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When is the chromatic quasisymmetric function symmetric?

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Abstract. We present several results towards the problem of determining when a chromatic quasisymmetric function (CQF) $X_G(x;q)$ of a graph G is symmetric. We first prove the remarkable fact that if a product of two quasisymmetric functions f and g in countably infinitely many variables is symmetric, then f and g must be symmetric. This allows the problem to be reduced to the case of connected graphs.

We then show that any labeled graph having more than one source or sink has a nonsymmetric CQF. As a corollary, we find that all trees other than a directed path have a nonsymmetric CQF. We also show that a family of graphs we call "mixed mountain graphs" always have symmetric CQF.

Keywords: Chromatic symmetric functions, quasisymmetric functions

1 Introduction

Chromatic symmetric functions of graphs are a topic of much recent study, especially in light of the Stanley–Stembridge and Shareshian–Wachs Conjectures. The **chromatic symmetric function** of a graph G on n vertices $1, 2, \ldots, n$ is defined to be

$$X_G(x) = X_G(x_1, x_2, \ldots) = \sum_{\substack{\kappa:[n]\to\mathbb{N} \\ \text{proper}}} x_{\kappa(1)} x_{\kappa(2)} \cdots x_{\kappa(n)}$$

where κ is a proper coloring of G with colors from $\mathbb{N}$. The **chromatic quasisymmetric function (CQF)** $X_G(x;q)$ is a q-analog of $X_G(x)$ defined as

$$X_G(x;q) = \sum_{\kappa} q^{\mathrm{asc}(\kappa)} x_{\kappa(1)} \cdots x_{\kappa(n)}$$

where $\mathrm{asc}(\kappa)$ is the number of pairs (i, j) of vertices with $i < j$ and $\kappa(i) < \kappa(j)$ (such pairs are called *ascents* of κ).

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In this extended abstract, which summarizes the results of [15], we provide several new results towards classifying which graphs have a CQF that is in fact symmetric. We make this more precise by recalling some definitions from symmetric function theory.

Definition 1.1. A **quasisymmetric function** (over $\mathbb{Z}$) in n variables is a polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$ such that for any composition $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $m \leq n$, the coefficient of $x_1^{\alpha_1} \cdots x_m^{\alpha_m}$ is the same as that of $x_{i_1}^{\alpha_1} \cdots x_{i_m}^{\alpha_m}$ for any $1 \leq i_1 < i_2 < \cdots < i_m \leq n$.

We write $\mathrm{QSym}_{\mathbb{Z}}[x_1, \ldots, x_n]$ for the ring of quasisymmetric functions in n variables, and write $\mathrm{QSym}_{\mathbb{Z}} = \mathrm{QSym}_{\mathbb{Z}}[x_1, x_2, \ldots]$ for the inverse limit as $n \to \infty$, under the restriction maps formed by setting $x_n = 0$. Thus, an element of $\mathrm{QSym}_{\mathbb{Z}}$ is a bounded-degree sum of monomials such that the coefficient of $x_{i_1}^{\alpha_1} \cdots x_{i_m}^{\alpha_m}$, for $i_1 < \cdots < i_m$, only depends on the composition α.

One natural basis for $\mathrm{QSym}_{\mathbb{Z}}$ consists of the **monomial quasisymmetric functions**

$$M_\alpha = \sum_{i_1 < \cdots < i_m} x_{i_1}^{\alpha_1} \cdots x_{i_m}^{\alpha_m}.$$

Definition 1.2. A function $f \in \mathbb{Z}[x_1, \ldots, x_n]$ is **symmetric** if for all permutations $\pi \in S_n$,

$$f(x_{\pi(1)}, \ldots, x_{\pi(n)}) = f(x_1, \ldots, x_n).$$

We write $\mathrm{Sym}_{\mathbb{Z}}[x_1, \ldots, x_n]$ for the ring of symmetric polynomials in n variables, and write $\mathrm{Sym}_{\mathbb{Z}} = \mathrm{Sym}_{\mathbb{Z}}[x_1, x_2, \ldots]$ for the inverse limit as above.

