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The s-permutahedron and its lattice quotients

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Abstract. For a tuple s of non-negative integers, the s-weak order is a lattice on s-trees, generalizing the weak order on permutations. We describe its join irreducible elements, its canonical join representations, and its forcing order in terms of combinatorial objects, generalizing the arcs, non-crossing arc diagrams, and subarc order for the weak order. We then extend the theory of shards and shard polytopes to construct geometric realizations of the s-weak order and all its lattice quotients as polyhedral complexes, generalizing the quotient fans and quotientopes of the weak order.

Résumé. Pour un uplet s d'entiers positifs ou nuls, le s-ordre faible est un treillis sur les s-arbres, généralisant l'ordre faible sur les permutations. Nous décrivons ses éléments sup irréductibles, ses représentations sup canoniques et son ordre de forçage par des objets combinatoires, généralisant les arcs, les diagrammes d'arcs non croisés et l'ordre des sous-arcs pour l'ordre faible. Nous étendons ensuite la théorie des tessons et des polytopes de tessons pour construire des réalisations géométriques du s-ordre faible et de tous ses treillis quotients sous forme de complexes polyédraux, généralisant les éventails quotients et les quotientopes de l'ordre faible.

1 Introduction

The structure of permutations and associations of an n-element set is a classical topic of algebraic and geometric combinatorics. In combinatorics, it is encoded by the Cayley graph of permutations under simple transpositions and the rotation graph on binary trees. In lattice theory, it materializes in the lattice morphism from the weak order on permutations to the Tamari lattice on binary trees. In polyhedral geometry, it appears in the braid arrangement and the sylvester fan, and their polar permutahedron and associahedron. See [12] for a survey on these connections and their influence in mathematics.

This prototype has motivated the study of all lattice congruences of the weak order, pioneered by N. Reading [15]. Combinatorially, he provided an elegant combinatorial model for the lattice theory of the weak order in [18]. Namely, the join irreducible permutations are encoded as certain arcs wiggling around the vertical axis, the canonical

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join representations of the permutations are encoded by non-crossing arc diagrams, and the forcing order on join irreducible permutations is encoded by the subarc order on these arcs. Geometrically, he showed in [16] that coarsening the braid fan according to the equivalence classes of any congruence of the weak order always yields a complete polyhedral fan. These quotient fans were shown to be normal fans of so-called quotientopes in [11]. Later, the problem was revisited in [8] using the theory of shard polytopes.

In [2, 3], C. Ceballos and V. Pons introduced the $\boldsymbol{s}$-weak order on $\boldsymbol{s}$-trees for an n-tuple $\boldsymbol{s}$ of non-negative integers, generalizing the classical weak order for $\boldsymbol{s} = (1, \dots, 1)$. An $\boldsymbol{s}$-tree is a rooted plane tree on $[n]$, where the node i has $s_i + 1$ children, all either leaves or nodes $j > i$. These $\boldsymbol{s}$-trees are ordered by inequalities between their inversion numbers, generalizing the definition of the weak order on permutations by inclusion of their inversion sets. See Section 2. [2, Section 1] proves that the $\boldsymbol{s}$-weak order is a lattice, describes its meets and joins, and establishes some lattice properties, in particular congruence uniformity. [2, Section 2] introduces the $\boldsymbol{s}$-Tamari lattice, a sublattice of the $\boldsymbol{s}$-weak order (also a lattice quotient when $\boldsymbol{s}$ contains no 0), which is isomorphic to the ν-Tamari lattice of [13, 1] for well-chosen $\boldsymbol{s}$ and ν. [3, Conjecture 3.1.2] conjectures that the Hasse diagram of the $\boldsymbol{s}$-weak order can be realized as an orientation of the skeleton of a polyhedral subdivision of the permutahedron, in the same spirit [1] realizes the ν-Tamari lattice. When $\boldsymbol{s}$ contains no 0, this conjecture was proved in [6] using flow polytopes.

Combining perspectives from combinatorics, lattice theory, polyhedral geometry and tropical geometry, we not only settle this conjecture for any $\boldsymbol{s}$ (including with some 0), but actually provide geometric realizations of all lattice quotients of the $\boldsymbol{s}$-weak order.

Theorem 1.1. *For any tuple $\boldsymbol{s}$ of non-negative integers (including with some 0), and any lattice congruence $\equiv$ of the $\boldsymbol{s}$-weak order $W_{\boldsymbol{s}}$, the lattice quotient $W_{\boldsymbol{s}}/{\equiv}$ is realized geometrically as*

- *the dual graph of the quotient foam $\mathcal{F}_{\equiv}$, a polyhedral complex extending quotient fans [16],*
- *the graph of the quotientoplex $\mathbb{Q}_{\equiv}$, a polytopal complex extending quotientopes [11, 8].*

To achieve this result, we first describe the join irreducibles of the $\boldsymbol{s}$-weak order in terms of $\boldsymbol{s}$-arcs. An $\boldsymbol{s}$-arc is an arc of [18] together with an integer bounded by $\boldsymbol{s}$. We extend the notions of crossings and subarcs of [18] to describe the canonical join representations of $\boldsymbol{s}$-trees as non-crossing $\boldsymbol{s}$-arc diagrams, and the forcing order of the $\boldsymbol{s}$-weak order as the subarc order on $\boldsymbol{s}$-arcs. Hence, the congruence lattice of the $\boldsymbol{s}$-weak order is isomorphic to the downset lattice of the subarc order on $\boldsymbol{s}$-arcs. See Sections 3 and 4.

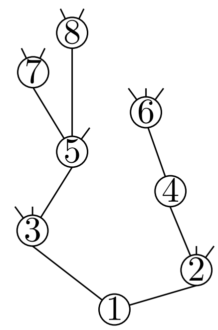
To each $\boldsymbol{s}$-arc, we then associate an $\boldsymbol{s}$-shard Σ_α (a polyhedral cone generalizing the shards of [14]) and a shardoplex $\mathbb{S}_\alpha$ (a polytopal complex constructed from and generalizing the shard polytopes of [8]). For a congruence $\equiv$ of the $\boldsymbol{s}$-weak order with $\boldsymbol{s}$-arcs $\mathcal{A}_{\equiv}$, the Hasse diagram of the quotient $W_{\boldsymbol{s}}/{\equiv}$ is then the oriented dual graph of the quotient foam $\mathcal{F}_{\equiv}$ whose union of walls is $\bigcup_{\alpha \in \mathcal{A}_{\equiv}} \Sigma_\alpha$, and the oriented graph of the quotientoplex $\mathbb{Q}_{\equiv}$ obtained as the Minkowski sum $\sum_{\alpha \in \mathcal{A}_{\equiv}} \mathbb{S}_\alpha$. See Sections 5 and 6.

