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Deforming the motivic Segre classes of Schubert cells in the Grassmannian

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Abstract. We β-deform the motivic Segre classes of Schubert cells in the d-step flag variety, conjecturing that the β-deformations come from analogues of stable envelopes in the equivariant connective K-ring of the cotangent bundle of the flag variety. Our main result is a combinatorial formula for the structure constants in the β-deformed basis in the $d = 1$ case using Knutson–Tao puzzles.

Keywords: divided difference operators, Knutson–Tao puzzles, connective K-theory

1 Introduction

Consider the d-step flag variety $\mathrm{Fl}_n(i_1, \dots, i_d)$ consisting of flags in $\mathbb{C}^n$ of the form

$$(0) \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{d-1} \subseteq V_d \subseteq \mathbb{C}^n, \quad \dim(V_j) = i_j.$$

When $d = 1$, the flag variety $\mathrm{Fl}_n(k)$ is the Grassmannian $\mathrm{Gr}(k, n)$ of k-planes in $\mathbb{C}^n$. The flag variety $\mathrm{Fl}_n(i_1, \dots, i_d)$ has a distinguished cell decomposition called the Bruhat decomposition, whose cells are indexed by $012 \cdots d$ sequences $\lambda = \lambda_1 \cdots \lambda_n$, where d appears i_1 times, $d - 1$ appears $i_2 - i_1$ times, and so on. We will denote the set of such sequences by Γ. The cells defining the Bruhat decomposition are called the Schubert cells X_λ°, $\lambda \in \Gamma$, and the closures of the Schubert cells are the Schubert varieties $X_\lambda := \overline{X_\lambda^\circ}$. There is a natural action of the n-dimensional complex torus T on $\mathrm{Fl}_n(i_1, \dots, i_d)$, and the Schubert varieties X_λ are invariant under this torus action. The varieties X_λ define natural classes S_λ, called Schubert classes, in the T-equivariant cohomology ring $H_T(\mathrm{Fl}_n(i_1, \dots, i_d))$. In fact, the Schubert classes form a basis for $H_T(\mathrm{Fl}_n(i_1, \dots, i_d))$ over $H_T(\mathrm{pt}) \simeq \mathbb{Z}[y_1, \dots, y_n]$, where pt denotes a point. The Littlewood–Richardson coefficients $c_{\lambda,\mu}^\nu$ are the structure constants in the Schubert basis:

$$S_\lambda \cdot S_\mu = \sum_\nu c_{\lambda,\mu}^\nu S_\nu.$$

In the paper [6], Knutson and Tao introduced the first manifestly positive formula for the Littlewood–Richardson coefficients $c_{\lambda,\mu}^\nu$ in the T-equivariant cohomology ring

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$H_T(\mathrm{Gr}(k,n))$ of the Grassmannian $\mathrm{Gr}(k,n)$. The combinatorial objects they used to compute the Littlewood–Richardson coefficients are called *puzzles*.

In [11] and [12], Maulik and Okounkov defined natural geometric classes, called *stable classes*, in the $(T \times \mathbb{C}^\times)$-equivariant cohomology ring and the $(T \times \mathbb{C}^\times)$-equivariant K-ring of the cotangent bundle $T^*(\mathrm{Fl}_n(i_1,\ldots,i_d))$ of $\mathrm{Fl}_n(i_1,\ldots,i_d)$. Dividing the stable classes in $H_{T\times\mathbb{C}^\times}(T^*(\mathrm{Fl}_n(i_1,\ldots,i_d)))$ and $K_{T\times\mathbb{C}^\times}(T^*(\mathrm{Fl}_n(i_1,\ldots,i_d)))$ by an appropriate constant class gives the *Segre–Schwartz–MacPherson (SSM) classes* and *motivic Segre classes* of Schubert cells, respectively. The SSM and motivic Segre classes of Schubert cells are both indexed by Γ. In [7] and [8], Knutson and Zinn-Justin proved combinatorial formulas for the structure constants in the bases of SSM classes and motivic Segre classes when $d = 1, 2$ using puzzles by applying the theory of integrable systems.

In this paper, we define a one-parameter (β-)deformation of the motivic Segre classes of Schubert cells for $T^*(\mathrm{Fl}_n(i_1,\ldots,i_d))$. Evaluating our deformed classes at $\beta = 1$ recovers the motivic Segre classes of Schubert cells, and evaluating at $\beta = 0$ recovers the SSM classes of Schubert cells. Our main result is Theorem 4.1, which is a combinatorial formula for the structure constants in the β-deformed basis in the case $d = 1$ using puzzles. In Conjecture 5.4, we conjecture that our β-deformed classes arise as analogues of stable classes in the $(T \times \mathbb{C}^\times)$-equivariant *connective* K-ring of $T^*(\mathrm{Fl}_n(i_1,\ldots,i_d))$.

2 Preliminaries and notation

2.1 Flag varieties

Let $G = \mathrm{GL}_n(\mathbb{C})$. Fix a Borel subgroup B in G, and a maximal torus T in B. Denote the weight lattice of T by $\Lambda := \mathrm{Hom}(T, \mathbb{C}^\times)$, and denote the root system of T by $\Sigma := A_{n-1}$. We denote the set of simple roots of Σ by $\Delta := \{\alpha_1, \ldots, \alpha_{n-1}\}$. The root system Σ decomposes as $\Sigma = \Sigma^+ \sqcup \Sigma^-$, where Σ^+ (resp. Σ^-) is the set of positive (resp. negative) roots of Σ. For a subset $\Theta := \{\alpha_{i_1}, \ldots, \alpha_{i_d}\}$, $i_1 < i_2 < \cdots < i_d$, of the simple roots Δ, we denote by P_Θ a parabolic subgroup of G that contains B and has (negative) simple roots $\Delta \setminus \Theta$. The d-step flag variety G/P_Θ is the variety of (partial) flags of vector spaces,

$$(0) \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_{d-1} \subseteq V_d \subseteq \mathbb{C}^n, \quad \dim(V_j) = i_j.$$

In the special case $\Theta = \Delta$, the flag variety G/P_Θ is the complete flag variety G/B. In the special case $\Theta = \{\alpha_k\}$, the group P_Θ is a maximal parabolic subgroup of G, and G/P_Θ is the Grassmannian $\mathrm{Gr}(k,n)$ of k-planes in $\mathbb{C}^n$. A **negative root** of G/P_Θ is a root in Σ that is *not* a root of P_Θ. Let Σ_Θ^- be the set of negative roots of G/P_Θ, and let $\Sigma_\Theta^+ := -\Sigma_\Theta^-$ be the set of **positive roots** of G/P_Θ. Define $\Sigma_\Theta := \Sigma_\Theta^+ \sqcup \Sigma_\Theta^-$. We will denote the Weyl group S_n of G by W. Let r_i be the simple transposition in W that swaps i and $i+1$ and fixes all other j. The subgroup W_Θ of W generated by $\langle r_j \rangle_{\alpha_j \in \Delta \setminus \Theta}$ is the Weyl group of P_Θ.

