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Failure of the Lefschetz Property for the Graphic Matroid

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Abstract. We consider the strong Lefschetz property for standard graded Artinian Gorenstein algebras. Such an algebra has a presentation of the quotient algebra of the ring of the differential polynomials modulo the annihilator of some homogeneous polynomial. There is a characterization of the strong Lefschetz property for such an algebra by the non-degeneracy of the higher Hessian matrix of the homogeneous polynomial. Maeno and Numata conjectured that if such an algebra is defined by the basis generating polynomial of any matroid, then it has the strong Lefschetz property. For this conjecture, we give counterexamples that are associated with graphic matroids. We prove the degeneracy of the higher Hessian matrix by constructing a non-zero element in the kernel of that matrix.

Keywords: strong Lefschetz property, Artinian Gorenstein algebra, higher Hessian matrix, graphic matroid, basis generating polynomial

1 Introduction

The Lefschetz property for Artinian Gorenstein algebras is inspired by the Hard Lefschetz Theorem on the cohomology of smooth complex projective varieties.

Let n be a positive integer. The polynomial algebra $\mathbb{R}[x_1, \ldots, x_n]$ is regarded as a module over the algebra $Q := \mathbb{R}[\partial_1, \ldots, \partial_n]$ where $\partial_i := \frac{\partial}{\partial x_i}$ for $i = 1, \ldots, n$.

For a homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree d, let

$$\operatorname{Ann}_Q(f) := \{\alpha \in Q \mid \alpha f = 0\},$$
$$A := Q/\operatorname{Ann}_Q(f).$$

Since f is homogeneous, $\operatorname{Ann}_Q(f)$ is a homogeneous ideal of Q. Thus the algebra A can be decomposed into

$$A = \bigoplus_{i=0}^{d} A_i$$

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as a graded Artinian algebra. Note that for all $i > d$, the homogeneous part of degree i of A is equal to $\{0\}$.

Furthermore, the algebra A is a *Poincaré duality algebra*. In other words, A_d is congruent to the ground field $\mathbb{R}$ and the bilinear pairing

$$A_i \times A_{d-i} \to A_d$$

is non-degenerate for all $i = 0, \ldots, d$. It is known that a graded Artinian algebra is Gorenstein if and only if it is a Poincaré duality algebra (see [3, Theorem 2.79]). Hence, the algebra A is Gorenstein. The number d is called the *socle degree* of A.

Conversely, according to [3, Lemma 3.74], any standard graded Artinian Gorenstein algebra has a presentation $Q/\mathrm{Ann}_Q(f)$ with some homogeneous polynomial f. This correspondence comes from *Macaulay's inverse system* [3, Section 2.4.2] and f is called the *Macaulay dual generator*.

We say that the algebra A has the *strong Lefschetz property* if there exists an element $\ell \in A_1$ such that the multiplication map

$$\times \ell^k \colon A_i \to A_{i+k}$$

has full rank for all $i = 0, \ldots, d-1$ and $k = 1, \ldots, d-i$. A weakening of this definition to only $k = 1$ is called the *weak Lefschetz property*.

Conjecture. The Lefschetz property is defined for algebras, but it has strong connections to combinatorics and represents an important concept in the field. Nevertheless, many questions remain unsolved. For example, the conditions on the polynomial f under which the algebra A has the Lefschetz property are not well understood.

In the papers [4, 5], Maeno and Numata conjectured that if f is the basis generating polynomial of any matroid, then the algebra A has the strong Lefschetz property. They proved this conjecture for matroids whose lattice of flats is modular geometric, and then provided an algebraic proof that every modular geometric lattice has the Sperner property, as an application of the Lefschetz property in combinatorics.