One natural basis of $\mathrm{Sym}_{\mathbb{Z}}$ consists of the **elementary symmetric functions** $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_k}$ for a partition λ of n, where e_d is defined as the sum of all square-free monomials of degree d. A symmetric function is ***e*-positive** if its expansion in the e_λ basis has all nonnegative integer coefficients. A final important basis is the **Schur basis** s_λ, which we do not define here; **Schur positivity** is always implied by e-positivity.

Example 1.3. The CQF of the path graph of length 1 shown in Figure 1 is

$$(1+q)(x_1x_2 + x_1x_3 + x_2x_3 + \cdots) = (1+q)e_2.$$

The Stanley–Stembridge conjecture [26, 27] states that certain chromatic symmetric functions are e-positive. In particular, the subset of graphs in this conjecture are the incomparability graphs of posets that are "3+1–avoiding", meaning they have no induced subposet isomorphic to the disjoint union of a chain of length 3 and a singleton. Gasharov [13] showed that for such graphs, $X_G(x)$ expands positively in the Schur basis, and a large body of modern work has shown that special sub-families of such graphs exhibit the desired e-positivity [1, 3, 5, 6, 7, 8, 9, 12, 14, 18, 21, 23, 26, 29, 30, 31]. Recently, a proof of the Stanley–Stembridge conjecture was proposed by Hikita in [20].

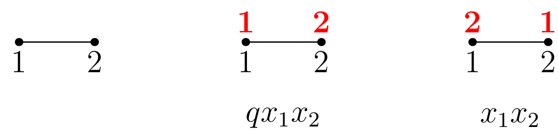

Figure 1: The labeled path graph of length 1 is shown at left; two colorings are shown at middle and right, with the middle having one ascent and the right having none.

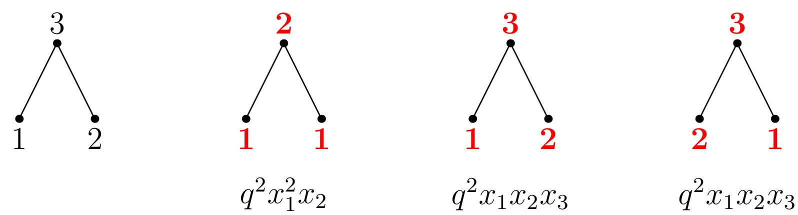

Figure 2: A graph G shown at left, for which $X_G(x;q)$ is not symmetric. Three of its colorings with maximal number of ascents are shown to the right.

The Shareshian–Wachs conjecture [25] refines this conjecture in a special case. An important example of graphs coming from $3+1$-free posets are **unit interval graphs**, whose vertex set is a collection of unit length intervals on the real line, and whose edges correspond to overlapping intervals. For such graphs, Shareshian and Wachs conjectured that $X_G(x;q)$ is also e-positive, in the sense that its coefficients in the e basis are in $\mathbb{Z}_+[q]$. They also conjectured a connection to the cohomology rings of *Hessenberg varieties*, which was later proven by Brosnan and Chow [4] and independently by Guay-Paquet [17]. Moreover, a result by Guay-Paquet [16] shows that the Shareshian–Wachs conjecture on unit interval graphs implies the Stanley–Stembridge conjecture for all $3+1$-free posets.

Both the Stanley–Stembridge conjecture and the Shareshian–Wachs conjecture have been proven in a number of notable special cases, involving various infinite subfamilies of graphs [1, 6, 7, 18, 21, 25]. There are also additional families of graphs whose CQF is known to be e-positive, most notably the cycle graphs C_n consisting of n vertices connected in a length n cycle [11]. However, not all CQF's are e-positive, and in particular they are not even necessarily symmetric functions, as shown in the following example.

Example 1.4. Consider the graph G on $\{1,2,3\}$ shown in Figure 2. The coefficient of the maximal power of q in $X_G(x;q)$, namely q^2, is

$$2e_3 + (x_1^2x_2 + x_1^2x_3 + x_2^2x_3 + \cdots) = 2e_3 + M_{2,1}$$

where $M_{2,1}$ is the monomial quasisymmetric function, which is not symmetric.