Let W^Θ be the set of minimal coset representatives of W/W_Θ in W. We will henceforth identify the set W^Θ with the quotient W/W_Θ. Define $p_j := i_j - i_{j-1}$. The set W^Θ is in bijection with the set Γ of lists of length n in the alphabet $\{0, \dots, d\}$, where d appears p_1 times, $d-1$ appears p_2 times, and so on. Under this identification, the longest word w_0 in W^Θ is identified with the list $d^{p_1}(d-1)^{p_2} \cdots 1^{p_d}0^{p_{d+1}}$, and $w \cdot w_0 \in W^\Theta$ is identified with the list $w(d^{p_1}(d-1)^{p_2} \cdots 1^{p_d}0^{p_{d+1}})$.

For each $w \in W^\Theta$, there is a Schubert cell $X_w^\circ = BwP_\Theta/P_\Theta$ in G/P_Θ. The Schubert cells form a cell decomposition for G/P_Θ. The closures of the Schubert cells $X_w := \overline{X_w^\circ}$ are Schubert varieties. Let $\mathrm{h}^*(-)$ be one of the following (generalized) cohomology theories: singular cohomology $H^*(-)$, T-equivariant cohomology $H_T(-)$, K-theory $K(-)$, and T-equivariant K-theory $K_T(-)$. Let pt be a point, so that $H^*(\mathrm{pt}) \simeq \mathbb{Z}$, $H_T(\mathrm{pt}) \simeq \mathbb{Z}[y_1, \dots, y_n]$, $K(\mathrm{pt}) \simeq \mathbb{Z}$, and $K_T(\mathrm{pt}) \simeq \mathbb{Z}[e^{\pm y_1}, \dots, e^{\pm y_n}]$. In $\mathrm{h}^*(G/P_\Theta)$ there are natural classes S_w, known as Schubert classes, that form a basis for $\mathrm{h}^*(G/P_\Theta)$, when we view $\mathrm{h}^*(G/P_\Theta)$ as a module over $\mathrm{h}^*(\mathrm{pt})$. In $K(G/P_\Theta)$ and $K_T(G/P_\Theta)$, the Schubert class S_w is the class of the structure sheaf $[\mathcal{O}_{X_w}]$ of X_w. In $H^*(G/P_\Theta)$ and $H_T(G/P_\Theta)$, the Schubert class S_w is the class $[X_w]$ defined by the Schubert variety X_w.

2.2 Cotangent bundles of flag varieties

The cotangent bundle $T^*(G/P_\Theta)$ of G/P_Θ has a natural action of $\widehat{T} := T \times \mathbb{C}^\times$, where $(t, z) \in \widehat{T}$ acts by $(t, z) \cdot (x, \vec{v}) := (t \cdot x, (t^{-1})^*(z \cdot \vec{v}))$ for all $(x, \vec{v}) \in T_x^*(G/P_\Theta)$. For any symplectic resolution $X \to Y$ with an algebraic $\widehat{T}$-action subject to certain conditions, Maulik and Okounkov defined natural geometric bases in $H_{\widehat{T}}^*(X)$ and $K_{\widehat{T}}(X)$ ([11], [12]), which arise from $\widehat{T}$-invariant cycles in X. These bases are called the cohomological and K-theoretic stable bases. When $X = T^*(G/P_\Theta)$, the cohomological and K-theoretic stable bases are indexed by W^Θ, and we will denote the stable basis in either case by $\{\mathrm{St}_\lambda\}_{\lambda \in W^\Theta}$. Consider the localizations $H_{\widehat{T}}^{\mathrm{loc}}(\mathrm{pt})$ and $K_{\widehat{T}}^{\mathrm{loc}}(\mathrm{pt})$ of the rings $H_{\widehat{T}}(\mathrm{pt}) \simeq H_T(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{Z}[\hbar]$ and $K_{\widehat{T}}(\mathrm{pt}) \simeq K_T(\mathrm{pt}) \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 2}]$ at $\{\hbar + \alpha\}_{\alpha \in \Sigma}$ and $\{1 - q^2 e^\alpha\}_{\alpha \in \Sigma}$, respectively. Here, $\hbar$ and q^2 denote the equivariant parameters coming from the $\mathbb{C}^\times$-factor of $\widehat{T}$ that dilates the cotangent fibres of $T^*(G/P_\Theta)$. Set $H_{\widehat{T}}^{\mathrm{loc}}(T^*(G/P_\Theta)) := H_{\widehat{T}}^{\mathrm{loc}}(\mathrm{pt}) \otimes_{H_{\widehat{T}}(\mathrm{pt})} H_{\widehat{T}}^*(T^*(G/P_\Theta))$ and $K_{\widehat{T}}^{\mathrm{loc}}(T^*(G/P_\Theta)) := K_{\widehat{T}}^{\mathrm{loc}}(\mathrm{pt}) \otimes_{K_{\widehat{T}}(\mathrm{pt})} K_{\widehat{T}}(T^*(G/P_\Theta))$. Let κ be the class of the zero section of $T^*(G/P_\Theta) \to G/P_\Theta$ in either $H_{\widehat{T}}(T^*(G/P_\Theta))$ or $K_{\widehat{T}}(T^*(G/P_\Theta))$. The elements $S_\lambda := \frac{\mathrm{St}_\lambda}{\kappa}$ live in the localization $H_{\widehat{T}}^{\mathrm{loc}}(T^*(G/P_\Theta))$ or the localization $K_{\widehat{T}}^{\mathrm{loc}}(T^*(G/P_\Theta))$, and $\{S_\lambda\}_{\lambda \in W^\Theta}$ forms a basis for $H_{\widehat{T}}^{\mathrm{loc}}(T^*(G/P_\Theta))$ or $K_{\widehat{T}}^{\mathrm{loc}}(T^*(G/P_\Theta))$ over the fraction field $\mathrm{Frac}(H_{\widehat{T}}(\mathrm{pt}))$ or the fraction field $\mathrm{Frac}(K_{\widehat{T}}(\mathrm{pt}))$, respectively. The elements S_λ in $H_{\widehat{T}}^{\mathrm{loc}}(T^*(G/P_\Theta))$ are called **Segre–Schwartz–MacPherson (SSM) classes**, and the elements S_λ in $K_{\widehat{T}}^{\mathrm{loc}}(T^*(G/P_\Theta))$ are called **motivic Segre classes**.