Previous work. The strong Lefschetz property at $i = 1$ is studied in [7, 8, 14] as it is related to the Hessian matrix of the polynomial f. As the most general result, Murai, Nagaoka, and Yazawa [7, Theorem 3.8 and Remark 3.9] showed that under the conjecture's condition (i.e., f is a basis generating polynomial of any matroid), the multiplication map $\times \ell^k \colon A_1 \to A_{1+k}$ has full rank for all $k = 1, \ldots, d-1$, where $\ell = a_1\partial_1 + \cdots + a_n\partial_n \in A_1$ for any $(a_1, \ldots, a_n) \in \mathbb{R}^n_{>0}$.

Our contributions. We try to verify this conjecture by computation with the mathematical software system `SageMath` [12]. We concentrate on a class of matroids called *graphic* *matroids*. For details of graphs and matroids, see [9]. By employing an enumeration of all simple graphs, we construct graphic matroids of them and check whether the conjecture holds.

Although the conjecture is true for all graphs of seven or fewer vertices, we found counterexamples associated with graphs of eight vertices. One of them (see Figure 1 in Section 3), the algebra $A = Q/\mathrm{Ann}_Q(f)$, with the smallest *codimension* $\dim_{\mathbb{R}} A_1$ has the following characteristic values: the number of variables $n = 13$, the socle degree $d = 7$, the minimal number of generators of $\mathrm{Ann}_Q(f)$ is 69, and the *Hilbert function* $(\dim_{\mathbb{R}} A_i)_{i=0}^{d} = (1, 13, 70, 166, 166, 70, 13, 1)$. The algebra A does not have the strong Lefschetz property at $i = 3$, i.e., there is no element $\ell \in A_1$ such that the multiplication map

$$\times\ell\colon A_3 \to A_4$$

has full rank. It means that the algebra A does not even have the weak Lefschetz property.

Whether there exists a case that fails at $i = 2$ remains unknown.

Organization. The rest of this paper is organized as follows. Section 2 contains detailed settings of the strong Lefschetz property and the conjecture. Section 3 describes our computation and the failure of the strong Lefschetz property at $i = 3$.

Finally, we note the partial failure of the strong Lefschetz property at $i = 2$ in Section 4. In contrast to the previous work, the element $\ell = \partial_1 + \cdots + \partial_n \in A_1$ is not a universal solution, despite the fact that $(1, \ldots, 1) \in \mathbb{R}^n_{>0}$.

This paper is the extended abstract of the paper [11]: see that paper for the details of counterexamples.

2 Preliminaries

Section 2.1 contains the details of the strong Lefschetz property for graded Artinian Gorenstein algebras. The conjecture is in Section 2.2.

2.1 Strong Lefschetz Property

Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree d. The algebra

$$A = A(f) := Q/\mathrm{Ann}_Q(f)$$

is a standard graded Artinian Gorenstein algebra and can be decomposed into

$$A = \bigoplus_{i=0}^{d} A_i.$$

Hereafter, we represent elements of A in terms of elements of Q without causing ambiguity.

The Poincaré duality algebra A satisfies the following properties:

- The linear map
$$[\bullet]\colon A_d \to \mathbb{R}\,; \quad [\alpha] := \alpha f$$
is the isomorphism from A_d to $\mathbb{R}$.
- For each $i = 0, \ldots, d$, the bilinear form
$$A_i \times A_{d-i} \to \mathbb{R}\,; \quad (\xi, \eta) \mapsto [\xi\eta]$$
is non-degenerate.

Next, we define the strong Lefschetz property of the algebra A.

Definition 2.1 (strong Lefschetz property (in the narrow sense))**.** *Let $k \leq d/2$ be a non-negative integer. We say that A has the* strong Lefschetz property at degree k*, shortly* SLP_k*, if there exists an element $\ell \in A_1$ such that the multiplication map*
$$\times \ell^{d-2k}\colon A_k \to A_{d-k}$$
is an isomorphism. In addition, if A has the SLP_k *for all $k \leq d/2$ with a common element $\ell \in A_1$, we say that A has the* strong Lefschetz property *and ℓ is a* Lefschetz element.

Since $\dim_{\mathbb{R}} A_i = \dim_{\mathbb{R}} A_{d-i}$ for all $i = 0, \ldots, d$, this definition of the strong Lefschetz property is equivalent to the one given in Section 1.