Due to space limitation, this extended abstract omits many details and proofs of [9].

2 The s-weak order

We first recall some definitions and results from [2] (with slightly different conventions). We fix an n-tuple $\boldsymbol{s} := (s_1, \dots, s_n)$ of non-negative integers (note that we allow $s_i = 0$).

Definition 2.1 ([2, Section 1.2]). An ***s*-tree** is a rooted plane tree with internal nodes bijectively labeled by $[n]$ such that the node j has $s_j + 1$ children which are all either leaves or nodes larger than j.



Definition 2.2 ([2, Definition 1.3]). For an $\boldsymbol{s}$-tree T and $1 \le i < j \le n$, the *position* $\mathrm{pos}(\mathsf{T}, i, j) \in [0, s_i]$ is the minimum of s_i and the number of outgoing edges of i to the right of the increasing path from the root of T to j.

For instance, the tree $\mathsf{T}_\circ$ on the right is a $(1,2,2,0,2,2,1,1)$-tree, where $\mathrm{pos}(\mathsf{T}_\circ, 1, 4) = 0$, $\mathrm{pos}(\mathsf{T}_\circ, 1, 5) = 1$, $\mathrm{pos}(\mathsf{T}_\circ, 2, 5) = 2$ and $\mathrm{pos}(\mathsf{T}_\circ, 5, 7) = 2$. The following definition is illustrated in Figure 1 (left) for $\boldsymbol{s} = (2, 1, 0)$.

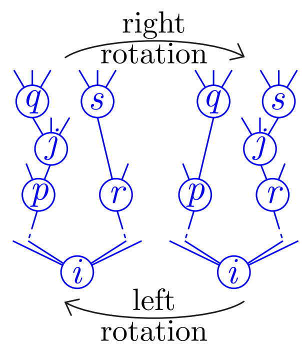
Definition 2.3 ([2, Definition 1.9]). The ***s*-weak order** $W_{\boldsymbol{s}}$ is the partially ordered set of $\boldsymbol{s}$-trees given by $\mathsf{T} \le \mathsf{T}'$ if and only if $\mathrm{pos}(\mathsf{T}, i, j) \le \mathrm{pos}(\mathsf{T}', i, j)$ for all $1 \le i < j \le n$.

Theorem 2.4 ([2, Theorems 1.21 & 1.40]). *The $\boldsymbol{s}$-weak order is a congruence uniform lattice.*

Definition 2.5 ([2, Definition 1.24]). A *descent* of an $\boldsymbol{s}$-tree T is a pair (i, j) of nodes where
- i is an ancestor of j and the increasing path from i to j takes the rightmost outgoing edge at each node, except at node i,
- either $s_j = 0$ or the rightmost edge of j is a leaf.

For instance, our $(1,2,2,0,2,2,1,1)$-tree $\mathsf{T}_\circ$ has descents $(1,5)$, $(2,4)$, $(2,6)$, $(5,7)$, $(5,8)$.

Definition 2.6 ([2, Definition 1.30]). Pick a descent (i, j) in an $\boldsymbol{s}$-tree T. Let p be the parent of j (p might be i). Let $j \to q$ be the leftmost outgoing edge of j (q might be a leaf). Let $r \to s$ be last edge whose source is smaller than j along the leftmost increasing path leaving i through the edge immediately to the right of the path from i to j (note that s might be a leaf). The *right rotation* of (i, j) transforms T by replacing the edges $p \to j$, $j \to q$ and $r \to s$ by new edges $p \to q$, $j \to s$ and $r \to j$ respectively.



Proposition 2.7 ([2, Theorem 1.32]). *The $\boldsymbol{s}$-trees covered by an $\boldsymbol{s}$-tree T in the $\boldsymbol{s}$-weak order $W_{\boldsymbol{s}}$ are precisely those obtained by the right rotation of a descent of T.*

Remark 2.8. When $\boldsymbol{s} = (1, \dots, 1)$, the $\boldsymbol{s}$-trees are just the increasing binary trees, in bijection with permutations of $[n]$. Namely, the permutation π_T corresponding to an increasing tree T is obtained by reading its nodes in infix order. Moreover, $\mathrm{pos}(\mathsf{T}, i, j) = 1$ if and only if (j, i) is an inversion of π_T, so that the $\boldsymbol{s}$-weak order is the classical weak order. More generally, if $\boldsymbol{s}$ contains no 0, then the $\boldsymbol{s}$-trees are in bijection with *Stirling $\boldsymbol{s}$-permutations*, *i.e.* permutations of the multiset $\{1^{s_1}, \dots, n^{s_n}\}$ avoiding the pattern 121 [2].

