Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #122, 12 pp.

Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

Higher Specht polynomials for the diagonal action

Maria M. Gillespie*[1]

[1]*Department of Mathematics, Colorado State University*

Abstract. We introduce higher Specht polynomials - analogs of Specht polynomials in higher degrees - in two sets of variables $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ under the diagonal action of the symmetric group S_n. This generalizes the classical Specht polynomial construction in one set of variables, as well as the higher Specht basis for the coinvariant ring R_n due to Ariki, Terasoma, and Yamada, which has the advantage of respecting the decomposition into irreducibles.

As our main application, we provide a higher Specht basis for the hook shape Garsia–Haiman modules. In the process, we obtain a new formula for their doubly graded Frobenius series in terms of new generalized cocharge statistics on tableaux.

Keywords: Diagonal coinvariants, Shuffle theorem, Schur positivity, Young tableaux, Garsia–Haiman modules, representation theory of the symmetric group

1 Introduction and notation

The polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$ comes with a natural action of the symmetric group S_n by permuting the variables. This gives the polynomial ring the structure of a graded S_n-representation, which naturally decomposes into irreducible representations. In [5], Ariki, Terasoma, and Yamada defined **higher Specht polyomials** that give a basis for $\mathbb{C}[x_1, \ldots, x_n]$ that respects the decomposition into irreducibles and that generalizes the ordinary Specht polynomial construction for the lowest-degree copy of each irreducible representation. In order to do so, they first found such a basis for the **coinvariant ring** $R_n = \mathbb{C}[x_1, \ldots, x_n]/(e_1, \ldots, e_n)$ where e_i is the ith *elementary symmetric polynomial* consisting of the sum of all square-free monomials of degree i in $x_1, \ldots, x_n$.

In this extended abstract, we extend the theory of higher Specht polynomials to the **diagonal action** of the symmetric group S_n on the polynomial ring

$$\mathbb{C}[\mathbf{x}, \mathbf{y}] := \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n],$$

defined by $\pi \cdot f(x_1, \ldots, x_n, y_1, \ldots, y_n) = f(x_{\pi(1)}, \ldots, x_{\pi(n)}, y_{\pi(1)}, \ldots, y_{\pi(n)})$. Full details and proofs on the results outlined here can be found in [12].

The coinvariant ring construction can be generalized to two variables as follows.

*maria.gillespie@colostate.edu. Partially supported by NSF DMS award number 2054391.

Definition 1.1. The ring of **diagonal coinvariants** is given by

$$\mathrm{DR}_n = \mathbb{C}[\mathbf{x},\mathbf{y}]/I_n$$

where I_n is generated by the positive-degree S_n-invariants under the diagonal action.

The ring DR_n arises naturally in the geometry of the Hilbert scheme of n points in the plane $\mathbb{C}^2$. Haiman [16] used this connection to prove the famous $(n+1)^{n-1}$ *Conjecture*, which states that $\dim_{\mathbb{C}}(\mathrm{DR}_n) = (n+1)^{n-1}$ as a complex vector space.

Haiman used similar methods to prove the $n!$ *Conjecture* [15]. This states that the *Garsia–Haiman modules* DR_μ (which are also quotients of $\mathbb{C}[\mathbf{x},\mathbf{y}]$, describing local information of a limit as the n points all approach 0 in the plane) have dimension $n!$ for any partition μ of size n. The proof of the $n!$ conjecture was the crucial step in the proof of the Macdonald Positivity Conjecture, which states that the transformed *Macdonald polynomials* [19] have a positive expansion in terms of the Schur symmetric functions.

Despite these advances, it remains open to understand the $n!$ and $(n+1)^{n-1}$ conjectures from a more combinatorial standpoint, in the following sense.

Problem 1.2. *Find an explicit basis of $n!$ polynomials for DR_μ, where μ is a partition of n.*

Problem 1.3. *Find an explicit basis of $(n+1)^{n-1}$ polynomials for DR_n.*

Problem 1.2 is open for general partitions μ, while Problem 1.3 has very recently been addressed by Carlsson and Oblomkov [7], who gave a construction of a basis for DR_n by establishing connections with affine Schubert calculus. However, their basis is not a higher Specht basis in the following sense.

Definition 1.4. A **higher Specht basis** for a (graded) S_n-module M is a basis $\mathbb{B}$ that admits a set partition $\bigsqcup \mathbb{B}_{\lambda,i}$ where λ and i represent a partition and positive integer respectively, such that:

1. Each $\mathbb{B}_{\lambda,i}$ spans one of the copies of an irreducible representation V_λ in the decomposition of M into irreducibles (there may be several such copies, so we distinguish with the subscript i, and every copy is one such span),

2. There is a bijection from $\mathbb{B}_{\lambda,i}$ to the set of ordinary Specht polynomials F_T (defined below) for shape λ, that preserves the S_n-action with respect to each basis.