To test the strong Lefschetz property, we employ the higher Hessian matrix.

Definition 2.2 (higher Hessian matrix)**.** *Let k be a non-negative integer and $B_k = \{\alpha_1, \ldots, \alpha_m\}$ be a set of homogeneous polynomials of degree k in Q. For polynomial $g \in \mathbb{R}[x_1, \ldots, x_n]$, we define an $m \times m$ polynomial matrix $\boldsymbol{H}_{B_k}(g)$ by*
$$\left(\boldsymbol{H}_{B_k}(g)\right)_{i,j} := (\alpha_i \alpha_j) g \quad (i, j = 1, \ldots, m).$$
This $\boldsymbol{H}_{B_k}(g)$ is called the k-th Hessian matrix *of g with respect to B_k.*

When $B_1 = \{\partial_1, \ldots, \partial_n\}$, the first Hessian matrix $\boldsymbol{H}_{B_1}(g)$ coincides with the usual Hessian matrix of g.

The strong Lefschetz property of the algebra A can be examined using this matrix.

Theorem 2.3 ([3, Theorem 3.76],[6, Theorem 3.1],[13, Theorem 4])**.** *Let $k \leq d/2$ be a non-negative integer and $B_k = \{\alpha_1, \ldots, \alpha_m\}$ be any $\mathbb{R}$-basis of A_k. For any $(a_1, \ldots, a_n) \in \mathbb{R}^n$, the algebra A has the* SLP_k *with an element $\ell = a_1\partial_1 + \cdots + a_n\partial_n \in A_1$ if and only if $\boldsymbol{H}_{B_k}(f)(a_1, \ldots, a_n)$ is non-degenerate where*
$$(\boldsymbol{H}_{B_k}(f)(a_1, \ldots, a_n))_{i,j} := ((\alpha_i \alpha_j) f)(a_1, \ldots, a_n) \quad (i, j = 1, \ldots, m).$$

Proof. Since the algebra A is a Poincaré duality algebra, A has the SLP_k with an element ℓ if and only if a bilinear form

$$A_k \times A_k \to \mathbb{R}\,; \quad (\xi, \eta) \mapsto [\ell^{d-2k}\xi\eta]$$

is non-degenerate. The representation matrix of this bilinear form with respect to the $\mathbb{R}$-basis B_k of A_k has the (i, j)-entry

$$[\ell^{d-2k}\alpha_i\alpha_j] = (\ell^{d-2k}\alpha_i\alpha_j)f = (d-2k)!\,((\alpha_i\alpha_j)f)(a_1, \dots, a_n) \quad (i, j = 1, \dots, m).$$

The last equality is due to Euler's homogeneous function theorem. Thus the representation matrix is $(d-2k)!\,\boldsymbol{H}_{B_k}(f)(a_1, \dots, a_n)$. □

Remark 2.4. *The algebra A has the* SLP_k *if and only if the polynomial* $\det \boldsymbol{H}_{B_k}(f)$ *is non-zero. Furthermore, if A has the* SLP_k *for all $k \leq d/2$, then there exists a common non-root $(a_1, \dots, a_n)$ of the polynomials* $\det \boldsymbol{H}_{B_k}(f)$*. In this case, the algebra A has the strong Lefschetz property with the Lefschetz element $\ell = a_1\partial_1 + \cdots + a_n\partial_n$.*

2.2 Basis Generating Polynomial

Let G be a connected graph with $d+1$ vertices and n edges. We number the edges one through n and identify the edges with the numbers.

A subgraph T of the graph G is called a *spanning tree* of G if T is connected graph on the same vertices as G without cycles.

Definition 2.5 (basis generating polynomial (for graphs))**.** *The* basis generating polynomial $f_G \in \mathbb{R}[x_1, \dots, x_n]$ *of the graph G is defined as the sum, over all spanning trees T of G, of the products of x_e for the edges e in T. More formally,*

$$f_G := \sum_T \prod_e x_e.$$

A spanning tree of G is a *basis* of the graphic matroid of G. Since the number of edges in every spanning tree of G is d, the polynomial f_G is a homogeneous polynomial of degree d.