2.3 Equivariant localization and divided difference operators

Let X be a smooth complex algebraic variety, and suppose E is a complex linear algebraic group acting on X algebraically with finitely many fixed points F. For both $h_E^*(-) = H_E^*(-)$ and $h_E^*(-) = K_E(-)$, the localization map

$$\iota\colon h_E^*(X) \to \bigoplus_{f\in F} h_E^*(\mathrm{pt}),$$

defined by restricting $h_E^*(X)$ to the corresponding fixed point f on each component, is injective. We will consider the case $X = G/P_\Theta$ and $E = T$, or the case $X = T^*(G/P_\Theta)$ and $E = \widehat{T}$. In either case, the E-fixed points of X are indexed by W^Θ. We will now describe how to obtain the images of the classes S_λ in the rings $H_T^*(G/P_\Theta)$, $H_{\widehat{T}}^{\mathrm{loc}}(T^*(G/P_\Theta))$, $K_T(G/P_\Theta)$, and $K_{\widehat{T}}^{\mathrm{loc}}(T^*(G/P_\Theta))$ under the localization map.

Given $\alpha_i \in \Delta$, there is a $\mathbb{Z}$-linear operator ∂_i that acts on $\bigoplus_{\lambda\in W^\Theta} h_E^*(\mathrm{pt})$, which we will define explicitly below. Any $w \in W^\Theta$ can be expressed as the reduced product of simple transpositions $w = r_{j_1}\cdots r_{j_k}$. The operator $\partial_w := \partial_{j_1}\circ\cdots\circ\partial_{j_k}$ also acts on $\bigoplus_{\lambda\in W^\Theta} h_E^*(\mathrm{pt})$ and is independent of the choice of reduced expression for w. We have $\iota(S_{w\cdot w_0}) = \partial_w(\iota(S_{w_0}))$, where w_0 is the longest word in W^Θ. We will now explicitly describe the restrictions $S_{w_0}|_\lambda$ of S_{w_0} to each fixed point $\lambda \in W^\Theta$, and we will give an explicit formula defining the operator ∂_i in each of the four cases $H_T^*(G/P_\Theta)$, $H_{\widehat{T}}^{\mathrm{loc}}(T^*(G/P_\Theta))$, $K_T(G/P_\Theta)$, and $K_{\widehat{T}}^{\mathrm{loc}}(T^*(G/P_\Theta))$. Given $\alpha = c_1\alpha_1 + \cdots + c_{n-1}\alpha_{n-1} \in \Sigma$, we denote by y_α the element $c_1(y_1 - y_2) + c_2(y_2 - y_3) + \cdots + c_{n-1}(y_{n-1} - y_n) \in \mathbb{Z}[y_1, \ldots, y_n]$.

1. $H_T^*(G/P_\Theta)$. Below are the formulas for S_{w_0} restricted to fixed points $\lambda \in W^\Theta$:

$$S_{w_0}|_\lambda = \begin{cases} \prod_{\alpha\in\Sigma_\Theta^+} y_{-\alpha}, & \lambda = w_0; \\ 0, & \lambda \neq w_0. \end{cases}$$

There is a natural action of W on the ring $H_T^*(\mathrm{pt}) \simeq \mathbb{Z}[y_1, \ldots, y_n]$, defined on generators by $w(y_i) := y_{w(i)}$ for all $w \in W$. This W-action extends to the ring $\bigoplus_{\lambda\in W^\Theta} H_T^*(\mathrm{pt})$ by $w(f_\lambda)_\lambda := (w(f_\lambda))_{w(\lambda)}$ for all $w \in W$. Consider the $\mathbb{Z}$-linear operators on $H_T^*(\mathrm{pt})$:

$$\partial_i(f) = \frac{f - r_i(f)}{y_{\alpha_i}}, \quad f \in H_T^*(\mathrm{pt}),\ \alpha_i \in \Delta.$$

The operators ∂_i naturally define $\mathbb{Z}$-linear operators on $\bigoplus_{\lambda\in W^\Theta} H_T^*(\mathrm{pt})$.

2. $H_{\widehat{T}}^{\mathrm{loc}}(T^*(G/P_\Theta))$. Below are the formulas for S_{w_0} restricted to fixed points $\lambda \in W^\Theta$:

$$S_{w_0}|_\lambda = \begin{cases} \prod_{\alpha\in\Sigma_\Theta^+} \frac{y_{-\alpha}}{y_{-\alpha}-\hbar}, & \lambda = w_0; \\ 0, & \lambda \neq w_0. \end{cases}$$

There is a natural action of W on the ring $H^*_{\widehat{T}}(\mathrm{pt}) \simeq \mathbb{Z}[y_1, \ldots, y_n, \hbar]$, defined on generators by $w(y_i) := y_{w(i)}$ and $w(\hbar) = \hbar$ for all $w \in W$. This W-action extends to the ring $\bigoplus_{\lambda \in W^\Theta} H^{\mathrm{loc}}_{\widehat{T}}(\mathrm{pt})$ by $w(f_\lambda)_\lambda := (w(f_\lambda))_{w(\lambda)}$ for all $w \in W$. Consider the $\mathbb{Z}[\hbar]$-linear operators on $H^{\mathrm{loc}}_{\widehat{T}}(\mathrm{pt})$:

$$\partial_i(f) = \frac{\hbar}{y_{\alpha_i}} f + \frac{y_{\alpha_i} - \hbar}{y_{\alpha_i}} r_i(f), \quad f \in H^{\mathrm{loc}}_{\widehat{T}}(\mathrm{pt}),\ \alpha_i \in \Delta.$$

The operators ∂_i naturally define $\mathbb{Z}[\hbar]$-linear operators on $\bigoplus_{\lambda \in W^\Theta} H^{\mathrm{loc}}_{\widehat{T}}(\mathrm{pt})$.

3. $K_T(G/P_\Theta)$. Below are the formulas for S_{w_0} restricted to fixed points $\lambda \in W^\Theta$:

$$S_{w_0}|_\lambda = \begin{cases} \prod_{\alpha \in \Sigma^+_\Theta} (1 - e^{y_{-\alpha}}), & \lambda = w_0; \\ 0, & \lambda \neq w_0. \end{cases}$$

There is a natural action of W on the ring $K_T(\mathrm{pt}) \simeq \mathbb{Z}[e^{\pm y_1}, \ldots, e^{\pm y_n}]$, defined on generators by $w(e^{\pm y_i}) := e^{\pm y_{w(i)}}$ for all $w \in W$. This W-action extends to the ring $\bigoplus_{\lambda \in W^\Theta} K_T(\mathrm{pt})$ by $w(f_\lambda)_\lambda := (w(f_\lambda))_{w(\lambda)}$ for all $w \in W$. Consider the $\mathbb{Z}$-linear operators on $K_T(\mathrm{pt})$:

$$\partial_i(f) = \frac{f}{1 - e^{-y_{\alpha_i}}} + \frac{r_i(f)}{1 - e^{y_{\alpha_i}}}, \quad f \in K_T(\mathrm{pt}),\ \alpha_i \in \Delta.$$

The operators ∂_i naturally define $\mathbb{Z}$-linear operators on $\bigoplus_{\lambda \in W^\Theta} K_T(\mathrm{pt})$.