Starting with the work in [5] for the coinvariant ring R_n, there have been several higher Specht bases constructed for related modules in recent years. In [11], the author and Rhoades found higher Specht bases for both the modules $R_{n,k}$ (the *Haglund–Rhoades–Shimozono modules* defined in the context of the $t=0$ specialization of the Delta conjecture [14]) and R_μ (the *Garsia–Procesi modules*, which are the cohomology rings of the fibers of the Springer resolution [8, 10, 27]). The construction was proven to be a basis in the former case and conjectured for the latter, with proof for μ having two parts.

This construction was similar to that of [3], in which Allen constructed a basis that respects the decomposition into irreducibles for R_n, and for R_μ for μ having two parts or being a hook shape. (Allen's basis is not a higher Specht basis in the strongest sense, since it does not satisfy condition 2 above). In [2] and [4], Allen also began an exploration the two-variable case, focusing on the diagonally symmetric subring of the polynomial ring in two variables and its quotients.

Most recently, Salois [22] defined a higher Specht basis for the cohomology rings of certain *Hessenberg varieties*. These rings also appear naturally in symmetric function theory, as they directly relate to the *Stanley–Stembridge Conjecture* [17, 25] and the *Shareshian–Wachs Conjecture* [23] on chromatic (quasi)symmetric functions.

Our main result is a higher Specht basis for the hook shape Garsia–Haiman modules:

Theorem 1.5. *Suppose $\mu = (n-k+1, 1^k)$ is a hook shape of height k. There is an explicitly constructed set of polynomials $\{F_T^S(\mathbf{x}, \mathbf{y})\}$ ranging over all pairs T, S of standard Young tableaux of the same shape that forms a higher Specht basis for* DR_μ.

We achieve this result by first developing a general theory of higher Specht polynomial constructions in two sets of variables (see Section 2 below).

1.1 Notation on Young tableaux and cocharge

We refer to [9] for notation on Young tableaux and partitions, with the caveat that we are using the "French" convention for Young diagrams in this paper, where the i-th row from the bottom of the Young diagram of a partition λ has λ_i squares. See Figure 1. We will also generalize the notion of a semistandard Young tableau (SSYT) to include entries from any ordered set, not just the natural numbers. We write $\mathrm{SYT}(\lambda)$ for the set of standard Young tableaux of shape λ, and $\mathrm{SYT}(n)$ for the set of Young tableaux of size n. We write $\mathrm{Tab}(\lambda)$ for the set of **standard general tableau** T of shape λ, which are ways of filling the squares of λ with the numbers $1, 2, \ldots, n$ each used once, with no restrictions on the rows or columns.

We recall the **RSK bijection**, which takes permutations of $1, \ldots, n$ to pairs (P, Q) of SYT's with the same shape as each other. Thus there are $n!$ such pairs.

Definition 1.6. The **cocharge labels** of a permutation π are defined by labeling the 1 with a subscript 0, then searching leftwards cyclically for the $2, 3, 4, \ldots$, each time incrementing the subscript label unless the search wraps around the word. The **cocharge** of π, written $\mathrm{cc}(\pi)$, is the sum of its cocharge labels.

Example 1.7. The permutation 25314 has cocharge labels $2_1 5_2 3_1 1_0 4_1$, so

$$\mathrm{cc}(25314) = 1 + 2 + 1 + 0 + 1 = 5.$$

5			
3	4		
2	2		
1	1	1	3

7			
4	8		
2	5		
1	3	6	9

7			
1	4		
6	2		
3	9	8	5

Figure 1: From left to right: A semistandard Young tableau, a standard Young tableau, and a standard general tableau T of shape $\mathrm{sh}(T) = \lambda = (4, 2, 2, 1)$. The tableau at left has content $(3, 2, 2, 1, 1)$, and the reading word of the tableau at middle is 748251369.

Lusztig and Stanley [24] showed (in a different notation, that is equivalent to the cocharge statistic defined by Lascoux and Schützenberger [18]) that the number of copies of the irreducible representation V_λ of S_n in the coinvariant ring R_n in degree d is equal to the number of standard Young tableaux of shape λ with cocharge d. We will express this fact in terms of the graded Frobenius character in the next section.

1.2 Symmetric functions and Frobenius series

The two bases of the ring of symmetric functions $\Lambda_{\mathbb{C}}(x_1, x_2, \ldots)$ that will be used are the **elementary** symmetric functions, defined by $e_d(x_1, x_2, \ldots) = \sum_{i_1 < i_2 < \cdots < i_d} x_{i_1} \cdots x_{i_d}$ and $e_\lambda = \prod e_{\lambda_i}$, and the **Schur functions**, defined by $s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} \prod_{c \in \lambda} x_{T(c)}$. The **Frobenius map** Frob is the additive map that sends the irreducible S_n-module V_ν to the Schur function s_ν. For instance, $\mathrm{Frob}(V_{(2)} \oplus V_{(1,1)} \oplus V_{(1,1)}) = s_{(2)} + 2s_{(1,1)}$.