Due to what is known about the cases of cycle graphs and unit interval graphs, we pose the following question.

Question 1.5. Is every CQF that is symmetric also e-positive?

In order to begin to address this question, here we explore the problem of determining when a CQF is symmetric. We first reduce the problem to the case of connected graphs using our first main result:

Theorem 1.6. *Suppose f and g are quasisymmetric functions in countably infinitely many variables and $f \cdot g$ is symmetric. Then both f and g are symmetric.*

Note that this does not hold for finitely many variables; we have that $x_1^2 x_2 \cdot x_1 x_2^2 = x_1^3 x_2^3$ is symmetric in two variables. Since CQF's are multiplicative across disjoint union of graphs, we immediately have the following.

Corollary 1.7. *Suppose G is a graph with two or more connected components, and $X_G(x;q)$ is symmetric. Then the CQF of each connected component is symmetric.*

Some prior progress on the classification of symmetric CQF's includes the observation in [25] that if a CQF is symmetric, then its coefficients in the basis of monomial quasisymmetric functions M_α are *palindromic* polynomials in q (when reading off the coefficients of each q^i). More recently, in the preprint [2], Aliniaeifard, Asgarli, Esipova, Shelburne, van Willigenburg, and Whitehead showed that for any non-standard orientation of the path graph, the resulting CQF is not symmetric. Our second main result generalizes this to all directed acyclic graphs with more than one sink or source.

Theorem 1.8. *The CQF of any connected, directed acyclic graph with more than one sink or source is not symmetric.*

As a corollary, we obtain an answer to an open question posed by Aliniaeifard, Asgarli, Esipova, Shelburne, van Willigenburg, and Whitehead [2] of which oriented trees have a symmetric CQF.

Corollary 1.9. *The CQF of a tree is symmetric if and only if the tree is a directed path.*

Our final main result is as follows. Define a k**-mountain** to be a complete k-clique, and a **bottomless** $k+1$**-mountain** to be a $k+1$-clique with one edge removed. To make a **mixed mountain graph**, we string together any sequence of k-mountains and bottomless $k+1$-mountains, where each pair of adjacent mountains shares a single vertex, and if it is a bottomless mountain the shared vertex is one of the vertices of the missing edge. Then we connect the final endpoints of the first and last mountains by a single extra edge, as shown in Figure 3. We orient all edges from left to right, corresponding to the left to right labeling of the vertices as shown with $1, 2, 3, \ldots, n$.

Theorem 1.10. *The CQF of every mixed mountain graph is symmetric.*

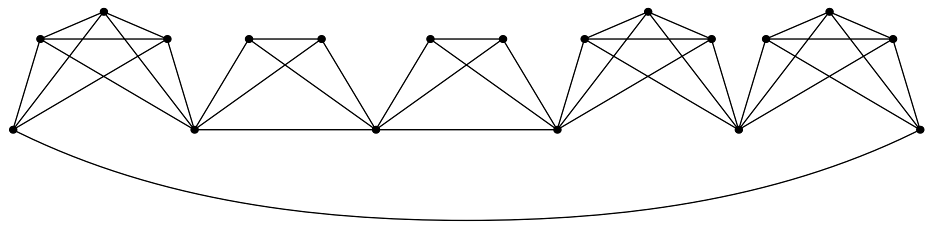

Figure 3: A mixed mountain graph for $k = 4$.

We have used Sage [28] to test all labeled, connected graphs up to 8 vertices, and have found that the mixed mountain graphs and unit interval orders describe all symmetric CQF's for these numbers. The mixed mountain graphs whose CQF we were able to compute in Sage also all had e-positive CQF's. We generally ask whether there are more families of graphs that have a symmetric CQF.

In Sections 2 to 4 below, we respectively give brief ideas and outlines of the proofs for each of the three main results (Theorems 1.6, 1.8 and 1.10).

2 Symmetric products of quasisymmetric functions

We begin with the following algebraic observation.