4. $K^{\mathrm{loc}}_{\widehat{T}}(T^*(G/P_\Theta))$. Below are the formulas for S_{w_0} restricted to fixed points $\lambda \in W^\Theta$:

$$S_{w_0}|_\lambda = \begin{cases} \prod_{\alpha \in \Sigma^+_\Theta} \frac{1 - e^{y_{-\alpha}}}{1 - q^2 e^{y_{-\alpha}}}, & \lambda = w_0; \\ 0, & \lambda \neq w_0. \end{cases}$$

There is a natural action of W on the ring $K_{\widehat{T}}(\mathrm{pt}) \simeq \mathbb{Z}[e^{\pm y_1}, \ldots, e^{\pm y_n}, q^{\pm 2}]$, defined on generators by $w(e^{\pm y_i}) := e^{\pm y_{w(i)}}$ and $w(q^{\pm 2}) = q^{\pm 2}$ for all $w \in W$. This W-action extends to the ring $\bigoplus_{\lambda \in W^\Theta} K^{\mathrm{loc}}_{\widehat{T}}(\mathrm{pt})$ by $w(f_\lambda)_\lambda := (w(f_\lambda))_{w(\lambda)}$ for all $w \in W$. Consider the $\mathbb{Z}[q^{\pm 2}]$-linear operators on $K^{\mathrm{loc}}_{\widehat{T}}(\mathrm{pt})$:

$$\partial_i(f) = \frac{1 - q^2}{1 - e^{-y_{\alpha_i}}} f + \frac{1 - q^2 e^{y_{\alpha_i}}}{1 - e^{y_{\alpha_i}}} r_i(f), \quad f \in K^{\mathrm{loc}}_{\widehat{T}}(\mathrm{pt}),\ \alpha_i \in \Delta.$$

The operators ∂_i naturally define $\mathbb{Z}[q^{\pm 2}]$-linear operators on $\bigoplus_{\lambda \in W^\Theta} K^{\mathrm{loc}}_{\widehat{T}}(\mathrm{pt})$.

3 Deformation of the motivic Segre classes

3.1 The Rees ring

Consider the polynomial ring $(\mathbb{Z}[\beta])[x_\lambda]_{\lambda\in\Lambda}$ over $\mathbb{Z}[\beta]$ in variables x_λ with $\lambda \in \Lambda$. Let J_Λ be the ideal $(x_0,\ x_{\lambda+\mu} - x_\lambda - x_\mu + \beta x_\lambda x_\mu,\ \lambda, \mu \in \Lambda)$ in $(\mathbb{Z}[\beta])[x_\lambda]_{\lambda\in\Lambda}$. The definition of *Rees ring* below is analogous to the definition in [10, Example 2.1].

Definition 3.1. *Set $S_\Lambda := ((\mathbb{Z}[\beta])[x_\lambda]_{\lambda\in\Lambda})/J_\Lambda$. The* ***Rees ring*** *$R_\Lambda$ is the ring*

$$R_\Lambda := \mathbb{Z}[q^{\pm 2}] \otimes_{\mathbb{Z}} S_\Lambda.$$

Lemma 3.2. *The ring R_Λ is an integral domain.*

Proof. As $R_\Lambda = \mathbb{Z}[q^{\pm 2}] \otimes_{\mathbb{Z}} S_\Lambda$, it is enough to show that S_Λ is an integral domain. Let $p\colon (\mathbb{Z}[\beta])[x_\lambda]_{\lambda\in\Lambda} \to S_\Lambda$ be the projection onto the quotient. Consider the ideal I_Λ of $(\mathbb{Z}[\beta])[x_\lambda]_{\lambda\in\Lambda}$ generated by the elements x_λ with $\lambda \in \Lambda$. Set $I_\Lambda^0 := S_\Lambda$. Then we have $\cap_{i\geq 0} I_\Lambda^i = (0)$. The ideal J_Λ is contained in I_Λ. Therefore, $\cap_{i\geq 0} p(I_\Lambda)^i = (0)$ as well. Form the associated graded ring $G := \oplus_{i\geq 0} p(I_\Lambda)^i / p(I_\Lambda)^{i+1}$. Let H be $H_T(\mathrm{pt})[\beta]$, the T-equivariant cohomology ring of a point extended by β. The argument in [1, Lemma 4.2] implies there is a ring isomorphism $H \to G$. As G is isomorphic to the integral domain H and $\cap_{i\geq 0} p(I_\Lambda)^i = (0)$, it follows that S_Λ is an integral domain. $\square$

It follows from Lemma 3.2 that the localization $(R_\Lambda)_\beta$ of R_Λ at β is an integral domain. There is an isomorphism $(R_\Lambda)_\beta \to K_{\widehat{T}}(\mathrm{pt})[\beta, \beta^{-1}]$ given by the following map:

$$(R_\Lambda)_\beta \to K_{\widehat{T}}(\mathrm{pt})[\beta, \beta^{-1}], \quad x_\lambda \mapsto \beta^{-1}(1 - e^\lambda), \quad q^2 \mapsto q^2, \quad \beta \mapsto \beta, \quad \lambda \in \Lambda. \tag{3.1}$$

The inverse sends $e^\lambda \mapsto 1 - \beta x_\lambda$ for all $\lambda \in \Lambda$. Interestingly, there is an isomorphism of $R_\Lambda/(\beta R_\Lambda)$ with $H_{\widehat{T}}(\mathrm{pt})[(1+\hbar)^{-1}]$ induced by the map

$$R_\Lambda/(\beta R_\Lambda) \to H^*_{\widehat{T}}(\mathrm{pt})[(1+\hbar)^{-1}], \quad x_\lambda \mapsto \lambda, \quad q^2 \mapsto 1 + \hbar, \quad \lambda \in \Lambda. \tag{3.2}$$

There is an action of W on R_Λ defined by generators by $w \cdot q^2 = q^2$, $w \cdot \beta = \beta$, and $w \cdot x_\lambda = x_{w(\lambda)}$ for $w \in W$ and $\lambda \in \Lambda$, which intertwines the isomorphisms (3.1) and (3.2).

3.2 Deformed divided difference operators

Definition 3.3. *For a simple root $\alpha \in \Delta$, we define the following $\mathbb{Z}[q^{\pm 2}][\beta]$-linear operator on R_Λ:*

$$\partial_\alpha = \frac{1 - q^2}{x_{-\alpha}} + \frac{1 - q^2(1 - x_\alpha)}{x_\alpha} r_\alpha.$$

Remark 3.4. *A similar "Demazure–Lusztig" operator is defined in [14, p. 61] (for more general oriented cohomology theories). The authors denote their operators by T^F_α.*

Write $w \in W$ as a reduced product of simple transpositions $w = r_{j_1} \cdots r_{j_k}$. We use the notation $\partial_w := \partial_{\alpha_{j_1}} \circ \cdots \circ \partial_{\alpha_{j_k}}$. The lemma below shows that ∂_w is independent of the choice of reduced decomposition for w.

