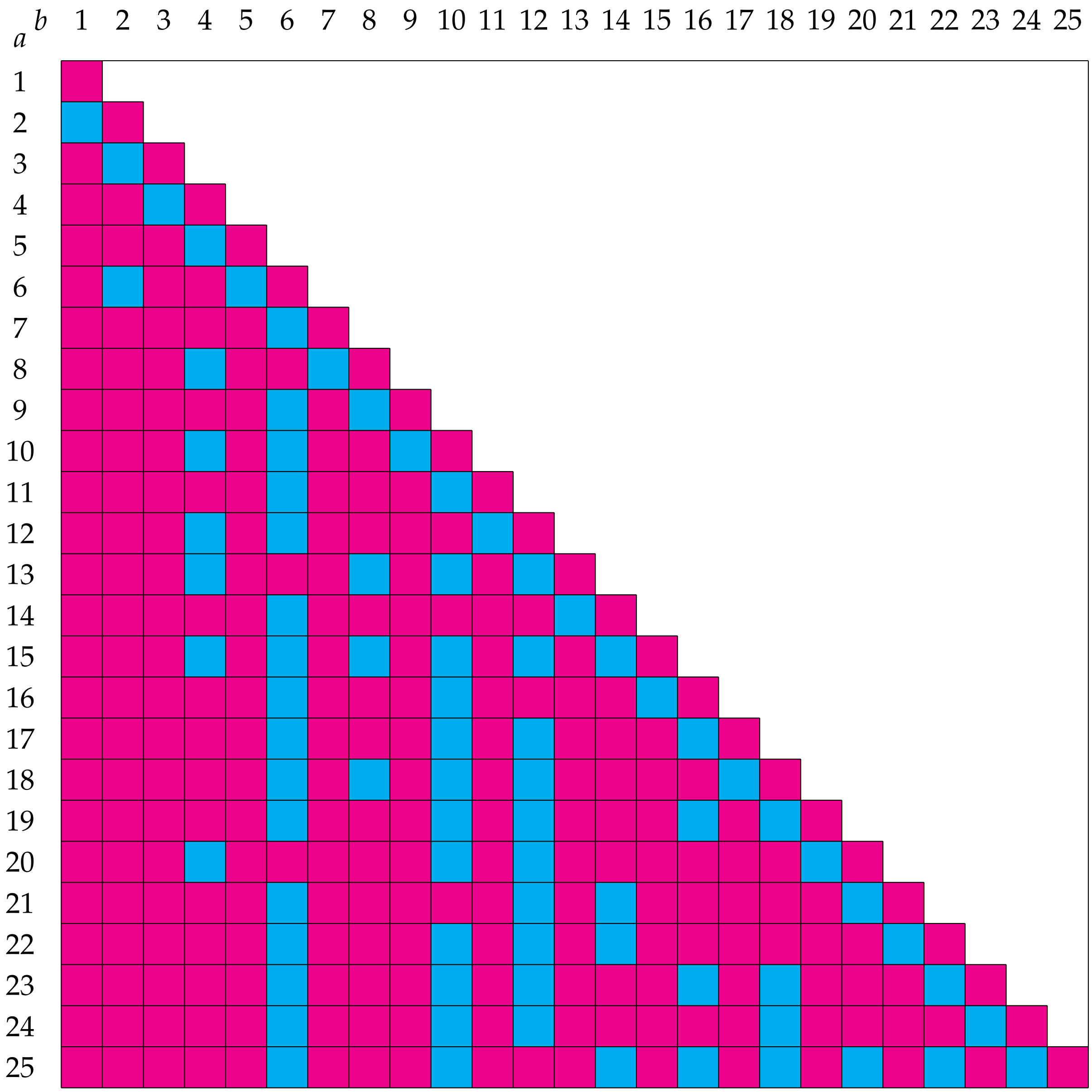
$$(a, b, c) \in \{(5, 3, 2), (6, 4, 2), (8, 6, 2), (9, 7, 2), (12, 4, 2), (8, 5, 3), (14, 9, 5)\}.$$

We checked Conjecture 2.16 up to $n = 95$ vertices.

Acknowledgements

The author would like to think José Aliste-Prieto, Richard Stanley, Stephanie van Willigenburg, and Kai Zheng for helpful discussions.

Figure 3: The e-positivity of spiders $S(a,b,1)$. Blue: e-positive. Red: not e-positive.



References

[1] A. Abreu and A. Nigro. "Splitting the cohomology of Hessenberg varieties and e-positivity of chromatic symmetric functions" (2023). arXiv:2304.10644.

[2] J. Aliste-Prieto, A. de Mier, R. Orellana, and J. Zamora. "Marked graphs and the chromatic symmetric function". *SIAM J. Discrete Math.* **37**.3 (2023), pp. 1881–1919. DOI.