Lemma 2.1. *Let R be any ring. If $f \in R[x_1, \ldots, x_n]$ is irreducible, then it is also irreducible in $R[x_1, \ldots, x_n, x_{n+1}, \ldots, x_m]$.*

We now use a theorem of [19] that establishes generators for QSym using λ-ring theory. A **Lyndon word** is a word $\alpha = \alpha_1, \ldots, \alpha_n$ where α is strictly lexicographically smaller than any cycling $\alpha_i, \ldots, \alpha_n, \alpha_1, \ldots, \alpha_{i-1}$. In [19], Hazewinkle defines quasisymmetric functions $\lambda_n(M_\alpha)$ for each Lyndon word α and positive integer n, of degree $n\,(\sum_i \alpha_i)$.

Theorem 2.2 ([19, Theorem 3.1])**.** *We have* $\mathrm{QSym}_{\mathbb{Z}} = \mathbb{Z}[\lambda_n(M_\alpha)]$ *where α ranges over all Lyndon words with* $\gcd(\alpha_i) = 1$*, and* $\lambda_n(M_1) = e_n$ *for all n.*

Corollary 2.3. *We have* $\mathrm{QSym}_{\mathbb{Z}} = \mathbb{Z}[e_1, e_2, \ldots, f_1, f_2, \ldots]$ *for some functions f_i such that there are finitely many f's of each degree.*

We can finally use the above corollary that $\mathrm{QSym}_{\mathbb{Z}}$ is freely generated to prove the main result. The proof is by induction on the degree of h, which allows us to restrict to a finite number of free generators and use properties of unique factorization domains.

Theorem 1.6. Suppose $f \cdot g = h$ where f and g are quasisymmetric and h is symmetric. Then f and g are symmetric.

Corollary 2.4. *Suppose G is a graph with multiple connected components $G_1, \ldots, G_k$. Then its chromatic quasisymmetric function $X_G(x;q)$ is symmetric if and only if each $X_{G_i}(x;q)$ is symmetric.*

We now use the following theorem from [22] to obtain a more general result. We follow their definition of the full ring K of bounded-degree formal sums of monomials in infinitely many variables:

Definition 2.5. We define $K = \mathbb{Z}[[x_1, x_2, \ldots]]$ to be the ring of formal power series of bounded degree in infinitely many variables x_i.

Lam and Pylyavskyy showed that K is a unique factorization domain, as well as showing the following theorem.

Theorem 2.6 ([22, Theorem 8.1])**.** *Suppose $f \in \mathrm{QSym}$ and $f = \prod_i f_i$ is a factorization of f into irreducibles in K. Then $f_i \in \mathrm{QSym}$ for each i.*

We can use this and our results above to conclude the following.

Corollary 2.7. *If h is a symmetric function and factors as $h = f \cdot g$ where f, g are both arbitrary power series in K, then f and g are both symmetric functions.*

3 Nonsymmetry of certain families of graphs

The proof of Theorem 1.8 relies on establishing some general results on conditions for nonsymmetry for CQF's. We work with vertex-labeled graphs G, with vertices labeled $1, 2, \ldots, n$, and we write $[n] = \{1, 2, \ldots, n\}$. The labeling induces an acyclic orientation on the edges of G, where an edge points from the smaller vertex label to the larger. Moreover, any directed acyclic graph admits a labeling that respects the orientation of the edges. Thus, it equivalent to define the CQF on a directed acyclic graph G where given a proper coloring κ of G, a directed edge (u, v) is said to have an ascent if $\kappa(u) < \kappa(v)$.

3.1 General results for directed acyclic graphs

Given a directed acyclic graph G, let G^{rev} denote the acyclic graph obtained from G by reversing the orientation of all of the edges.

Lemma 3.1. *Let G be a directed acyclic graph. Then $X_G(x;q)$ is symmetric if and only if $X_{G^{\mathrm{rev}}}(x;q)$ is symmetric.*

Any vertex of a directed acyclic G that only has edges exiting the vertex is called a **source**. Likewise, any vertex with only incoming edges is said to be a **sink**. The following observation is due to Liu [24].