Lemma 3.5. *The operators ∂_α satisfy the following relations:*

1. $\partial_{\alpha_i} \circ \partial_{\alpha_j} = \partial_{\alpha_j} \circ \partial_{\alpha_i}$ *whenever* $|i-j| > 1$.
2. $\partial_{\alpha_i} \circ \partial_{\alpha_{i+1}} \circ \partial_{\alpha_i} = \partial_{\alpha_{i+1}} \circ \partial_{\alpha_i} \circ \partial_{\alpha_{i+1}}$ *for all* $i = 1, \ldots, n-2$.
3. $(\partial_{\alpha_i} + q^2) \circ (\partial_{\alpha_i} + (q^2\beta - q^2 - \beta)) = 0$ *for all* $i = 1, \ldots, n-1$.

Remark 3.6. *The operator ∂_α has an "inverse"*

$$\widetilde{\partial}_\alpha = \frac{1}{q^2(\beta + q^2 - \beta q^2)} \left(\frac{q^2 - 1}{x_\alpha} + \frac{1 - q^2(1 - x_\alpha)}{x_\alpha} r_\alpha \right)$$

in the sense that $\partial_\alpha \circ \widetilde{\partial}_\alpha = \widetilde{\partial}_\alpha \circ \partial_\alpha = 1$. The operators $\widetilde{\partial}_\alpha$ satisfy the following relations:

1. $\widetilde{\partial}_{\alpha_i} \circ \widetilde{\partial}_{\alpha_j} = \widetilde{\partial}_{\alpha_j} \circ \widetilde{\partial}_{\alpha_i}$ *whenever* $|i-j| > 1$.
2. $\widetilde{\partial}_{\alpha_i} \circ \widetilde{\partial}_{\alpha_{i+1}} \circ \widetilde{\partial}_{\alpha_i} = \widetilde{\partial}_{\alpha_{i+1}} \circ \widetilde{\partial}_{\alpha_i} \circ \widetilde{\partial}_{\alpha_{i+1}}$ *for all* $i = 1, \ldots, n-2$.
3. $\left(\widetilde{\partial}_{\alpha_i} + \frac{1}{q^2}\right) \circ \left(\widetilde{\partial}_{\alpha_i} + \frac{1}{q^2\beta - q^2 - \beta}\right) = 0$ *for all* $i = 1, \ldots, n-1$.

3.3 Deformed motivic Segre classes

Let R^{loc}_Λ be the localization of R_Λ at the elements $\{1 - q^2(1 - x_\lambda)\}_{\lambda \in \Sigma}$. Consider the rings $\widetilde{R}_\Lambda := \bigoplus_{\lambda \in W^\Theta} R_\Lambda$ and $\widetilde{R}^{\mathrm{loc}}_\Lambda := \bigoplus_{\lambda \in W^\Theta} R^{\mathrm{loc}}_\Lambda$, both with pointwise addition and multiplication. There is a natural action of W on $\widetilde{R}_\Lambda$ given by $w \cdot (f_\lambda)_\lambda = (w(f_\lambda))_{w(\lambda)}$ for all $w \in W$. Moreover, the action of W on R_Λ preserves the set $\{1 - q^2(1 - x_\lambda)\}_{\lambda \in \Sigma}$. Therefore, the action of W on $\widetilde{R}_\Lambda$ induces an action of W on $\widetilde{R}^{\mathrm{loc}}_\Lambda$. The action of W on $\widetilde{R}^{\mathrm{loc}}_\Lambda$ induces an action of ∂_w on $\widetilde{R}^{\mathrm{loc}}_\Lambda$ for all $w \in W$. Consider the element $S_{w_0} \in \widetilde{R}^{\mathrm{loc}}_\Lambda$:

$$S_{w_0}|_\lambda = \begin{cases} \prod_{\alpha \in \Sigma^+_\Theta} \frac{x_{-\alpha}}{1 - q^2(1 - x_{-\alpha})}, & \text{if } \lambda = w_0; \\ 0, & \text{otherwise.} \end{cases}$$

Define $S_{w \cdot w_0} := \partial_w(S_{w_0})$. The set $\{S_\lambda\}_{\lambda \in W^\Theta}$ forms a basis for $\widetilde{R}^{\mathrm{loc}}_\Lambda$ over the fraction field $\mathrm{Frac}(R_\Lambda)$. The image of S_{w_0} in $\oplus_{\lambda \in W^\Theta} \mathrm{Frac}(K_{\widehat{T}}(\mathrm{pt})[\beta, \beta^{-1}])$ induced by (3.1) is

$$S_{w_0}|_\lambda = \begin{cases} \prod_{\alpha \in \Sigma^+_\Theta} \frac{1 - e^{-\alpha}}{\beta(1 - q^2) + q^2(1 - e^{-\alpha})}, & \text{if } \lambda = w_0; \\ 0, & \text{otherwise.} \end{cases}$$

Example 3.7. *After applying the isomorphism* (3.1) *and evaluating* $\beta = 1$, *the operator* ∂_α *and class* S_{w_0} *are the following:*

$$\partial_\alpha = \frac{1-q^2}{1-e^{-\alpha}} + \frac{1-q^2e^{\alpha}}{1-e^{\alpha}} r_\alpha \quad \textit{and} \quad S_{w_0}|_\lambda = \begin{cases} \prod_{\alpha \in \Sigma_\Theta^+} \frac{1-e^{-\alpha}}{1-q^2e^{-\alpha}}, & \textit{if } \lambda = w_0; \\ 0, & \textit{otherwise.} \end{cases}$$

Therefore, $\{S_\lambda\}_{\lambda \in W^\Theta}$ *is the set of images of motivic Segre classes of Schubert cells in the ring* $K_{\widehat{T}}^{loc}(T^*(G/P_\Theta))$ *under the localization map* ι.

Example 3.8. *After applying the isomorphism* (3.2), *the operator* ∂_α *and class* S_{w_0} *are the following:*

$$\partial_\alpha = \frac{\hbar}{\alpha} + \frac{\hbar\alpha + \alpha - \hbar}{\alpha} r_\alpha \quad \textit{and} \quad S_{w_0}|_\lambda = \begin{cases} \prod_{\alpha \in \Sigma_\Theta^+} \frac{\alpha}{\hbar\alpha + \hbar + \alpha}, & \textit{if } \lambda = w_0; \\ 0, & \textit{otherwise.} \end{cases}$$