For S_n-modules with a grading (such as the coinvariant ring, which is graded by degree), its **graded Frobenius** is the generating function $\mathrm{grFrob}_q(R) := \sum_d \mathrm{Frob}(R_d) q^d$ where R_d is the d-th graded piece and q is a formal variable. For doubly graded S_n-modules (by the x-degree and y-degree here) the **bi-graded Frobenius map** is

$$\mathrm{grFrob}_{q,t}(R) := \sum_d \mathrm{Frob}(R_{d_1,d_2}) q^{d_1} t^{d_2}.$$

The theorem of Lusztig and Stanley [24] mentioned above on the decomposition of R_n into irreducibles can therefore be stated as

$$\mathrm{grFrob}_q(R_\mu) = \sum_{T \in \mathrm{SSYT}_\mu} q^{\mathrm{cc}(T)} s_{\mathrm{sh}(T)}$$

where SSYT_μ is the set of SSYT's of content μ. We will not define Garsia–Haiman modules in full in this extended abstract, but we discuss their properties when μ is a **hook shape** of the form $(n - k + 1, 1^{k-1})$. Several monomial bases for DR_μ for hook shapes μ were established in [1]. The graded Frobenius was derived by Stembridge [26], and makes use of "maj" and "comaj" statistics:

$$\mathrm{grFrob}_{q,t}(\mathrm{DR}_\mu) = \sum_{T \in \mathrm{SYT}(n)} q^{\mathrm{maj}_{1,n-k+1}(T)} t^{\mathrm{comaj}_{n-k+1,n}(T)} s_{\mathrm{sh}(T)}. \tag{1.1}$$

1.3 Specht polynomial constructions

The standard construction of the *Specht modules* V_λ, which are the irreducible representations of S_n where λ ranges over all partitions of n, is often presented via *Young tabloids* (see [21]). This construction is equivalent (see [19, p. 125, Exercise 15(c)]) to defining a submodule of $\mathbb{C}[x_1, \ldots, x_n]$ spanned by *Specht polynomials* as follows.

Definition 1.8. The **Specht polynomial** F_T corresponding to a Young tableau T whose entries are $1, 2, 3, \ldots, n$ is given by

$$F_T = \prod_{C \in \mathrm{col}(T)} \prod_{\substack{i,j \in C \\ i \text{ above } j}} (x_i - x_j)$$

where $\mathrm{col}(T)$ is the set of columns of T. (See Figure 2.)

One can then define the Specht module as

$$M_\lambda = \mathrm{span}\{F_T : T \in \mathrm{Tab}(\lambda)\} \subseteq \mathbb{C}[x_1, \ldots, x_n],$$

and we have $M_\lambda \cong V_\lambda$ with a basis given by the F_T such that T is a **standard Young tableau (SYT)**, in which the entries are increasing along rows and up columns. The **Garnir relations** (see [21]) give a straightening algorithm for expressing any F_T in terms of the standard Young tableau basis, and thereby gives a rule for computing with the S_n-action on the Specht module.

In [5], Ariki, Terasoma, and Yamada noted that F_T may also be defined (up to a constant) as a Young idempotent operator applied to a monomial. In particular, define

$$\varepsilon_T = \sum_{\tau \in C(T)} \sum_{\sigma \in R(T)} \mathrm{sgn}(\tau) \tau \sigma$$

where $C(T) \subseteq S_n$ is the group of *column permutations* that preserve the columns of T, and $R(T) \subseteq S_n$ is the group of *row permutations*. Then it is not hard to check that F_T is a scalar multiple of $\varepsilon_T(x_T^r)$ where $x_T^r = \prod x_i^{\mathrm{row}(i)-1}$ with $\mathrm{row}(i)$ denoting the row that i occurs in in T, indexed from bottom to top.

Example 1.9. We have

$$\varepsilon_{\begin{smallmatrix} 3 & \\ 1 & 2 \end{smallmatrix}} = \mathrm{id} + (12) - (13) - (123)$$

and $F_{\begin{smallmatrix} 3 & \\ 1 & 2 \end{smallmatrix}} = x_3 - x_1$.

$$T = \begin{array}{|c|c|c|c|} \hline 3 \\ \cline{1-1} 7 & 1 & 6 \\ \cline{1-3} 2 & 8 & 4 & 5 \\ \hline \end{array} \qquad F_T = (x_3 - x_7)(x_7 - x_2)(x_3 - x_2) \cdot (x_1 - x_8) \cdot (x_6 - x_4)$$

Figure 2: A tableau $T \in \mathrm{Tab}(4, 3, 1)$ in French notation, and the Specht polynomial F_T.