From the above, the conjecture we mentioned in Section 1 is as follows.

Conjecture 2.6 (Maeno–Numata conjecture (for graphs) [4])**.** *The algebra $A(f_G)$ has the strong Lefschetz property for any connected graph G.*

Remark 2.7. *If edges i and j are multiple edges, then $\partial_i - \partial_j \in \mathrm{Ann}_Q(f_G)$. If an edge k is a self loop, then $\partial_k \in \mathrm{Ann}_Q(f_G)$. Thus for Conjecture 2.6 we can ignore self loops and multiple edges; and focus only on* simple *graphs, i.e., graphs without such edges.*

In the following, we say that the graph G has the strong Lefschetz property or the SLP_k if the algebra $A(f_G)$ has the strong Lefschetz property or the SLP_k, respectively.

3 Failure of the SLP_3

In this section, we provide a planar graph and a non-planar graph without the SLP_3 as the counterexamples to Conjecture 2.6. Our probabilistic and deterministic methods are described in Sections 3.1 and 3.2, respectively.

3.1 Screening

To prove that the algebra $A = A(f_G)$ does not have the SLP_3, it is necessary to verify that the third Hessian matrix $\boldsymbol{H}_{B_3}(f_G)$ is degenerate for an $\mathbb{R}$-basis B_3 of A_3. This problem is a variant of *Edmonds' problem* [2]. Unfortunately, computing the determinant of that large matrix of multivariate polynomials is difficult. For this reason, we first employ a randomized algorithm based on Lemma 3.1.

Lemma 3.1 (Schwartz–Zippel lemma [1, 10, 15]). *Let $g \in \mathbb{R}[x_1, \ldots, x_n]$ be a non-zero polynomial. Suppose that S is a finite subset of $\mathbb{R}$ and $r_1, \ldots, r_n$ are selected at random independently and uniformly from S. Then,*

$$\Pr[g(r_1, \ldots, r_n) = 0] \leq \frac{\deg g}{|S|}.$$

Since the SLP_3 is trivial or undefined for graphs of seven or fewer vertices, let G be a graph of eight vertices. Every entry of $\boldsymbol{H}_{B_3}(f_G)$ is the sixth-order partial derivative of the polynomial f_G of degree seven. Thus if the polynomial $g := \det \boldsymbol{H}_{B_3}(f_G)$ is non-zero, then $\deg g = \dim_{\mathbb{R}} A_3$.

We repeated the following check 100 times: select $r_1, \ldots, r_n$ at random independently and uniformly from the set $S = \{1, \ldots, 10^9\}$ and assure whether $g(r_1, \ldots, r_n) = 0$. If the polynomial g is non-zero, then the probability that g passes our check is less than $\left(\frac{\dim_{\mathbb{R}} A_3}{|S|}\right)^{100}$. Because our computation showed $\dim_{\mathbb{R}} A_3 \leq 500$, this probability is smaller than 10^{-630}.

We found 152 counterexample candidates out of $11,117$ simple connected graphs of eight vertices, up to isomorphism of graphs. Among them, for the one which has the smallest number of edges (Figure 1) and one of which is a planar graph (Figure 2), we verify that $\boldsymbol{H}_{B_3}(f_G)$ is degenerate for an $\mathbb{R}$-basis B_3 of A_3. The details of Figure 1 are in Section 3.2 and [11].