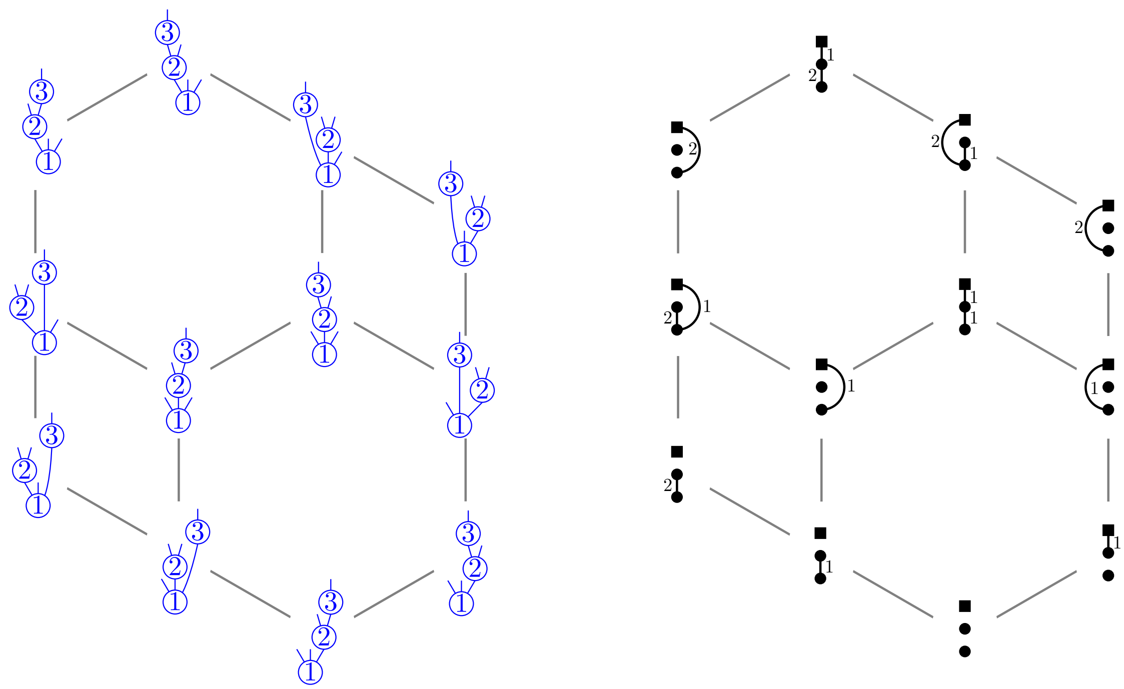


Figure 1: The s-weak order on s-trees and on s-arc diagrams, for $s = (2, 1, 0)$.

3 Canonical representations in the s-weak order

We now develop lattice properties of the s-weak order. Generalizing the arcs and non-crossing arc diagrams of [18], we describe the join irreducible s-trees and the canonical join representations in the s-weak order in terms of the following combinatorial objects.

Definition 3.1. An *s-arc* is a quintuple (i, j, A, B, r) where $1 \le i < j \le n$, the sets A and B form a partition of $\{k \in \,]i, j[\mid s_k \neq 0\}$, and $r \in [s_i]$.

Definition 3.2. Consider two s-arcs $\alpha := (i, j, A, B, r)$ and $\alpha' := (i', j', A', B', r')$. Assume without loss of generality that $j \le j'$ (otherwise, exchange α and α'). Then α and α' are *non-crossing* if $j < j'$, and one of the following holds:

1. $j \le i'$,
2. $i < i' < j$ and $i' \in A$ and $j \notin A'$ and $A' \cap \,]i, j[\,\subseteq A \cap \,]i', j'[$,
3. $i < i' < j$ and $i' \in B$ and $j \notin B'$ and $A' \cap \,]i, j[\,\supseteq A \cap \,]i', j'[$,
4. $i = i'$ and $r < r'$ and $j \notin A'$ and $A' \cap \,]i, j[\,\subseteq A \cap \,]i', j'[$,
5. $i = i'$ and $r = r'$ and $s_j = 0$ and $A' \cap \,]i, j[\,= A \cap \,]i', j'[$,
6. $i = i'$ and $r > r'$ and $j \notin B'$ and $A' \cap \,]i, j[\,\supseteq A \cap \,]i', j'[$,
7. $i' < i$ and $i \in A'$ and $j \notin B'$ and $A' \cap \,]i, j[\,\supseteq A \cap \,]i', j'[$,
8. $i' < i$ and $i \in B'$ and $j \notin A'$ and $A' \cap \,]i, j[\,\subseteq A \cap \,]i', j'[$.

A *non-crossing s-arc diagram* is a set δ of pairwise non-crossing s-arcs.

Graphically, consider n points on the vertical axis, labeled by $[n]$ from bottom to top. The $\boldsymbol{s}$-arc (i, j, A, B, r) is represented by a curve wiggling around these points, starting at i and ending at j, passing on the right of the points in A and on the left of the points in B (we do not care if it passes on the left or right of the points in $\{k \in \,]i, j[\mid s_k = 0\}$), and with a label r close to it. Two arcs are non-crossing if the corresponding curves are not crossing in their interior, plus some boundary conditions at their endpoints (the starting points may coincide according to rules 4, 5, 6 of Definition 3.2, while the ending points must be distinct). See Figure 1 (right), and the figure on the bottom of this page.

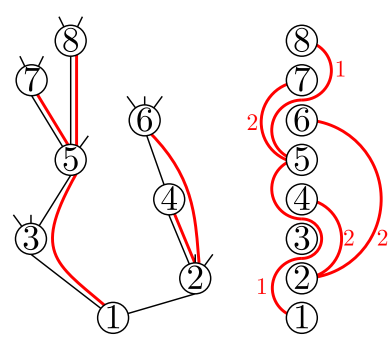
Remark 3.3. When $\boldsymbol{s} = (1, \dots, 1)$, the $\boldsymbol{s}$-arcs are the original arcs of [18]. An arc is a quadruple (i, j, A, B) where $1 \leq i < j \leq n$ and $]i, j[\, = A \sqcup B$, or equivalently a curve joining i to j and passing right of A and left of B. Two arcs are non-crossing if they do not cross in their interior and have distinct starting points and distinct ending points.

Let $(L, \leq, \vee)$ be a join semilattice. A *join representation* of $x \in L$ is a subset $J \subseteq L$ such that $x = \bigvee J$. Such a representation is *irredundant* if $x \neq \bigvee J'$ for any strict subset $J' \subsetneq J$. The irredundant join representations of $x \in L$ are ordered by $J \leq J'$ if and only if for any $y \in J$ there is $y' \in J'$ with $y \leq y'$. The *canonical join representation* of x is the unique minimal irredundant join representation of x for this order when it exists. Note that the canonical joinands form an antichain of *join irreducible* elements of L (those which cover a single element). Canonical join representations always exist if and only if L is *join semidistributive* (that is, $x \vee y = x \vee z$ implies $x \vee (y \wedge z) = x \vee y$ for any $x, y, z \in L$). It follows from Theorem 2.4 and [4] that the $\boldsymbol{s}$-weak order is join semidistributive. We now provide bijections from join irreducible $\boldsymbol{s}$-trees to $\boldsymbol{s}$-arcs, and from all $\boldsymbol{s}$-trees to non-crossing $\boldsymbol{s}$-arc diagrams, which enable to read the canonical join representations in the $\boldsymbol{s}$-weak order.