Ariki, Terasoma, and Yamada then generalized this construction to define

$$F_T^S(\mathbf{x}) = \varepsilon_T(x_T^S) \tag{1.2}$$

where T, S are a pair of standard Young tableaux of the same shape, and x_T^S is defined as follows. Let $\mathrm{ccTab}(S)$ be the tableau consisting of the cocharge subscripts of the reading word of S, written in the corresponding squares of the diagram of λ. Define

$$x_T^S = \prod_c x_{T(c)}^{\mathrm{ccTab}(S)(c)}$$

where the product is over all squares c in the diagram of λ.

Theorem 1.10 ([5]). *The polynomials $F_T^S(\mathbf{x})$ form a higher Specht basis for the one-variable coinvariant ring $R_n = \mathbb{C}[x_1, \dots, x_n]/(e_1, \dots, e_n)$.*

2 General constructions in two sets of variables

For a tableau $T \in \mathrm{Tab}(\lambda)$, fix an ordering on the squares of λ and write $T_1, T_2, \dots, T_n$ to denote the values of T in each of those squares. Also, for any tuples $c = (c_1, \dots, c_n)$ and $d = (d_1, \dots, d_n)$ of nonnegative integers, we write

$$x_T^c = x_{T_1}^{c_1} x_{T_2}^{c_2} \cdots x_{T_n}^{c_n} \qquad \text{and} \qquad y_T^d = y_{T_1}^{d_1} y_{T_2}^{d_2} \cdots y_{T_n}^{d_n}.$$

Definition 2.1. We write

$$F_T^{c,d} = \varepsilon_T(x_T^c y_T^d).$$

Remark 2.2. All proofs in this section hold for any number of variables; for instance, we could have three sets of variables $\mathbf{x}, \mathbf{y}, \mathbf{z}$ and three exponent sequences c, d, e and define $F_T^{c,d,e} = \varepsilon_T(\mathbf{x}^c \mathbf{y}^d \mathbf{z}^e)$ and we would obtain analogous results.

We will first show the general statement that, assuming the polynomials $F_T^{c,d}$ are independent for T standard of shape λ, the submodule $V^{c,d} \subseteq \mathbb{C}[\mathbf{x}, \mathbf{y}]$ spanned by these polynomials is a copy of the irreducible S_n-module V_λ. We first recall (see, for instance, [20, 21]) that the Garnir relations govern the S_n-module structure of V_λ with respect to the standard Specht basis.

Our first preliminary result is that Lemma 3.3 in [20] (or equivalently [11, Lemma 3.13]) generalizes to this setting (see [12] for the proof):

Proposition 2.3. *We have $\pi F_T^{c,d} = F_{\pi T}^{c,d}$ for any $\pi \in S_n$, and the $F_T^{c,d}$ elements satisfy the Garnir relations.*

It follows that the span M_λ of the polynomials $F_T^{c,d}$ is an S_n-module. The Garnir straightening algorithm (see [21]) along with Proposition 2.3 then implies the following.

Proposition 2.4. *The subspace $M_\lambda = \mathrm{span}(F_T^{c,d} : T \in \mathrm{Tab}(\lambda)) \subseteq \mathbb{C}[\mathbf{x},\mathbf{y}]$ is generated by the subset $\mathbb{B} = \{F_T^{c,d} : T \in \mathrm{SYT}(\lambda)\}$. Moreover, if the polynomials in $\mathbb{B}$ are independent, then $M_\lambda \cong V_\lambda$ as an S_n-module, and there is an action-preserving isomorphism induced by the bijection $F_T^{c,d} \to F_T$ that sends the basis $\mathbb{B}$ to the ordinary Specht basis.*

2.1 Conditions for independence

Proposition 2.4 shows that if the basis elements are independent, then M_λ is a copy of the irreducible Specht module. We now state some conditions for determining independence in quotients by S_n-invariant ideals or in the full polynomial ring (see [12] for proofs).

Proposition 2.5. *The polynomials in $\mathbb{B} = \{F_T^{c,d} : T \in \mathrm{SYT}(\lambda)\}$ are independent in $\mathbb{C}[\mathbf{x},\mathbf{y}]/I$ if and only if some particular element $F_T^{c,d}$ is nonzero in the quotient (i.e., not in the ideal I).*

We now state a sufficient condition for one of the $F_T^{c,d}$ elements to be nonzero in the full polynomial ring $\mathbb{C}[\mathbf{x},\mathbf{y}]$ that we will use below.