[3] J. Aliste-Prieto and J. Zamora. "Proper caterpillars are distinguished by their chromatic symmetric function". *Discrete Math.* **315** (2014), pp. 158–164. DOI.

[4] J. Aliste-Prieto, J. L. Martin, J. D. Wagner, and J. Zamora. "Chromatic symmetric functions and polynomial invariants of trees". *Bull. Lond. Math. Soc.* **56**.11 (2024), pp. 3452–3476. DOI.

[5] P. Brosnan and T. Y. Chow. "Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties". *Adv. Math.* **329** (2018), pp. 955–1001. DOI.

[6] S. Clearman, M. Hyatt, B. Shelton, and M. Skandera. "Evaluations of Hecke algebra traces at Kazhdan-Lusztig basis elements". *Electron. J. Combin.* **23**.2 (2016), Paper 2.7, 56 pp. DOI.

[7] S. Dahlberg, A. She, and S. van Willigenburg. "Schur and e-positivity of trees and cut vertices". *Electron. J. of Combin.* **27**.1 (2020), Paper No. 1.2, 22 pp. DOI.

[8] S. Dahlberg, A. Foley, and S. van Willigenburg. "Resolving Stanley's e-positivity of claw-contractible-free graphs". *J. Eur. Math. Soc. (JEMS)* **22**.8 (2020), pp. 2673–2696. DOI.

[9] A. Ganesan, N. Narayanan, B. V. R. Rao, and S. S. Sawant. "Proper q-caterpillars are distinguished by their chromatic symmetric functions". *Discrete Math.* **347**.11 (2024), Paper No. 114162, 8 pp. DOI.

[10] M. Guay-Paquet. "A second proof of the Shareshian–Wachs conjecture, by way of a new Hopf algebra" (2016). arXiv:1601.05498.

[11] M. Harada and M. Precup. "The cohomology of abelian Hessenberg varieties and the Stanley–Stembridge conjecture". *Algebr. Comb.* **2**.6 (2019), pp. 1059–1108. DOI.

[12] S. Heil and C. Ji. "On an algorithm for comparing the chromatic symmetric functions of trees". *Australas. J. Comb.* **75** (2019), pp. 210–222.

[13] T. Hikita. "A proof of the Stanley–Stembridge conjecture" (2024). arXiv:2410.12758.

[14] M. Loebl and J.-S. Sereni. "Isomorphism of weighted trees and Stanley's isomorphism conjecture for caterpillars". *Ann. Inst. Henri Poincaré D* **6**.3 (2019), pp. 357–384. DOI.

[15] J. L. Martin, M. Morin, and J. D. Wagner. "On distinguishing trees by their chromatic symmetric functions". *J. Combin. Theory Ser. A* **115**.2 (2008), pp. 237–253. DOI.

[16] R. Orellana and G. Scott. "Graphs with equal chromatic symmetric functions". *Discrete Math.* **320** (2014), pp. 1–14. DOI.

[17] J. Shareshian and M. Wachs. "Chromatic quasisymmetric functions". *Adv. Math.* (2016), pp. 497–551. DOI.

[18] J. Shareshian and M. L. Wachs. "Chromatic quasisymmetric functions and Hessenberg varieties". *Configuration spaces*. Vol. 14. CRM Series. Ed. Norm., Pisa, 2012, pp. 433–460. DOI.

[19] M. Skandera. "Characters and chromatic symmetric functions". *Electron. J. Combin.* **28**.2 (2021), Paper No. 2.19, 39 pp. DOI.

[20] R. Stanley. "A symmetric function generalization of the chromatic polynomial". *Adv. Math.* **111**.1 (1995), pp. 166–194. DOI.

[21] R. Stanley and J. Stembridge. "On immanants of Jacobi–Trudi matrices and permutations with restricted position". *J. Combin. Theory Ser. A* **62**.2 (1993), pp. 261–279. DOI.

[22] J. Sylvester. "On subinvariants, i.e. Semi-Invariants to Binary Quantics of an Unlimited Order". *Am. J. Math.* **5**.1 (1882), pp. 79–136. DOI.

[23] D. Tang, D. Wang, and M. Wang. "The spiders $S(4m+2,2m,1)$ are e-positive" (2024). arXiv:2405.04915.

[24] F. Tom. "On e-positivity of trees and connected partitions" (2024). arXiv:2409.12934.

[25] D. G. L. Wang and M. M. Y. Wang. "The e-positivity and Schur positivity of some spiders and broom trees". *Discrete Appl. Math.* **325** (2023), pp. 226–240. DOI.

[26] H. Wolfgang. "Two interactions between combinatorics and representation theory: Monomial immanants and Hochschild cohomology". PhD thesis. Massachusetts Institute of Technology, 1997.

[27] K. Zheng. "On the e-positivity of trees and spiders". *J. Combin. Theory Ser. A* **189** (2022), Paper No. 105608, 22 pp. DOI.