Remark 3.2. *It is sufficient to consider only* biconnected graphs*, that is, graphs whose connectivity is preserved when any one vertex is deleted. The reason is as follows. First, the polynomial f_G is a product of the basis generating polynomials of each* biconnected components *of G, that are maximal biconnected subgraphs of G. In consequence, according to [3, Theorem 3.34 and Proposition 3.77], the strong Lefschetz property of each biconnected component derives the strong Lefschetz property of the whole graph G.*

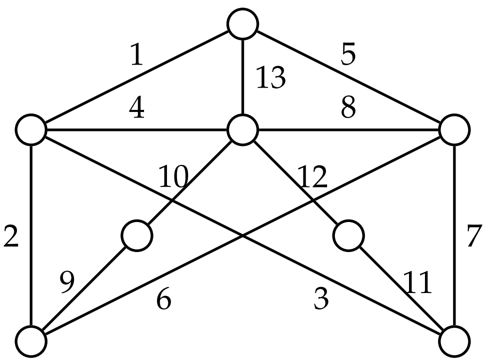

Figure 1: The graph with the smallest number of edges without SLP_3

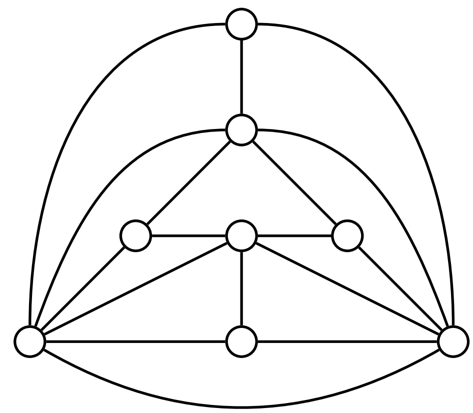

Figure 2: A planar graph without SLP_3

The number of biconnected graphs of eight vertices is $7,123$*, up to isomorphism.*

3.2 Verification

Let G be the graph shown in Figure 1. The numbering of each edge is also shown in Figure 1. The algebra $A = A(f_G)$ is mentioned in Section 1: the number of variables $n = 13$, the socle degree $d = 7$, the minimal number of generators of $\mathrm{Ann}_Q(f_G)$ is 69, and the Hilbert function $(\dim_{\mathbb{R}} A_i)_{i=0}^{d} = (1, 13, 70, 166, 166, 70, 13, 1)$. Let $m := \dim_{\mathbb{R}} A_3 = 166$.

We fix the $\mathbb{R}$-basis B_3 of A_3 as follows. Let $(i, j, k) := \partial_i \partial_j \partial_k$ for $1 \le i < j < k \le n$. The set of monomials $\{(i, j, k) \mid 1 \le i < j < k \le n\}$ is a generating set of A_3 because the polynomial f_G is a square-free polynomial. Enumerate this set in the lexicographic order, as $\beta_1 = (1, 2, 3), \ldots, \beta_{\binom{n}{3}} = (n-2, n-1, n)$. We use

$$B_3 := \left\{ \beta_i \;\middle|\; 1 \le i \le \binom{n}{3}, \langle \beta_1, \ldots, \beta_{i-1} \rangle \subsetneq \langle \beta_1, \ldots, \beta_i \rangle \right\}.$$

The third Hessian matrix $\boldsymbol{H}_{B_3}(f_G)$ contains $8,450$ non-zero entries. To verify the degeneracy of $\boldsymbol{H}_{B_3}(f_G)$, we construct a non-zero vector of polynomials $\boldsymbol{F} = (F_1, \ldots, F_m)^{\mathrm{T}} \in \mathbb{R}[x_1, \ldots, x_n]^m$ such that $\boldsymbol{H}_{B_3}(f_G)\boldsymbol{F} = \boldsymbol{0}$. Such an $\boldsymbol{F}$ satisfies the following conditions:

Theorem 3.3. *Let $k \le d/2$ be a non-negative integer, $B_k = \{\alpha_1, \ldots, \alpha_m\}$ be any $\mathbb{R}$-basis of A_k, and $\boldsymbol{F} = (F_1, \ldots, F_m)^{\mathrm{T}} \in \mathbb{R}[x_1, \ldots, x_n]^m$ with $\boldsymbol{H}_{B_k}(f)\boldsymbol{F} = \boldsymbol{0}$. The following hold.*

1. *For any $(a_1, \ldots, a_n) \in \mathbb{R}^n$, let $\ell := a_1\partial_1 + \cdots + a_n\partial_n$ and*

$$\xi := \sum_{i=1}^{m} F_i(a_1, \ldots, a_n)\alpha_i.$$

The element $\xi \in A_k$ is in the kernel of the multiplication map $\times \ell^{d-2k} \colon A_k \to A_{d-k}$.