Proposition 2.6. *Suppose that $c = (c_1, \ldots, c_n)$ and $d = (d_1, \ldots, d_n)$ are tuples of nonnegative integers, with their ordering corresponding to a chosen ordering of the boxes of the Young diagram of a partition λ of n. Suppose further that there exists a total ordering on the possible pairs of values (m,n) that arise among the pairs c_i, d_i such that, when the boxes of λ are filled in order with (c_i, d_i), it forms a semistandard Young tableau with respect to the ordering on the pairs.*

Then for any standard tableau T of shape λ, the polynomial $F_T^{c,d}$ is nonzero in $\mathbb{C}[x,y]$.

Example 2.7. Suppose we choose the ordering on pairs (m,n) by $(m,n) > (x,y)$ if and only if either $m > x$ or $m = x$ and $n > y$. Consider the following tableau below at left, where pairs (m,n) are written as mn:

22	22			
21	21	22	22	
01	10	10	13	
00	00	00	01	10

12	13			
10	11	14	15	
4	6	7	9	
1	2	3	5	8

Also, for simplicity let T be the tableau above at right that standardizes the above (the choice of T does not matter in the proof). Then $x_T^c y_T^d = x_{15}^2 y_{15}^2 \cdot x_{14}^2 y_{14}^2 \cdot x_{13}^2 y_{13}^2 \cdot x_{12}^2 y_{12}^2 \cdot x_{11}^2 y_{11} \cdots$. In the proof [12], this leading term does not cancel after applying ε_T.

3 New basis for hook-shape Garsia–Haiman modules

Throughout this subsection, we set $\mu = (n-k+1, 1^{k-1})$ to be the *hook shape* of height k and size n. We now define our higher Specht polynomials for DR_μ.

Definition 3.1. For $S \in \mathrm{SYT}(n)$, define the **μ-cocharge tableau** of S, written $\mathrm{ccTab}_\mu(S)$, to be the tableau of shape $\mathrm{sh}(S)$ that has 0 in the squares occupied by $1, 2, \ldots, n-k+1$ in S, and where for $n-k+1, n-k+2, \ldots, n$ we apply the cocharge algorithm in reverse reading order, incrementing the cocharge label whenever we reach an entry $i+1$ that is above i. The **μ-cocharge** of S, written $\mathrm{cc}_\mu(S)$, is the sum of the entries of $\mathrm{ccTab}_\mu(S)$.

Definition 3.2. For $S \in \mathrm{SYT}(n)$, define the **reverse μ-cocharge tableau**, which we write $\mathrm{ccTab}'_\mu(S)$, to be the tableau of shape $\mathrm{sh}(S)$ that has 0 in the squares occupied by $n-k+1, \ldots, n$ in S, and where for $n-k+1, n-k, n-k-1, \ldots, 1$ we calculate cocharge in *reverse*, that is, in forward reading order and labeling the numbers from biggest to smallest, incrementing the label on i when it is below $i+1$. The **μ-reverse cocharge**, written $\mathrm{cc}'_\mu(S)$, is the sum of the entries of the reverse $\mathrm{ccTab}'_\mu(S)$.

Example 3.3. Suppose $n = 8$, S is the tableau below at left, and $k = 5$. Then $\mathrm{ccTab}_\mu(S)$ is the tableau below at middle and $\mathrm{ccTab}'_\mu(S)$ is at right:

$$\begin{array}{|c|c|c}\hline 5 & 7 & \\ \hline 3 & 6 & \multicolumn{1}{c|}{8} \\ \hline 1 & 2 & \multicolumn{1}{c|}{4} \\ \hline \end{array} \qquad \begin{array}{|c|c|c}\hline 1 & 2 & \\ \hline 0 & 1 & \multicolumn{1}{c|}{2} \\ \hline 0 & 0 & \multicolumn{1}{c|}{0} \\ \hline \end{array} \qquad \begin{array}{|c|c|c}\hline 0 & 0 & \\ \hline 0 & 0 & \multicolumn{1}{c|}{0} \\ \hline 1 & 1 & \multicolumn{1}{c|}{0} \\ \hline \end{array}$$

Thus we have $\mathrm{cc}'_\mu(S) = 1 + 1 = 2$ and $\mathrm{cc}_\mu(S) = 1 + 1 + 2 + 2 = 6$.

In [12], we show that these statistics agree with maj and comaj from Equation (1.1):

Lemma 3.4. *We have that* $\mathrm{cc}_\mu(S) = \mathrm{comaj}_{n-k+1,n}(S)$ *and* $\mathrm{cc}'_\mu(S) = \mathrm{maj}_{1,n-k+1}(S)$ *for any standard Young tableau S.*

Definition 3.5. Let $(T, S) \in \mathrm{Tab}(\lambda) \times \mathrm{SYT}(\lambda)$. The μ-**monomial** of (T, S) is

$$\mathbf{xy}_T^S := \prod_{b \in \lambda} x_{T(b)}^{\mathrm{ccTab}_\mu(S)(b)} y_{T(b)}^{\mathrm{ccTab}'_\mu(S)(b)}.$$

Then we define $F_T^S(x, y) = \varepsilon_T \mathbf{xy}_T^S$.