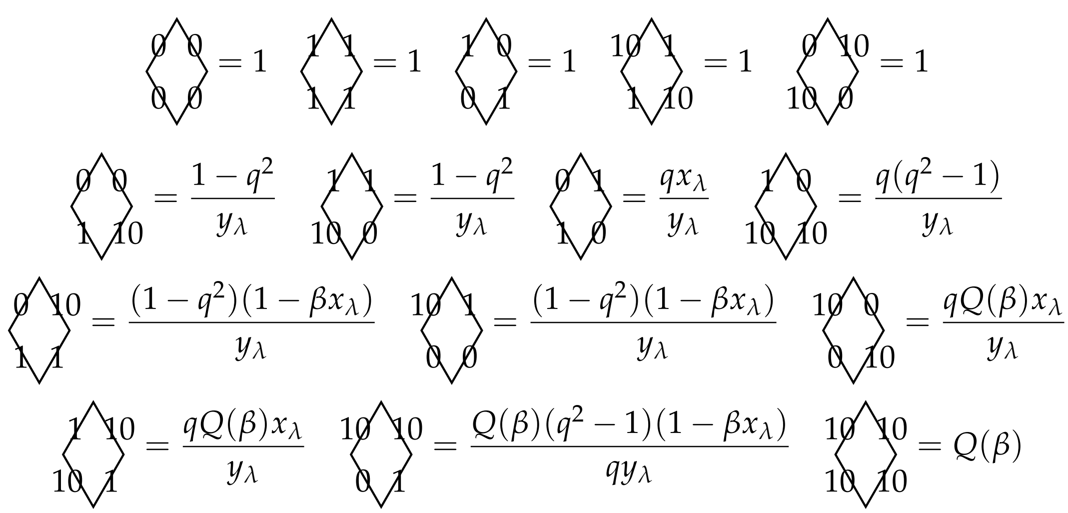
Replace ∂_α *with* $\partial_\alpha^H := \frac{\partial_\alpha}{\hbar+1}$, *replace* S_{w_0} *with* $S_{w_0}^H := (\hbar+1)^{l(w_0)} S_{w_0}$, *and set* $\overline{\hbar} := \frac{\hbar}{\hbar+1}$. *Then*

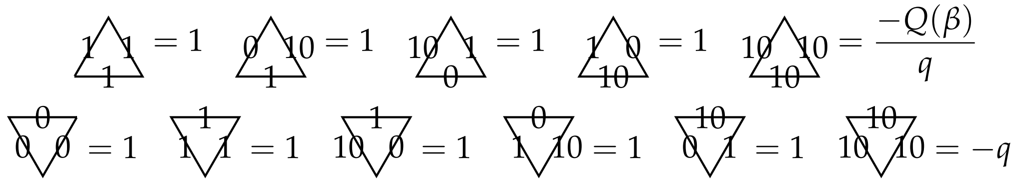
$$\partial_\alpha^H = \frac{\overline{\hbar}}{\alpha} + \frac{\alpha - \overline{\hbar}}{\alpha} r_\alpha \quad \textit{and} \quad S_{w_0}^H|_\lambda = \begin{cases} \prod_{\alpha \in \Sigma_\Theta^+} \frac{\alpha}{\alpha + \overline{\hbar}}, & \textit{if } \lambda = w_0; \\ 0, & \textit{otherwise.} \end{cases}$$

Thus, we can view the set of homogenizations $\{(\hbar+1)^{l(\lambda)} S_\lambda\}_{\lambda \in W^\Theta}$ *as the set of images of the SSM classes of Schubert cells in* $H_{\widehat{T}}^{loc}(T^*(G/P_\Theta))$ *under the localization map* ι.

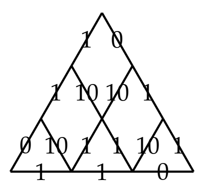
4 Structure constants in the $d = 1$ case

Assume $d = 1$. Define $Q(\beta) := q^2 + \beta - q^2\beta$. For $\lambda \in \Lambda$, we will set $y_\lambda := 1 - q^2(1 - x_\lambda)$. A **puzzle** with side labels λ, μ, ν in W^Θ is a triangle with side labels λ, μ, ν that is tiled by the following **puzzle pieces** with edge labels 0, 1, and 10. Each puzzle piece is equipped with a function from Λ to the fraction field $\mathrm{Frac}(\mathbb{Z}[q^{\pm 1}] \otimes_{\mathbb{Z}} S_\Lambda)$ known as its **fugacity**.



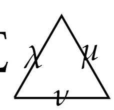


The bottom row of a puzzle is always tiled by the triangle puzzle pieces, and the rest of the puzzle is tiled by (not rotated) rhombus puzzle pieces. The **fugacity of a puzzle** is the product of the fugacities of the rhombi and triangles that tile it. The fugacity of a triangle tile is constant, independent of its position in the puzzle, whereas the fugacity of a rhombus tile depends on its position in the puzzle. A rhombus tile that lies in the i-th southwest-to-northeast diagonal and $(j-1)$-th northwest-to-southeast diagonal depends on $\lambda = -\sum_{t=i}^{j} \alpha_t$. For example, the fugacity of the puzzle below is $\frac{qQ(\beta)x_{-\alpha_1}}{y_{-\alpha_1}} \cdot \frac{q(q^2-1)}{y_{-\alpha_1-\alpha_2}} \cdot 1$.



We will use the notation  to mean the sum over the fugacities of all possible puzzles with the prescribed boundary labels. The **homogenization** of S_λ by a factor q is $q^{l(\lambda)}S_\lambda$, where $l(\lambda)$ is the length of $\lambda \in W^\Theta$. We can now state our main theorem:

Theorem 4.1. *The product of two classes $q^{l(\lambda)}S_\lambda$ and $q^{l(\mu)}S_\mu$ is given by the "puzzle" formula*

$$(q^{l(\lambda)}S_\lambda)(q^{l(\mu)}S_\mu) = \sum_\nu$$  $$(q^{l(\nu)}S_\nu). \tag{4.1}$$

Proof. The divided difference operators ∂_α and $\widetilde{\partial}_\alpha$ produce an R-matrix recursion that defines the classes $q^{l(\lambda)}S_\lambda$. The fugacities of the rhombus and triangle tiles can be used to define R, U, and D matrices that satisfy various Yang–Baxter type equations [8, Proposition 3.4]. The rest of the proof is essentially the same as that of [8, Theorem 3.8]. □

Remark 4.2. *Following [8, Section 6], we define a **positivity monoid** M to be a submonoid of R^{loc}_Λ under addition such that $M \cap (-M) = 0$. Consider the submonoid M of R^{loc}_Λ, defined as the set of sums of products of the factors over all $\alpha \in \Sigma^+$:*

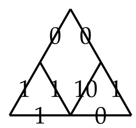
$$-q^{\pm} \qquad Q(\beta) \qquad 1-\beta x_{-\alpha} \qquad \frac{1-q^2}{y_{-\alpha}} \qquad -\frac{x_{-\alpha}}{y_{-\alpha}}$$

Then M is a positivity monoid of R^{loc}_Λ. To see it, view M as a submonoid of $\mathrm{Frac}(K_{\widehat{T}}(\mathrm{pt})[\beta,\beta^{-1}])$ *via the isomorphism* (3.1)*, and then evaluate $\beta = 1$, $e^{-\alpha_i} = 2^i$, and $q = -2^{-n/2}$ to see that every factor is positive. As the structure constants for $\{q^{l(\lambda)}S_\lambda\}_{\lambda\in W^\Theta}$ lie in the positivity monoid M, it is in this sense that we consider the structure constants and puzzle formula for them **positive**.*