2.
$$\sum_{i=1}^{m} F_i \cdot (\alpha_i f) = 0.$$

Algorithm for Constructing $\boldsymbol{F}$. The vector of polynomials $\boldsymbol{F}$ is constructed through the following steps using *polynomial interpolation*.

1. Determine the maximum degree
$$D_i := \max_{j=1,\dots,m} \deg F_j(1,\dots,x_i,\dots,1),$$
where $F_j(1,\dots,x_i,\dots,1)$ denotes the univariate polynomial in x_i obtained by setting all x_k with $k \neq i$ equal to one in F_j.

2. Construct the polynomials $\boldsymbol{F}$ using multivariate polynomial interpolation based on the values of $\boldsymbol{F}$ at each point $(a_1,\dots,a_n) \in X_1 \times \cdots \times X_n$, where X_i is a set of $D_i + 1$ points in $\mathbb{Z}$.

The first step also employs univariate polynomial interpolation. We found that the maximum degrees are $(D_1,\dots,D_{13}) = (1,1,1,0,1,1,1,0,2,2,2,2,1)$.

Algorithm for Evaluating $\boldsymbol{F}$. In both steps, it is necessary to determine the values of $\boldsymbol{F}$ at certain points $(a_1,\dots,a_n) \in \mathbb{R}^n$. Since $\boldsymbol{F}(a_1,\dots,a_n) \in \ker \boldsymbol{H}_{B_3}(f_G)(a_1,\dots,a_n)$, we can obtain information about the value from the kernel, a linear subspace of $\mathbb{R}^m$.

Let $(a_1,\dots,a_n) \in \mathbb{Z}^n_{>0}$. The value $\boldsymbol{F}(a_1,\dots,a_n) \in \mathbb{Z}^m$ is computed using the following steps.

1. Verify that the kernel $\ker \boldsymbol{H}_{B_3}(f_G)(a_1,\dots,a_n)$ is of dimension one.

2. Choose a non-zero vector $\boldsymbol{F}'$ from $\ker \boldsymbol{H}_{B_3}(f_G)(a_1,\dots,a_n)$.

3. Determine the coefficient $c \neq 0$ such that every component of the vector $c\boldsymbol{F}'$ is an integer, the greatest common divisor of the components of $c\boldsymbol{F}'$ is one, and $cF'_{i_0} > 0$ for a predetermined index i_0.

4. Lastly, this $c\boldsymbol{F}'$ will be the value of $\boldsymbol{F}(a_1,\dots,a_n)$.

By the verification of the first step, the vectors $\boldsymbol{F}'$ and $\boldsymbol{F}(a_1,\dots,a_n)$ are parallel. In some counterexample candidates, this confirmation fails because the kernel of the third Hessian matrix is most likely of dimension two.

The third step is a form of normalization. We anticipated the greatest common divisor of the components of $\boldsymbol{F}(a_1,\dots,a_n)$ to be one by choosing a_i to be a prime power. We

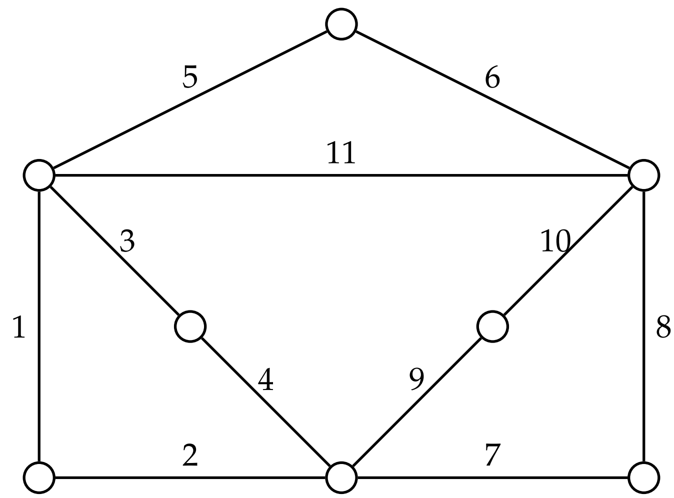

Figure 3: The graph does not have the SLP_2 with the element $\ell = \partial_1 + \cdots + \partial_n$