Example 3.6. Keeping S as in Example 3.3 and $k = 5$, we set

$$T = \begin{array}{|c|c|c}\hline 6 & 8 & \\ \hline 2 & 4 & \multicolumn{1}{c|}{7} \\ \hline 1 & 3 & \multicolumn{1}{c|}{5} \\ \hline \end{array} \qquad \text{and} \qquad S = \begin{array}{|c|c|c}\hline 5 & 7 & \\ \hline 3 & 6 & \multicolumn{1}{c|}{8} \\ \hline 1 & 2 & \multicolumn{1}{c|}{4} \\ \hline \end{array}$$

and we find $\mathbf{xy}_T^S = x_4 x_6 x_7^2 x_8^2 \cdot y_1 y_3$.

In [12], we show that the values of $c := \mathrm{ccTab}_\mu(S)$ and $d := \mathrm{ccTab}'_\mu(S)$ form an SSYT according to a particular ordering on the pairs of values, so that it satisfies the conditions of Proposition 2.6. Then, we obtain the following by Proposition 2.4.

Proposition 3.7. *The polynomials $F_T^S(\mathbf{x},\mathbf{y})$ for $T \in \mathrm{SYT}(\lambda)$ span a copy of V_λ in $\mathbb{C}[\mathbf{x},\mathbf{y}]$.*

We now note that we can recover the one-variable higher Specht polynomials from our construction by a substitution and a degree shift.

Lemma 3.8. *Consider the map to the Laurent polynomial ring (also defined in [1])*

$$\psi : \mathbb{C}[x_1,\ldots,x_n,y_1,\ldots,y_n] \to \mathbb{C}[x_1,\ldots,x_n,x_1^{-1},\ldots,x_n^{-1}]$$

that sends $y_i \mapsto x_i^{-1}$. Then if q is the largest entry in $\mathrm{ccTab}'_\mu(S)$, we have

$$\psi(F_T^S(\mathbf{x},\mathbf{y})) \cdot x_1^q x_2^q \cdots x_n^q = F_T^S(\mathbf{x})$$

where $F_T^S(\mathbf{x})$ is the one-variable higher Specht polynomial defined in [5] (see Equation (1.2)).

For notational brevity, we define M_λ^S to be the copy of V_λ inside $\mathbb{C}[\mathbf{x},\mathbf{y}]$ generated by the polynomials F_T^S over all $T \in \mathrm{SYT}(\mathrm{sh}(S))$. We can use Lemma 3.8 and the results of [5] to show these copies are distinct for different S.

Lemma 3.9. *If S_1 and S_2 are two distinct standard Young tableaux of size n and shapes λ and ρ respectively, then $M_\lambda^{S_1}$ and $M_\rho^{S_2}$ are disjoint submodules of $\mathbb{C}[\mathbf{x},\mathbf{y}]$.*

We now recall a generating set for the ideal I_μ defining DR_μ in the hook shape case, and a sub-ideal defined in [1].

Proposition 3.10 ([1]). *If $\mu = (n-k+1, 1^{k-1})$, the ideal I_μ is generated by:*

1. *The elementary symmetric functions $e_1(\mathbf{x}),\ldots,e_n(\mathbf{x})$,*
2. *The elementary symmetric functions $e_1(\mathbf{y}),\ldots,e_n(\mathbf{y})$,*
3. *All products $x_{i_1}\cdots x_{i_k}$ of k distinct x variables,*
4. *All products $y_{j_1}\cdots y_{j_{n-k+1}}$ of $n-k+1$ distinct y variables.*
5. *The products $x_i y_i$ for $i = 1,\ldots,n$.*

Definition 3.11 ([1]). Define $\mathcal{I}_k$ to be the ideal generated by the last three bullet points in the definition above, that is, just the products $x_{i_1}\cdots x_{i_k}$, $y_{j_1}\cdots y_{j_{n-k+1}}$, and $x_i y_i$. Define

$$\mathcal{P}_n^{(k)} = \mathbb{C}[\mathbf{x},\mathbf{y}]/\mathcal{I}_k.$$

It is not hard to see that the elements $F_T^S(\mathbf{x},\mathbf{y})$ are nonzero in $\mathcal{P}_n^{(k)}$. To show they descend to a basis of DR_μ, we recall the S_n-invariant basis $e_\nu^{(k)}$ of $(\mathcal{P}_n^{(k)})^{S_n}$ from [1].