Lemma 3.2. *If G has a different number of sources and sinks, then $X_G(x;q)$ is not symmetric.*

Definition 3.3. An **antichain** in a directed acyclic graph is a set of vertices $\{v_1, \ldots, v_k\}$ such that there is not a directed path from any v_i to v_j.

Note that this matches the definition of antichain where we consider the directed acyclic graph as a poset. The following is a key lemma in our proofs.

Lemma 3.4. *Let G be a directed acyclic graph. If G has an antichain whose size is larger than the number of sinks (and the number of sources), then $X_G(x;q)$ is not symmetric.*

3.2 Directed acyclic graphs and trees

Consider a connected, directed acyclic graph G with at least two sources. We show that the chromatic quasisymmetric function of G is not symmetric. As before, we will view G as a poset where a vertex w is less than v if and only if there is a directed path from w to v. Let n be the number of vertices in G and $a \geq 2$ be its number of sources (and therefore its number of sinks by Lemma 3.2). Let $S(G)$ be the set of all vertices v having at least two sources that are smaller than v. As G has at least two sources, $S(G)$ is nonempty.

Given $v \in S(G)$, define $\mathrm{stat}(v)$ to be the number of nonsource vertices smaller than v. Let $k = 1 + \min_{v \in S(G)} \mathrm{stat}(v)$. Denote by $K_\alpha^{|E|}$ the set of proper colorings of G having weight α and $|E|$ ascents. We will construct a strictly injective map from $K^{|E|}_{(1^k,a,1^{n-k-a})}$ to $K^{|E|}_{(a,1^{n-a})}$. Before defining this map, we recall Dilworth's Theorem.

Theorem 3.5 (Dilworth's Theorem, [10])**.** *Let P be a finite poset. Then the largest antichain of P is equal to the minimum number of disjoint chains needed to cover P.*

By Lemma 3.4, we may assume that the largest antichain of G has size a. Thus, G can be covered by a disjoint chains, each of which contains exactly one source and sink. Fix such a minimal chain decomposition R of the graph G.

Definition 3.6. Let $\varphi_{G,R} \colon K^{|E|}_{(1^k,a,1^{n-k-a})} \to K^{|E|}_{(a,1^{n-a})}$ where $\varphi_{G,R}(\kappa)$ is the coloring obtained by iterating over every chain in R that does not contain a vertex colored 1 as follows: Recolor the vertex colored $k+1$ with the color 1. Then sort the colors in the chain such that they are increasing from the chain's source to the chain's sink.

Lemma 3.7. *The map $\varphi_{G,R}$ is well-defined and injective, but not surjective.*

Lemma 3.7 gives us the desired result on directed acyclic graphs.

Theorem 1.8. Let G be a connected, directed acyclic graph. If G has at least two sources (or sinks), then $X_G(x;q)$ is not symmetric.

As a corollary, we obtain the following characterization of all directed trees with symmetric chromatic quasisymmetric functions, thereby settling an open question in [2].

Corollary 3.8. *Let T be a directed tree. Then $X_T(x;q)$ is symmetric if and only if T is a directed path.*

We can also characterize which directed acyclic cycles have a symmetric chromatic quasisymmetric function. A directed acyclic cycle is said to be naturally oriented if the cycle has exactly one source and sink with an edge going from its source to sink.

Corollary 3.9. *Let C be a directed acyclic cycle. Then $X_C(q;t)$ is symmetric if and only if C is naturally oriented.*

4 Symmetry of mixed mountain graphs

We define the (p,k)**-mountain graph** $M_{p,k}$ by replacing all but one edge in a cycle of length $p+1$ with a k-clique (we require $p \geq 2$ and $k \geq 2$). We call each of the k-cliques the **mountains** of $M_{p,k}$. We draw the graph so that the unchanged edge connects the leftmost and rightmost vertices - we call this edge the **bottom edge**. Then we order the vertices from left to right, as seen in Figure 4.

Mountain graphs are not natural unit interval graphs unless $p = 2$ and $k = 2$ (or a relabeling of this graph when $p = 2$ and $k = 3$). In the case of $(p,2)$-mountain graphs, our work recovers the result that the chromatic quasisymmetric functions of naturally labeled cycles are symmetric [11]. We further generalize this definition as follows.