Example 4.3. *We will compute the product $(q \cdot S_{10}) \cdot S_{01}$ using puzzles. First, we compute*

$$q \cdot S_{10} = q \cdot \left[0, \tfrac{x_{-\alpha_1}}{y_{-\alpha_1}}\right]; \qquad S_{01} = (\tfrac{1}{q}\partial_{\alpha_1})(q \cdot S_{10}) = \left[1, \tfrac{1-q^2}{y_{-\alpha_1}}\right].$$

Here, α_1 is the positive root of $\mathrm{Gr}(1,2)$. Pointwise multiplication of these classes yields the equation $(q \cdot S_{10}) \cdot S_{01} = \frac{1-q^2}{y_{-\alpha_1}} \cdot (q \cdot S_{10})$. The puzzle rule (4.1) gives the same result:



$$= \frac{1-q^2}{y_{-\alpha_1}}.$$

5 Towards a geometric interpretation of the classes

Let E be any linear algebraic group over $\mathbb{C}$. Let Sm_E be the category of smooth algebraic varieties over $\mathbb{C}$ equipped with an algebraic action of E, with morphisms being E-equivariant morphisms of varieties. An E-equivariant oriented cohomology theory h_E^* is a functor from Sm_E to the category of commutative unital rings satifying several "cohomological-type" axioms. See [14] or [2] for detailed exposition. Let Ω_E be the E-equivariant algebraic cobordism of Krishna [9] and Malagón-López and Heller [4]. The functor Ω_E is an example of an E-equivariant oriented cohomology theory.

Example 5.1. *The E-equivariant Chow theory CH_E of Edidin–Graham [3] is an E-equivariant oriented cohomology theory. Let $\widehat{\mathrm{CH}}_E(\mathrm{pt})$ be the completion of $\mathrm{CH}_E(\mathrm{pt})$ at the augmentation ideal (i.e., the ideal generated by algebraic cycles of positive codimensions). Let $X \in \mathrm{Sm}_E$. In [9, Section 7.1] it is shown there is a canonical map $\Omega_E(\mathrm{pt}) \to \widehat{\mathrm{CH}}_E(\mathrm{pt})$ and a ring isomorphism*

$$\Omega_E(X) \otimes_{\Omega_E(\mathrm{pt})} \widehat{\mathrm{CH}}_E(\mathrm{pt}) \simeq \mathrm{CH}_E(X) \otimes_{\mathrm{CH}_E(\mathrm{pt})} \widehat{\mathrm{CH}}_E(\mathrm{pt}).$$

Example 5.2. *The E-equivariant algebraic K-theory $K_E := K_E^0$ of Thomason [13] is an E-equivariant oriented cohomology theory. Let $\widehat{K}_E(\mathrm{pt})$ be the completion of $K_E(\mathrm{pt})$ at the augmentation ideal (i.e., the ideal of virtual representations of rank 0). Let $X \in \mathrm{Sm}_E$. In [9, Section 7.2] it is shown that there is a canonical map $\Omega_E(\mathrm{pt}) \to \widehat{K}_E(\mathrm{pt})$ and a ring isomorphism*

$$\Omega_E(X) \otimes_{\Omega_E(\mathrm{pt})} \widehat{\mathrm{K}}_E(\mathrm{pt}) \simeq K_E(X) \otimes_{K_E(\mathrm{pt})} \widehat{K}_E(\mathrm{pt}).$$

For any $X \in \mathrm{Sm}_E$, we define

$$\widehat{CH}_E(X) := \Omega_E(X) \otimes_{\Omega_E(\mathrm{pt})} \widehat{\mathrm{CH}}_E(\mathrm{pt}) \quad \text{and} \quad \widehat{K}_E(X) := \Omega_E(X) \otimes_{\Omega_E(\mathrm{pt})} \widehat{K}_E(\mathrm{pt}).$$

Let I_Λ be the ideal in $(\mathbb{Z}[\beta])[x_\lambda]_{\lambda \in \Lambda}$ generated by the elements x_λ, with $\lambda \in \Lambda$. Consider the I_Λ-adic completion $(\mathbb{Z}[\beta])[\![x_\lambda]\!]_{\lambda \in \Lambda}$ of $(\mathbb{Z}[\beta])[x_\lambda]_{\lambda \in \Lambda}$. Let $\mathcal{J}_\Lambda$ be the closure of the ideal in $(\mathbb{Z}[\beta])[\![x_\lambda]\!]_{\lambda \in \Lambda}$ generated by x_0 and $x_{\lambda+\mu} - x_\lambda - x_\mu + \beta x_\lambda x_\mu$ over all $\lambda, \mu \in$

Λ. Set $\mathcal{S}_\Lambda := (\mathbb{Z}[\beta])[\![x_\lambda]\!]_{\lambda\in\Lambda}/\mathcal{J}_\Lambda$. It follows from [1, Corollary 2.13] that $\mathcal{S}_\Lambda$ is an integral domain. It follows from [1, Section 2.5] and [2, Theorem 3.3] that there are maps from $\mathcal{S}_\Lambda$ to both $\widehat{CH}_T(\mathrm{pt})$ and $\widehat{K}_T(\mathrm{pt})$, which are induced by maps of formal group laws, and that there is a map $\Omega_T(\mathrm{pt}) \to \mathcal{S}_\Lambda$. The kernel of the composition $(\mathbb{Z}[\beta])[x_\lambda]_{\lambda\in\Lambda} \hookrightarrow (\mathbb{Z}[\beta])[\![x_\lambda]\!]_{\lambda\in\Lambda} \to \mathcal{S}_\Lambda$ is precisely the ideal J_Λ defined in Section 3.1. Thus, we can view S_Λ as a subring of $\mathcal{S}_\Lambda$. Define $\widehat{R}_\Lambda := \mathbb{Z}[\beta][\![x_q]\!] \otimes_{\mathbb{Z}[\beta]} \mathcal{S}_\Lambda$. We can view $\mathbb{Z}[q^{\pm 2}] \otimes_{\mathbb{Z}} S_\Lambda$ as a subring of $\widehat{R}_\Lambda$ via the map $q^2 \mapsto 1 - x_q$ (observe that $q^{-2} \mapsto 1 + x_q + x_q^2 + \cdots$.) As $\Omega_{\widehat{T}}(\mathrm{pt}) \simeq \Omega_T(\mathrm{pt}) \otimes_{\mathbb{L}} \mathbb{L}[\![x]\!]$, the map $\Omega_T(\mathrm{pt}) \to \mathcal{S}_\Lambda$ induces a map $\Omega_{\widehat{T}}(\mathrm{pt}) \to \widehat{R}_\Lambda$, $x \mapsto x_q$.

We expect the definition below to agree with that of [5]. We plan to explore this.