Definition 3.4. For a descent (i, j) of an $\boldsymbol{s}$-tree T, let $r := \operatorname{pos}(\mathsf{T}, i, j)$ and A (resp. B) the set of nodes $i < k < j$ with $s_k \neq 0$ and weakly on the left (resp. strictly on the right) of the path from i to j. Define $\alpha_\vee(\mathsf{T}, i, j) := (i, j, A, B, r)$ and $\delta_\vee(\mathsf{T}) := \{\alpha_\vee(\mathsf{T}, i, j) \mid (i, j) \text{ descent of } \mathsf{T}\}$.

Theorem 3.5. *The map $\alpha_\vee$ is a bijection from join irreducible $\boldsymbol{s}$-trees to $\boldsymbol{s}$-arcs. The map $\delta_\vee$ is a bijection from all $\boldsymbol{s}$-trees to non-crossing $\boldsymbol{s}$-arc diagrams. The canonical join representation of an $\boldsymbol{s}$-tree T is $\mathsf{T} = \bigvee_{\alpha \in \delta_\vee(\mathsf{T})} \alpha_\vee^{-1}(\alpha)$.*

As illustrated on the right, the non-crossing $\boldsymbol{s}$-arc diagram $\delta_\vee(\mathsf{T})$ of an $\boldsymbol{s}$-tree T can be obtained graphically by drawing the path joining i to j in T for each descent (i, j) of T, perturbing all these paths so that they pass slightly to the right of their intermediate nodes, and flattening the picture horizontally, allowing the arcs to bend but not to cross nor to pass a node.



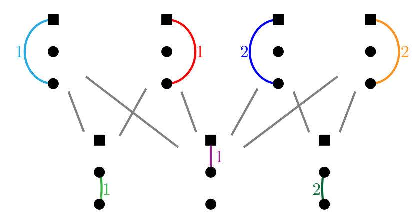
For instance, Figure 1 (right) represents the $\boldsymbol{s}$-weak order where the $\boldsymbol{s}$-trees are replaced by their non-crossing $\boldsymbol{s}$-arc diagrams, for $\boldsymbol{s} = (2, 1, 0)$.

4 Quotients of the s-weak order

A *congruence* $\equiv$ on a finite lattice $(L, \leq, \wedge, \vee)$ is an equivalence relation on L such that $x \equiv x'$ and $y \equiv y'$ implies $x \vee y \equiv x' \vee y'$ and $x \wedge y \equiv x' \wedge y'$. The *lattice quotient* $L/\!\equiv$ is the lattice where the join $X \vee Y$ (resp. meet $X \wedge Y$) of two congruence classes is the congruence class of the join $x \vee y$ (resp. meet $x \wedge y$) of any representatives $x \in X$ and $y \in Y$. A congruence $\equiv$ is determined by its contracted join irreducibles, *i.e.* its relations $j \equiv j_\star$ where j covers only $j_\star$. If any congruence contracting j also contracts j', then j *forces* j'. This defines a preorder on join irreducibles of L whose lattice of downsets is isomorphic to the *congruence lattice* of L. This preposet is a poset when L is congruence uniform. Understanding the congruences of L then boils down to understanding the forcing poset on join irreducibles of L. Generalizing the subarc order of [18], we now describe the forcing order of the $\boldsymbol{s}$-weak order in terms of the subarc order on $\boldsymbol{s}$-arcs.

Definition 4.1. Consider two $\boldsymbol{s}$-arcs $\alpha := (i, j, A, B, r)$ and $\alpha' := (i', j', A', B', r')$. We say that α is a *subarc* of α' if all the following conditions hold:

- $i' \leq i < j \leq j'$,
- $A \subseteq A'$ and $B \subseteq B'$,
- if $s_j = 0$ then $j = j'$,
- if $i' = i$ then $r = r'$,
- if $i' < i$ then $i \in A'$ and $r = 1$, or $i \in B'$ and $r = s_i$.



Remark 4.2. When $\boldsymbol{s} = (1, \ldots, 1)$, an arc (i, j, A, B) is a subarc of an arc (i', j', A', B') if and only if $i' \leq i < j \leq j'$ and $A \subseteq A'$ and $B \subseteq B'$, as defined in [18].

Theorem 4.3. *The forcing order on join irreducible $\boldsymbol{s}$-trees is the subarc order on $\boldsymbol{s}$-arcs: If T and T' are two join irreducible $\boldsymbol{s}$-trees, then T forces T' if and only if $\alpha_\vee(\mathsf{T})$ is a subarc of $\alpha_\vee(\mathsf{T}')$.*

Corollary 4.4. *The congruence lattice of the $\boldsymbol{s}$-weak order $W_{\boldsymbol{s}}$ is isomorphic to the lattice of downsets of the subarc order on $\boldsymbol{s}$-arcs. We denote by $\mathcal{A}_\equiv$ the downset of a congruence $\equiv$.*

Remark 4.5. We can exploit Corollary 4.4 to define relevant congruences and quotients:

- The *$\boldsymbol{s}$-Tamari lattice* is the quotient of the $\boldsymbol{s}$-weak order by the *$\boldsymbol{s}$-sylvester congruence* $\equiv_{\mathrm{sylv}}$ where $\mathcal{A}_{\equiv_{\mathrm{sylv}}}$ is the set of *right $\boldsymbol{s}$-arcs* $(i, j, \{k \in \,]i, j[\, \mid s_k \neq 0\}, \varnothing, r)$ for some $1 \leq i < j \leq n$ and $r \in [s_i]$. When $\boldsymbol{s}$ contains no 0, this quotient was considered in [2, Section 2.4] in connection to the ν-Tamari lattice [13, 1].
- For any $\boldsymbol{s}$-arc α, the *α-Cambrian lattice* is the quotient of the $\boldsymbol{s}$-weak order by the *α-Cambrian congruence* $\equiv_\alpha$, where $\mathcal{A}_{\equiv_\alpha}$ is the downset of subarcs of α. See [17].
- For a map $\delta : \{i \in [n] \mid s_i \neq 0\} \rightarrow \{⦶, ☮, ⦷, ⊗\}$, the *$\delta$-permutree lattice* is the quotient of the $\boldsymbol{s}$-weak order by the *δ-permutree congruence* $\equiv_\delta$, where $\mathcal{A}_{\equiv_\delta}$ is the set of arcs which do not pass on the right of a point j with $\delta(j) \in \{⦷, ⊗\}$ nor on the left of the points j with $\delta(j) \in \{☮, ⊗\}$. See [10].