Definition 4.1. A (p,k)**-mixed mountain graph** is a (p,k)-mountain graph where some number of the k-cliques are replaced with $k+1$-cliques with the edge between the two bottom vertices removed ("bottomless mountains"). See Figure 3.

Given a proper coloring $\kappa : V(G) \to \mathbb{N}$, define the $(a, a+1)$**-colored subgraph** to be the induced subgraph on the vertex set $\kappa^{-1}(\{a, a+1\})$. To show that $X_G(x;q)$ is symmetric, we wish to show that for all colors a, there is an ascent-preserving bijection on colorings of G that swaps the number of instances of a and $a+1$, but preserves the number of instances of all other colors. We do so via a number of composite bijections that apply in different cases. Our proof outline is as follows.

Step 1. Reduce to the case that $a, a+1$ label the bottom edge. If the two vertices on the bottom edge are not colored a and $a+1$, then the $(a, a+1)$-colored subgraph does not include the bottom edge. In this case, the work of Shareshian and Wachs [25] on unit interval graphs determines the map on the coloring that swaps $a, a+1$. In particular we can reduce to the case that the two vertices on the bottom edge have consecutive colors.

Step 2. Reduce to $a = 1$. We define a map cycle for $a > 1$ that cycles the color labels as follows (see Figure 4). Starting with a coloring κ, change all vertices colored 1 to color $c+1$ where c is the largest color appearing in κ. Then, for each vertex colored $c+1$:

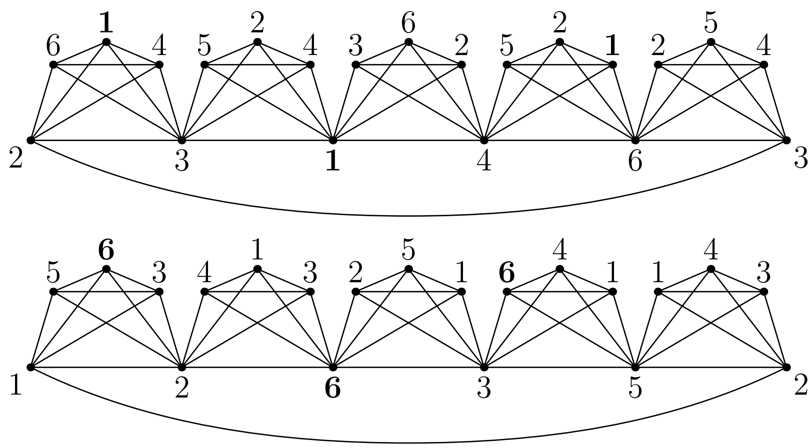

Figure 4: Above, a coloring κ of G, and below, the output cycle(κ).

- If the vertex is a common vertex between two cliques, do nothing.
- Otherwise, suppose that it is the i-th upper vertex from the left in its clique. Then reorder the upper vertices in the clique so that this vertex is the i-th upper vertex from the right, while the relative order of the other vertices is preserved.

Finally, reduce the value of all colors by 1. We call the output cycle(κ), and note that the bottom edge of cycle(κ) has vertices labeled $a-1$ and a. After applying the cycle map $a-1$ times, the vertices on the bottom edge are colored 1 and 2.

Lemma 4.2. *The colorings κ and* cycle(κ) *have the same number of ascents.*

Step 3: Reflect. We next define a map reflect that gives a proper coloring on G^{rev}. We do not give the full definition here, but instead list its key properties.

Lemma 4.3. *Let G be a mixed mountain graph, and let κ be a coloring of G with maximal color c such that the vertices on the bottom edge are colored* 1 *and* 2*. Then* reflect(κ) *is a coloring of G^{rev} with the following properties (see Figure 5):*

- *The map* reflect *swaps the number of* 1*'s and* 2*'s, and swaps the number of* 3*'s and c's,* 4*'s and $(c-1)$'s, and so on.*
- *The colorings κ and* reflect(κ) *have the same number of ascents.*

For ordinary mountain graphs, the above two maps are sufficient to prove that $X_G(x;q)$ is symmetric. For mixed mountain graphs, we need one more.