Definition 3.12. For $d = 1, \ldots, n$, define

$$e_d^{(k)} = \begin{cases} e_d(x_1, \ldots, x_n) & d \leq k-1 \\ e_{n-d}(y_1, \ldots, y_n) & d \geq k \end{cases}$$

and $e_\nu^{(k)} = \prod_i e_{\nu_i}^{(k)}$.

As a stepping stone to proving our main result, we show the following [12].

Proposition 3.13. *The set of polynomials $\{F_T^S(\mathbf{x}, \mathbf{y}) \cdot e_\nu^{(k)}\}$, ranging over all pairs (T, S) of SYT's of the same shape and all (possibly empty) ν with $\nu_1 \leq n$, forms a higher Specht basis of $\mathcal{P}_n^{(k)}$.*

We finally conclude our main result.

Theorem 1.5. The set $\{F_T^S(\mathbf{x}, \mathbf{y})\}$ is a higher Specht basis for DR_μ.

4 Future directions and observations

One corollary of Theorem 1.5 is that we can express the graded Frobenius series for hook shapes $\mu = (n - k + 1, 1^{k-1})$ as

$$\mathrm{grFrob}_{q,t}(\mathrm{DR}_\mu) = \sum_{S \in \mathrm{SYT}(n)} q^{\mathrm{cc}_\mu(S)} t^{\mathrm{cc}'_\mu(S)} s_{\mathrm{sh}(T)}.$$

It is a more general open problem than Problem 1.2 to find a combinatorial Schur expansion for the Frobenius series of DR_μ, and one possible route would be to generalize these new cc_μ and cc'_μ statistics to more general shapes μ.

For the diagonal coinvariant ring (Problem 1.3), it would be interesting to see if a higher Specht basis could be constructed by combining the general tools from Section 2 with the new basis of Carlsson and Oblomkov [7], or by interpreting parking functions as certain pairs of tableaux.

To conclude, we provide higher Specht bases for DR_2 and DR_3. The Shuffle theorem (conjectured in [13], proven in [6]) gives a combinatorial formula for the graded Frobenius series of DR_n, which is also equal to a Macdonald eigenoperator ∇ applied to the elementary symmetric function e_n. We used Sage [28] to expand ∇e_2 and ∇e_3 in terms of Schur functions, giving us the decompositions of DR_2 and DR_3 into irreducibles, and used that data to guess and verify higher Specht bases.

Example 4.1. For DR_2, a higher Specht basis is: $1, x_2 - x_1, y_2 - y_1$.

Example 4.2. For DR_3, its graded Frobenius series from the Shuffle theorem is

$$\nabla e_3 = s_{(3)} + (q+t)s_{(2,1)} + (q^2 + qt + t^2)s_{(2,1)} + qts_{(1,1,1)} + (q^3 + t^3 + q^2t + t^2q)s_{(1,1,1)}.$$

The following set of polynomials, corresponding to each term in this expansion left to right, form a higher Specht basis for DR_3:

$$1, \quad x_2 - x_1,\, x_3 - x_1, \quad y_2 - y_1,\, y_3 - y_1, \quad \varepsilon_{\begin{smallmatrix}3\\1\,2\end{smallmatrix}} x_3x_2,\, \varepsilon_{\begin{smallmatrix}2\\1\,3\end{smallmatrix}} x_2x_3, \quad \varepsilon_{\begin{smallmatrix}3\\1\,2\end{smallmatrix}} x_3y_2,\, \varepsilon_{\begin{smallmatrix}2\\1\,3\end{smallmatrix}} x_2y_3$$

$$\varepsilon_{\begin{smallmatrix}3\\1\,2\end{smallmatrix}} y_3y_2,\, \varepsilon_{\begin{smallmatrix}2\\1\,3\end{smallmatrix}} y_2y_3, \quad \varepsilon_{\begin{smallmatrix}3\\2\\1\end{smallmatrix}} x_3y_1, \quad (x_3 - x_2)(x_3 - x_1)(x_2 - x_1),$$

$$(y_3 - y_2)(y_3 - y_1)(y_2 - y_1), \quad \varepsilon_{\begin{smallmatrix}3\\2\\1\end{smallmatrix}} x_3^2y_1, \quad \varepsilon_{\begin{smallmatrix}3\\2\\1\end{smallmatrix}} x_3y_1^2$$

Acknowledgements

The author thanks Kelvin Chan, Bryan Gillespie, Sean Griffin, Jake Levinson, and Brendon Rhoades for helpful mathematical conversations pertaining to this project. Special thanks to Jake Levinson for his support and encouragement that made it possible.

References

[1] R. M. Adin, J. B. Remmel, and Y. Roichman. "The Combinatorics of the Garsia-Haiman Modules for Hook Shapes". *Electron. J. Comb.* **15**.1 (2007). DOI.

[2] E. E. Allen. "The descent monomials and a basis for the diagonally symmetric polynomials". *J. Algebraic Combin.* **3**.1 (1994), pp. 5–16. DOI.