- A congruence $\equiv$ is *regular* if any minimal (in subarc order) arc (i, j, A, B) of the complement of the set $\mathcal{A}_{\equiv}$ is either a left arc ($A = \varnothing$) or a right arc ($B = \varnothing$). See [7].

5 Quotient foams

We now realize the $\mathbf{s}$-weak order and its lattice quotients as dual graphs of polyhedral complexes. Generalizing the shards of [14], we first associate codimension 1 polyhedral cones to the $\mathbf{s}$-arcs, and use them to generalize the braid fan. See Figure 2 (left).

Definition 5.1. The *$\mathbf{s}$-shard* of an $\mathbf{s}$-arc $\alpha := (i, j, A, B, r)$ is the polyhedron Σ_α of $\mathbb{R}^n$ given by

- the equality $x_i - x_j = r - 1 + \sum_{k \in B} \max(0, s_k - 1)$,
- the inequalities $x_i - x_a \geq r - 1 + \sum_{k \in B \cap]i,a[} \max(0, s_k - 1)$ for all $a \in A$, and
- the inequalities $x_i - x_b \leq r - 1 + \sum_{k \in B \cap]i,b[} \max(0, s_k - 1)$ for all $b \in B$.

Definition 5.2. The *$\mathbf{s}$-foam* $\mathcal{F}_\mathbf{s}$ is the polyhedral complex of $\mathbb{R}^n$ whose maximal cells are the closures of the connected components of the complement of the union of all $\mathbf{s}$-shards.

Theorem 5.3. *The Hasse diagram of the $\mathbf{s}$-weak order is isomorphic to the dual graph of the $\mathbf{s}$-foam $\mathcal{F}_\mathbf{s}$ oriented in the direction $\boldsymbol{\omega} := (1, 2, \dots n) - (n, \dots, 2, 1) = \sum_{i \in [n]} (2i - n - 1)\, \mathbf{e}_i$.*

Remark 5.4. When $\mathbf{s} = (1, \dots, 1)$, the shard of an arc (i, j, A, B) is the polyhedral cone defined by $x_b \leq x_i = x_j \leq x_a$ for all $a \in A$ and $b \in B$. The union of the shards is the braid arrangement, and the $\mathbf{s}$-foam is the classical braid fan.

Remark 5.5. The cells of the $\mathbf{s}$-foam $\mathcal{F}_\mathbf{s}$ can also be obtained as the closures of the fibers of an insertion algorithm, generalizing the insertion algorithm of permutations in increasing trees. This perspective is the most convenient to prove Theorem 5.3, but its

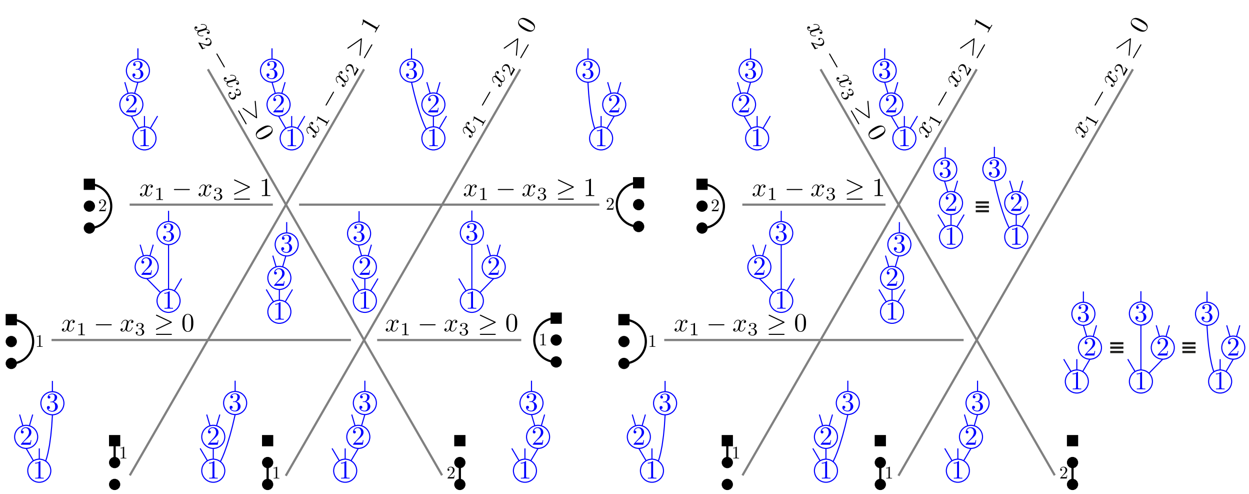


Figure 2: The $\mathbf{s}$-foam and its sylvester quotient foam, for $\mathbf{s} = (2, 1, 0)$. See Figure 4.

presentation is more technical, and unnecessary here. We refer to [9, Section 1] for details.

Proposition 5.6. *The vertices of $\mathcal{F}_{\boldsymbol{s}}$ form a $(s_0 \times \cdots \times s_{n-1})$-grid $\boldsymbol{\Lambda_s} := \prod_{i \in [n-1]} [s_i](\boldsymbol{e}_1 - \boldsymbol{e}_{i+1})$.*

Generalizing [16], we can then either glue some regions of the $\boldsymbol{s}$-foam or delete some $\boldsymbol{s}$-shards according to a congruence of the $\boldsymbol{s}$-weak order, as illustrated in Figure 2 (right).