Definition 5.3. *The (completed) $\widehat{T}$-**equivariant connective** K**-ring** of $T^*(G/P_\Theta)$ is*

$$CK_{\widehat{T}}(T^*(G/P_\Theta)) := \Omega_{\widehat{T}}(T^*(G/P_\Theta)) \otimes_{\Omega_{\widehat{T}}(\mathrm{pt})} \widehat{R}_\Lambda.$$

There are ring homomorphisms,

$$CK_{\widehat{T}}(T^*(G/P_\Theta)) \to \widehat{\mathrm{CH}}_{\widehat{T}}(T^*(G/P_\Theta)) \quad \text{and} \quad CK_{\widehat{T}}(T^*(G/P_\Theta)) \to \widehat{K}_{\widehat{T}}(T^*(G/P_\Theta)),$$

where $x_q \mapsto -x_\hbar$ on the left and $x_q \mapsto x_q$ on the right. Consider the localization $\widehat{R}_\Lambda^{\mathrm{loc}}$ of $\widehat{R}_\Lambda$ at the set $\{x_q + x_\alpha - x_q x_\alpha\}_{\alpha\in\Sigma}$. Let $\widehat{\mathrm{CH}}^{\mathrm{loc}}_{\widehat{T}}(\mathrm{pt})$ be the localization of $\widehat{\mathrm{CH}}_{\widehat{T}}(\mathrm{pt})$ at the set $\{x_\hbar + x_\alpha + x_\hbar x_\alpha\}_{\alpha\in\Sigma}$, and let $\widehat{K}^{\mathrm{loc}}_{\widehat{T}}(\mathrm{pt})$ be the localization of $\widehat{K}_{\widehat{T}}(\mathrm{pt})$ at the set $\{x_q + x_\alpha - x_q x_\alpha\}_{\alpha\in\Sigma}$. Define $\widehat{\mathrm{CH}}^{\mathrm{loc}}_{\widehat{T}}(T^*(G/P_\Theta)) := \widehat{\mathrm{CH}}_{\widehat{T}}(T^*(G/P_\Theta)) \otimes_{\widehat{\mathrm{CH}}_{\widehat{T}}(\mathrm{pt})} \widehat{\mathrm{CH}}^{\mathrm{loc}}_{\widehat{T}}(\mathrm{pt})$ and $\widehat{K}^{\mathrm{loc}}_{\widehat{T}}(T^*(G/P_\Theta)) := \widehat{K}_{\widehat{T}}(T^*(G/P_\Theta)) \otimes_{\widehat{K}_{\widehat{T}}(\mathrm{pt})} \widehat{K}^{\mathrm{loc}}_{\widehat{T}}(\mathrm{pt})$.

Conjecture 5.4. *We conjecture:*

1. *There are classes $\overline{S}_\lambda \in \mathrm{CK}_{\widehat{T}}(T^*(G/P_\Theta)) \otimes_{\widehat{R}_\Lambda} \widehat{R}^{loc}_\Lambda$ indexed by $\lambda \in W^\Theta$ whose images in $\widehat{\mathrm{CH}}^{loc}_{\widehat{T}}(T^*(G/P_\Theta))$ and $\widehat{\mathrm{K}}^{loc}_{\widehat{T}}(T^*(G/P_\Theta))$ can be identified with SSM and motivic Segre classes, respectively. The set $\{\overline{S}_\lambda\}_{\lambda\in W^\Theta}$ is an $\widehat{R}^{loc}_\Lambda$-basis for $\mathrm{CK}_{\widehat{T}}(T^*(G/P_\Theta)) \otimes_{\widehat{R}_\Lambda} \widehat{R}^{loc}_\Lambda$.*

2. *Under the "localization map" $\iota\colon \mathrm{CK}_{\widehat{T}}(T^*(G/P_\Theta)) \otimes_{\widehat{R}_\Lambda} \widehat{R}^{loc}_\Lambda \to \oplus_{\lambda\in W^\Theta} \widehat{R}^{loc}_\Lambda$, the image $\iota(\overline{S}_\lambda)$ is the class S_λ defined in Section 3.3.*

Remark 5.5. *The classes S_λ satisfy a "GKM-type" property. We believe this property, together with a connective K-theory analogue of a theorem of Chang and Skjelbred, imply that there exist classes $\overline{S}_\lambda \in \mathrm{CK}_{\widehat{T}}(T^*(G/P_\Theta)) \otimes_{\widehat{R}_\Lambda} \widehat{R}^{loc}_\Lambda$ satisfying Conjecture 5.4. This is work in progress.*

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References

[1] B. Calmès, V. Petrov, and K. Zainoulline. "Invariants, torsion indices and oriented cohomology of complete flags". *Ann. Sci. Éc. Norm. Supér. (4)* **46**.3 (2013), 405–448 (2013). DOI.

[2] B. Calmès, K. Zainoulline, and C. Zhong. "Equivariant oriented cohomology of flag varieties". *Doc. Math.* Extra vol.: Alexander S. Merkurjev's sixtieth birthday (2015), pp. 113–144.

[3] D. Edidin and W. Graham. "Equivariant intersection theory". *Invent. Math.* **131**.3 (1998), pp. 595–634. DOI.

[4] J. Heller and J. Malagón-López. "Equivariant algebraic cobordism". *J. Reine Angew. Math.* **684** (2013), pp. 87–112. DOI.

[5] N. A. Karpenko and A. S. Merkurjev. "Equivariant connective K-theory". *J. Algebraic Geom.* **31**.1 (2022), pp. 181–204. DOI.

[6] A. Knutson and T. Tao. "Puzzles and (equivariant) cohomology of Grassmannians". *Duke Math. J.* **119**.2 (2003), pp. 221–260. DOI.

[7] A. Knutson and P. Zinn-Justin. "Schubert puzzles and integrability I: Invariant trilinear forms". 2017. arXiv:1706.10019.

[8] A. Knutson and P. Zinn-Justin. "Schubert puzzles and integrability II: Multiplying motivic Segre classes". 2021. arXiv:2102.00563.

[9] A. Krishna. "Equivariant cobordism of schemes". *Doc. Math.* **17** (2012), pp. 95–134.

[10] C. Lenart and K. Zainoulline. "Towards generalized cohmology Schubert calculus via formal root polynomials". *Math. Res. Lett.* **24**.3 (2017), pp. 839–877. DOI.

[11] D. Maulik and A. Okounkov. "Quantum groups and quantum cohomology". *Astérisque* 408 (2019), pp. ix+209. DOI.

[12] A. Okounkov. "Lectures on K-theoretic computations in enumerative geometry". *Geometry of moduli spaces and representation theory*. Vol. 24. IAS/Park City Math. Ser. Amer. Math. Soc., Providence, RI, 2017, pp. 251–380. DOI.

[13] R. W. Thomason. "Algebraic K-theory of group scheme actions". *Algebraic topology and algebraic K-theory (Princeton, N.J., 1983)*. Vol. 113. Ann. of Math. Stud. Princeton Univ. Press, Princeton, NJ, 1987, pp. 539–563.

[14] G. Zhao and C. Zhong. "Geometric representations of the formal affine Hecke algebra". *Adv. Math.* **317** (2017), pp. 50–90. DOI.