also predict that $F_{i_0}(a_1, \ldots, a_n)$ is always positive. We used $i_0 = 2$. The corresponding basis element is $\alpha_2 = \partial_1\partial_2\partial_4$. Finally it was revealed that $F_2 = (x_1 + x_5 + x_{13})(x_2 + x_6)x_{10}^2 x_{11} x_{12}$.

The vector of polynomials $\boldsymbol{F}$ has 90 zeros and 76 homogeneous polynomials of degree six. We observed that a vector of polynomials $\boldsymbol{F}$, defined similarly for the graph shown in Figure 2, contains only 154 zeros and 137 homogeneous polynomials of degree six as well.

4 Partial Failure of the SLP_2

First of all, every graph of eight or fewer vertices has the SLP_2. However, only one graph (Figure 3) does not have the SLP_2 with fixed element $\ell = \partial_1 + \cdots + \partial_n$. This graph also does not have the SLP_3 with the same element ℓ, but has the strong Lefschetz property with other elements.

This ℓ is typically one of the Lefschetz elements of the strong Lefschetz property. A previous work [14] showed that the complete or complete bipartite graph has the SLP_1 with this element ℓ. Besides, $\det \boldsymbol{H}_{B_k}(f)(1, \ldots, 1)$ is calculated in [5]. For any set of edges I, $\left(\left(\prod_{i \in I} \partial_i\right) f_G\right)(1, \ldots, 1)$ is equal to the number of spanning trees in G which contains all of edges in I.

Let G be the graph shown in Figure 3. The number of variables $n = 11$, the socle degree $d = 7$, the minimal number of generators of $\mathrm{Ann}_Q(f_G)$ is 42, and the Hilbert function $(\dim_{\mathbb{R}} A_i)_{i=0}^d = (1, 11, 51, 112, 112, 51, 11, 1)$. As in Section 3, we construct an $\mathbb{R}$-basis B_2 of A_2 by the same method and find a non-zero vector $\boldsymbol{F} = (F_1, \ldots, F_{51})^{\mathrm{T}} \in \mathbb{R}^{51}$ such that $\boldsymbol{H}_{B_2}(f_G)(1, \ldots, 1)\boldsymbol{F} = \boldsymbol{0}$.

Table 1: The basis B_2 and the vector $\boldsymbol{F}$

	∂_1	∂_2	∂_3	∂_4	∂_5	∂_6	∂_7	∂_8	∂_9	∂_{10}	∂_{11}
∂_1		10	0	-4	0	0	0	0	0	0	-3
∂_2			-4		0	0	0	0	0	0	-3
∂_3				10	0	0	0	0	0	0	-3
∂_4					0	0	0	0	0	0	-3
∂_5						0	0	0	0	0	
∂_6							0	0	0	0	
∂_7								-10	0	4	3
∂_8									4		3
∂_9										-10	3
∂_{10}											3
∂_{11}											

The vector $\boldsymbol{F}$ has only 16 non-zero components. Our B_2 and $\boldsymbol{F}$ are on Table 1. Empty cells mean that the corresponding monomials are not in B_2: $\partial_2\partial_4$, $\partial_5\partial_{11}$, $\partial_6\partial_{11}$, and $\partial_8\partial_{10}$. Non-empty cells contain corresponding F_i, e.g., $F_1 = 10$ for $\alpha_1 = \partial_1\partial_2$. Table 1 shows symmetries of the squares $\{\partial_1, \dots, \partial_9\} \times \{\partial_2, \dots, \partial_{10}\}$ and of the rightmost column.

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