Definition 5.7. For any congruence $\equiv$ of the $\boldsymbol{s}$-weak order, the *$\equiv$-quotient foam* $\mathcal{F}_\equiv$ is the complete polyhedral complex defined by the following equivalent descriptions:

(i) the maximal cells of $\mathcal{F}_\equiv$ are obtained by glueing together the maximal cells of the $\boldsymbol{s}$-foam $\mathcal{F}_{\boldsymbol{s}}$ corresponding to $\boldsymbol{s}$-trees in the same congruence class of $\equiv$,

(ii) the union of the walls of $\mathcal{F}_\equiv$ is the union of the $\boldsymbol{s}$-shards Σ_α of the $\boldsymbol{s}$-arcs α in $\mathcal{A}_\equiv$.

Theorem 5.8. *For any congruence $\equiv$ of the $\boldsymbol{s}$-weak order $W_{\boldsymbol{s}}$, the two descriptions of Definition 5.7 coincide and define a polyhedral complex $\mathcal{F}_\equiv$, whose dual graph, oriented in the direction $\boldsymbol{\omega}$, is isomorphic to the Hasse diagram of the quotient $W_{\boldsymbol{s}}/\!\equiv$.*

Remark 5.9. When $\boldsymbol{s} = (1, \ldots, 1)$, the $\boldsymbol{s}$-quotient foams are the quotient fans of [16].

6 Quotientoplexes

We now realize the $\boldsymbol{s}$-weak order and its lattice quotients as graphs of polytopal complexes. Our construction relies on the shard polytopes of [8], which are elementary pieces whose Minkowski sums enable to construct quotientopes for all lattice congruences of the weak order. We first recall their definition.

Definition 6.1 ([8, Definitions 39 & 40])**.** Fix an original arc $\alpha := (i, j, A, B)$. An *α-alternating matching* μ is a sequence $i \le i_1 < j_1 < i_2 < j_2 < \cdots < i_q < j_q \le j$ such that $i_p \in \{i\} \cup A$ and $j_p \in \{j\} \cup B$ for all $p \in [q]$. Its *characteristic vector* is $\boldsymbol{\chi}_\mu := \sum_{p \in [q]} \boldsymbol{e}_{i_p} - \boldsymbol{e}_{j_p}$. The *shard polytope* $\mathbb{SP}_\alpha$ is the convex hull of the characteristic vectors of all α-alternating matchings.

Given an $\boldsymbol{s}$-arc α, we now associate a face $\mathbb{S}^{\boldsymbol{q}}_\alpha$ of a shard polytope to each vertex $\boldsymbol{q} \in \boldsymbol{\Lambda_s}$ of the $\boldsymbol{s}$-foam, and consider the polytopal complex formed by the collection $(\mathbb{S}^{\boldsymbol{q}}_\alpha)_{\boldsymbol{q} \in \boldsymbol{\Lambda_s}}$.

Definition 6.2. Consider an $\boldsymbol{s}$-arc $\alpha := (i, j, A, B, r)$, the arc $\tilde{\alpha} := (i, j, A, B)$, and a vertex $\boldsymbol{q} \in \boldsymbol{\Lambda_s}$ of the $\boldsymbol{s}$-foam $\mathcal{F}_{\boldsymbol{s}}$. The *local shard polytope* $\mathbb{S}^{\boldsymbol{q}}_\alpha$ is the face of the shard polytope $\mathbb{SP}_{\tilde{\alpha}}$ maximizing the scalar product with $\sum_{\ell \in]i,j]} \big(q_i - q_\ell + r - 1 + \sum_{k \in B \cap]i,\ell[} \max(0, s_k - 1)\big) \boldsymbol{e}_\ell$.

Proposition 6.3. *For any $\boldsymbol{s}$-arc α, the collection of all local shard polytopes $\mathbb{S}^{\boldsymbol{q}}_\alpha$ for $\boldsymbol{q} \in \boldsymbol{\Lambda_s}$, together with all their faces, form a polyhedral complex $\mathbb{S}_\alpha$ that we call the shardoplex of α.*

We now construct quotientoplexes as Minkowski sums of shardoplexes, see Figure 4.