[3] E. E. Allen. "A conjecture of Procesi and the straightening algorithm of Rota". *Proc. Nat. Acad. Sci. U.S.A.* **89**.9 (1992), pp. 3980–3984. DOI.

[4] E. E. Allen. "Bitableaux bases for the diagonally invariant polynomial quotient rings". *Adv. Math.* **130**.2 (1997), pp. 242–260. DOI.

[5] S. Ariki, T. Terasoma, and H.-F. Yamada. "Higher Specht polynomials". *Hiroshima Math. J.* **27**.1 (1997), pp. 177–188. Link.

[6] E. Carlsson and A. Mellit. "A proof of the shuffle conjecture". *J. Amer. Math. Soc.* **31**.3 (2018), pp. 661–697. DOI.

[7] E. Carlsson and A. Oblomkov. "Affine Schubert calculus and double coinvariants". 2018. arXiv:1801.09033.

[8] C. De Concini and C. Procesi. "Symmetric functions, conjugacy classes and the flag variety". *Invent. Math.* **64**.2 (1981), pp. 203–219. DOI.

[9] W. Fulton. *Young tableaux*. Vol. 35. London Mathematical Society Student Texts. With applications to representation theory and geometry. Cambridge University Press, Cambridge, 1997, pp. x+260.

[10] A. M. Garsia and C. Procesi. "On certain graded S_n-modules and the q-Kostka polynomials". *Adv. Math.* **94**.1 (1992), pp. 82–138. DOI.

[11] M. Gillespie and B. Rhoades. "Higher Specht bases for generalizations of the coinvariant ring". *Ann. Comb.* **25**.1 (2021), pp. 51–77. DOI.

[12] M. Gillespie. "Higher Specht polynomials under the diagonal action". *Algebr. Comb.* **7**.6 (2024), pp. 1731–1750. DOI.

[13] J. Haglund, M. Haiman, N. Loehr, J. B. Remmel, and A. Ulyanov. "A combinatorial formula for the character of the diagonal coinvariants". *Duke Math. J.* **126**.2 (2005), pp. 195–232. DOI.

[14] J. Haglund, B. Rhoades, and M. Shimozono. "Ordered set partitions, generalized coinvariant algebras, and the Delta Conjecture". *Adv. Math.* **329** (2018), pp. 851–915. DOI.

[15] M. Haiman. "Hilbert schemes, polygraphs and the Macdonald positivity conjecture". *J. Amer. Math. Soc.* **14**.4 (2001), pp. 941–1006. DOI.

[16] M. Haiman. "Vanishing theorems and character formulas for the Hilbert scheme of points in the plane". *Invent. Math.* **149**.2 (2002), pp. 371–407. DOI.

[17] T. Hikita. "A proof of the Stanley–Stembridge conjecture". 2024. arXiv:2410.12758.

[18] A. Lascoux and M.-P. Schützenberger. "Sur une conjecture de H. O. Foulkes." *C. R. Acad. Sci. Paris Sér. I Math.* **288** (1979), 95–98.

[19] I. G. Macdonald. *Symmetric functions and Hall polynomials*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1979, pp. viii+180.

[20] M. H. Peel. "Specht modules and symmetric groups". *J. Algebra* **36**.1 (1975), pp. 88–97. DOI.

[21] B. E. Sagan. *The symmetric group*. Second. Vol. 203. Graduate Texts in Mathematics. Representations, combinatorial algorithms, and symmetric functions. Springer-Verlag, New York, 2001, pp. xvi+238. DOI.

[22] K. Salois. "Higher Specht bases and q-series for the cohomology rings of certain Hessenberg varieties". 2024. arXiv:2410.08366.

[23] J. Shareshian and M. L. Wachs. "Chromatic quasisymmetric functions". *Adv. Math.* **295** (2016), pp. 497–551. DOI.

[24] R. P. Stanley. "Invariants of finite groups and their applications to combinatorics". *Bull. Amer. Math. Soc. (N.S.)* **1**.3 (1979), pp. 475–511. DOI.

[25] R. P. Stanley and J. R. Stembridge. "On immanants of Jacobi-Trudi matrices and permutations with restricted position". *J. Combin. Theory Ser. A* **62**.2 (1993), pp. 261–279. DOI.

[26] J. R. Stembridge. "Some particular entries of the two-parameter Kostka matrix". *Proc. Amer. Math. Soc.* **121**.2 (1994), pp. 367–373. DOI.

[27] T. Tanisaki. "Defining ideals of the closures of the conjugacy classes and representations of the Weyl groups". *Tohoku Math. J. (2)* **34**.4 (1982), pp. 575–585. DOI.

[28] The Sage Developers. *SageMath (Version 9.0)*. `https://www.sagemath.org`. 2020.