Step 4. Swap. For a mixed mountain graph, the k-cliques and bottomless $k+1$-cliques appear in reverse order in G^{rev}, so reflect does not preserve the graph. We remedy this issue by showing that the CQF is independent of the ordering of the k-cliques and bottomless $k+1$-cliques, and instead only depends on the number of each.

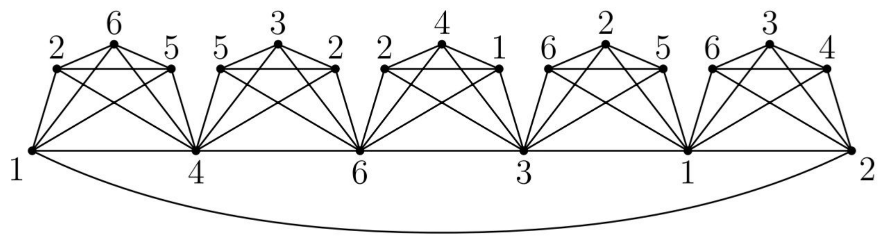

Figure 5: The output reflect(cycle(κ)) for κ as in Figure 4.

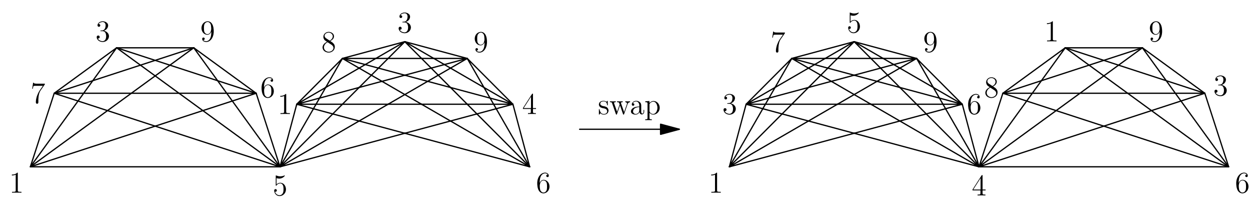

Figure 6: An example of swap, showing that we can swap a k-clique with an adjacent bottomless $k+1$-clique without changing its chromatic quasisymmetric function.

Indeed, let G be a mixed mountain graph, with a k-clique $\mathcal{A}$ to the left of a bottomless $(k+1)$-clique $\mathcal{B}$. Let G' be the graph obtained by swapping $\mathcal{A}$ and $\mathcal{B}$. For a proper coloring κ of G, we define a proper coloring swap(κ) of G' with the following properties:

- The colors on vertices outside of $\mathcal{A}$ and $\mathcal{B}$ are unchanged.
- The colorings κ and swap(κ) have the same number of instances of each color, and $\mathrm{asc}(\kappa) = \mathrm{asc}(\mathrm{swap}(\kappa))$.
- The map $\kappa \mapsto \mathrm{swap}(\kappa)$ is a bijection between colorings of G and G'.

See Figure 6 for an example of the swap map, and [15] for the full definition.

Corollary 4.4. *If G is a (p,k)-mixed mountain graph, and G' is the (p,k)-mixed mountain graph obtained by swapping an adjacent k-clique and bottomless $k+1$-clique, then $X_G(x;q) = X_{G'}(x;q)$.*

As a consequence of the above result, it suffices to show that chromatic quasisymmetric functions of (p,k)-mixed mountain graphs are symmetric where every k-mountain is to the left of every $(k+1)$-bottomless mountain. Using the properties of cycle, reflect, and swap, we have the following:

Proposition 4.5. *There is an ascent-preserving automorphism on the set of colorings of κ, which color vertices on the bottom edge a and $a+1$, and this automorphism swaps the number of occurrences of the colors a and $a+1$.*

Proposition 4.5 and Corollary 4.4 imply the desired symmetry of the chromatic quasisymmetric function of (p,k)-mixed mountain graphs.

Theorem 1.10. Let G be a (p,k)-mixed mountain graph. Then $X_G(x;q)$ is symmetric.

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