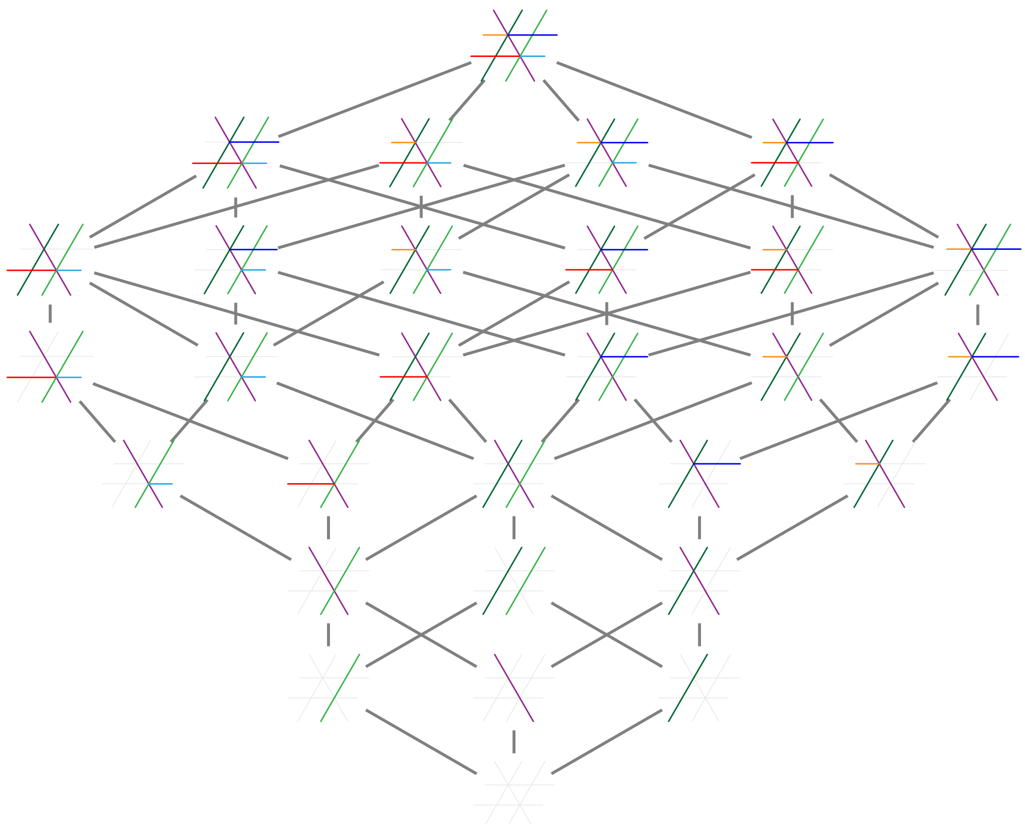


Figure 3: The congruence lattice of the $(2,1,0)$-weak order, where each congruence $\equiv$ is replaced by its quotient foam $\mathcal{F}_{\equiv}$. See also Figure 5.

Definition 6.4. For a congruence $\equiv$ of the s-weak order W_s, the *quotientoplex* $\mathbb{Q}_{\equiv}$ is the polytopal complex obtained as the coordinatewise Minkowski sum $(\sum_{\alpha \in \mathcal{A}_{\equiv}} \mathbb{S}_{\alpha}^{q})_{q \in \Lambda_s}$ of the shardoplexes $\mathbb{S}_{\alpha}$ over all s-arcs α in $\mathcal{A}_{\equiv}$.

Proposition 6.5. *There is an inclusion reversing bijection ψ from the faces of the quotient foam $\mathcal{F}_{\equiv}$ to the faces of the quotientoplex $\mathbb{Q}_{\equiv}$ such that $\mathbb{F}$ and $\psi(\mathbb{F})$ are orthogonal.*

Theorem 6.6. *For any congruence $\equiv$ of the s-weak order W_s, the Hasse diagram of the quotient $W_s/{\equiv}$ is isomorphic to the skeleton of the quotientoplex $\mathbb{Q}_{\equiv}$.*

Remark 6.7. When $s = (1, \dots, 1)$, the shardoplexes are the shard polytopes of [8], and the quotientoplexes are the quotientopes of [8].

Our next result states that quotientoplexes are polytopal subdivisions of quotientopes. To be precise, we need to manipulate weighted Minkowski sums of shard polytopes and shardoplexes. Note that the combinatorics of the resulting quotientopes and quotientoplexes does not depend on these weights, as long as they are positive.

Proposition 6.8. *For an $\boldsymbol{s}$-arc $\alpha := (i, j, A, B, r)$, denote by $\tilde{\alpha} := (i, j, A, B)$ the corresponding original arc. For a congruence $\equiv$ of the $\boldsymbol{s}$-weak order $W_{\boldsymbol{s}}$, denote by $\tilde{\equiv}$ the corresponding congruence of the weak order W_n, with downset of arcs $\mathcal{A}_{\tilde{\equiv}} := \{\tilde{\alpha} \mid \alpha \in \mathcal{A}_{\equiv}\}$. Consider $\boldsymbol{\lambda} := (\lambda_\alpha)_{\alpha \in \mathcal{A}_{\equiv}}$ with $\lambda_\alpha > 0$, and let $\tilde{\boldsymbol{\lambda}} := (\tilde{\lambda}_{\tilde{\alpha}})_{\tilde{\alpha} \in \mathcal{A}_{\tilde{\equiv}}}$ with $\tilde{\lambda}_{\tilde{\alpha}} := \sum_\alpha \lambda_\alpha$ where the sum ranges over all $\boldsymbol{s}$-arcs α which project to $\tilde{\alpha}$. Then the quotientoplex $(\sum_{\alpha \in \mathcal{A}_{\equiv}} \lambda_\alpha \, \mathbb{S}_\alpha^{\boldsymbol{q}})_{\boldsymbol{q} \in \boldsymbol{\Lambda}_{\boldsymbol{s}}}$ is a polytopal subdivision of (a translate of) the quotientope $\sum_{\tilde{\alpha} \in \mathcal{A}_{\tilde{\equiv}}} \tilde{\lambda}_{\tilde{\alpha}} \, \mathbb{SP}_{\tilde{\alpha}}$.*

For instance, Figure 6 show the $(2, 1, 0, 1)$-permutahedron and the $(2, 1, 0, 1)$-associahedron, which are polytopal subdivisions of a permutahedron and an associahedron.

Applying Theorem 6.6 and Proposition 6.8 to the trivial congruence (where each congruence class contains a single $\boldsymbol{s}$-tree), we obtain the following statement, answering a question of C. Ceballos and V. Pons [2, 3]. We note that this question was partially solved in [6] in the case when $\boldsymbol{s}$ contains no 0 entry, with a very different method based on a combination of flow polytopes, tropical geometry, and Cayley embedding.

Corollary 6.9. *For any $\boldsymbol{s}$ (including with some 0 entries), the Hasse diagram of the $\boldsymbol{s}$-weak order $W_{\boldsymbol{s}}$ is isomorphic to the oriented skeleton of a polytopal subdivision of a polytope combinatorially equivalent to the zonotope $\mathbb{Z}\mathrm{ono}(\boldsymbol{s}) := \sum_{1 \le i < j \le n} s_i \operatorname{conv}\{\boldsymbol{e}_i, \boldsymbol{e}_j\}$.*

We conclude with some intriguing conjectures on the special congruences discussed in Remark 4.5.

Conjecture 6.10. *Up to isomorphism, the face lattice of the α-Cambrian quotientoplex $\mathbb{Q}_{\equiv_\alpha}$ only depends on the endpoints of α.*

Conjecture 6.11. *For any fixed $\boldsymbol{s}$, changing any Ⓨ to ☮ in δ does not affect the cardinality of the δ-permutree lattice $W_{\boldsymbol{s}}/\!\equiv_\delta$, although it may change the isomorphism class of the face lattice of the δ-permutree quotientoplex $\mathbb{Q}_{\equiv_\delta}$.*

Conjecture 6.12. *All faces of the quotientoplex $\mathbb{Q}_{\equiv}$ are simple polytopes if and only $\equiv$ is regular.*

Remark 6.13. When $\boldsymbol{s} = (1, \dots, 1)$, these conjectures hold by [17], [10], and [7, 5].

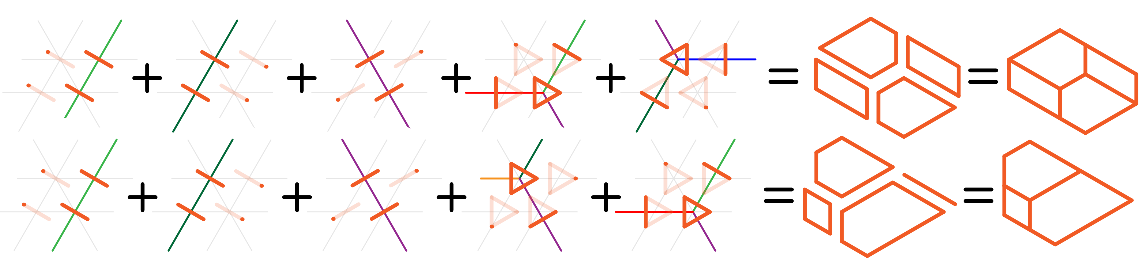


Figure 4: The $\boldsymbol{s}$-permutahedron (top) and $\boldsymbol{s}$-associahedron (bottom) obtained as Minkowski sums of shardoplexes, for $\boldsymbol{s} = (2, 1, 0)$. See Figure 2.

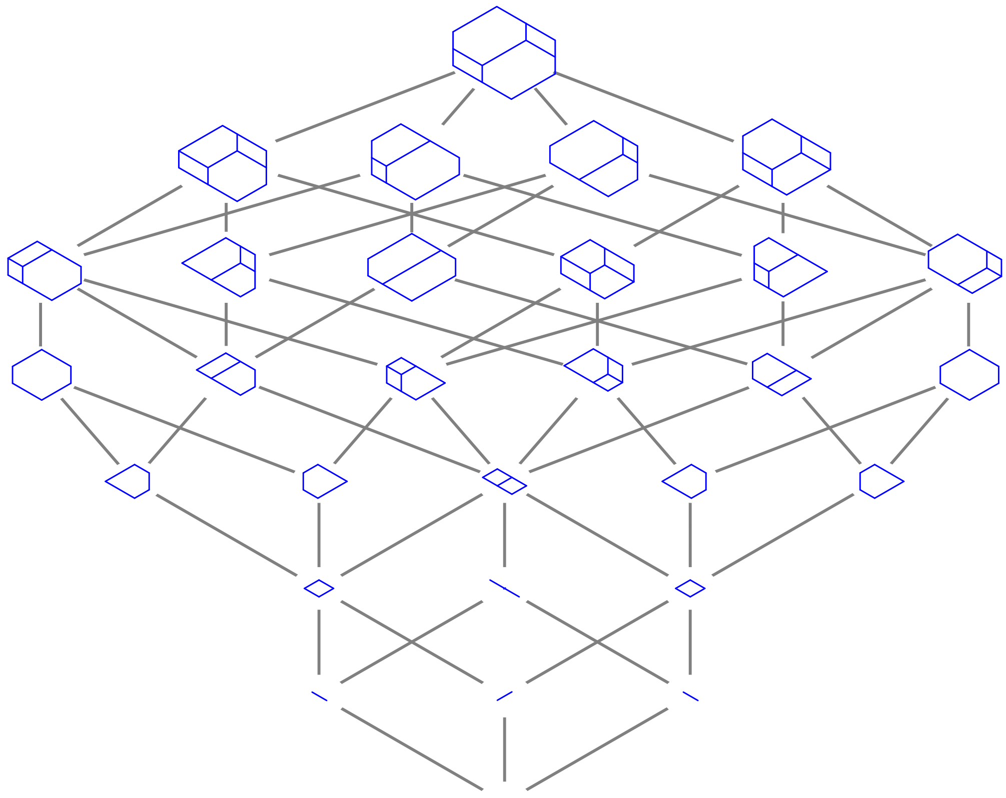


Figure 5: The congruence lattice of the $(2, 1, 0)$-weak order, where each congruence $\equiv$ is replaced by its quotientoplex $\mathbb{Q}_{\equiv}$. See also Figure 3.

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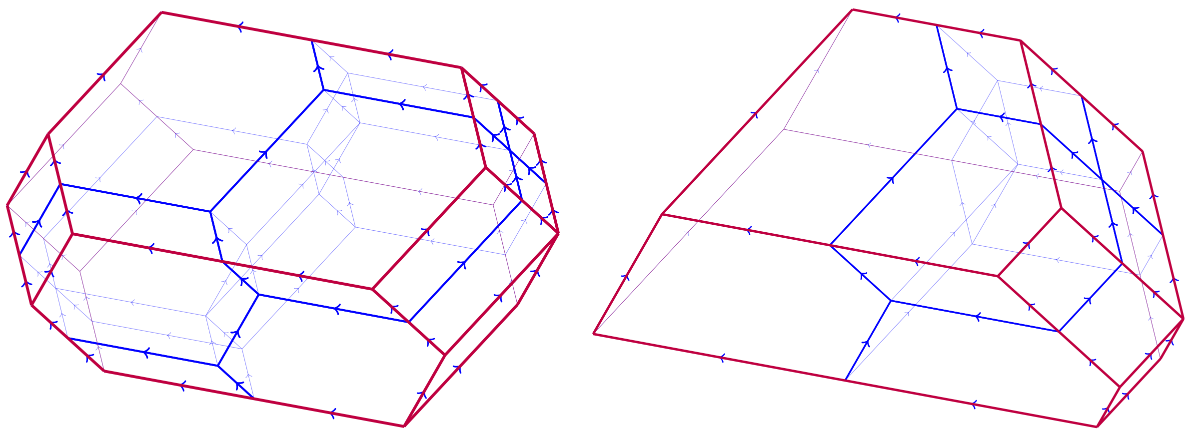

Figure 6: The $(2,1,0,1)$-permutahedron and the $(2,1,0,1)$-associahedron.

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