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Orthogonal roots, Macdonald representations, and quasiparabolic W-sets

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Abstract. Let W be a finite Weyl group with root system Φ and of rank $n > 1$. We study the maximal sets of orthogonal positive roots of Φ with cardinality n, which exist if and only if W has type E_7, E_8, or D_n for n even. We show that in these types, the set X of all such maximal orthogonal sets forms a quasiparabolic W-set in the sense of Rains–Vazirani. The quasiparabolic structure can be described in terms of certain quadruples of orthogonal roots that we call crossings, nestings, and alignments. This leads to noncrossing and nonnesting bases of a suitable irreducible representation of W known as a Macdonald representation, as well as some highly structured partially ordered sets, including the strong Bruhat poset of symmetric groups. In type E_8, we use the set X to give a concise description of a graph that is known to be non-isomorphic but quantum isomorphic to the orthogonality graph of the E_8 root system.

Keywords: root system, Macdonald representation, canonical basis, quasiparabolic W-set, Specht module

1 Introduction

Root systems and Weyl groups are both fundamental objects in algebraic combinatorics. In 1972, Macdonald [9] described a simple yet powerful method to construct an irreducible $\mathbb{Q}$-representation $j_\Psi^\Phi(\text{sgn})$ of a finite Weyl group W from the root system Φ of W and any subsystems Ψ of Φ. This paper studies such representations arising from a proper subsystem Ψ of Φ of type nA_1 where n is the rank of Φ, whence Ψ is of the form $\{\pm\alpha_1\} \cup \{\pm\alpha_2\} \cup \cdots \cup \{\pm\alpha_n\}$ for pairwise orthogonal roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ in Φ. Such proper subsystems exist precisely when W has type E_7, E_8 or D_n for n even (Figure 1), and in these cases the corresponding Macdonald representation, which we henceforth denote by $j_{nA_1}^\Phi(\text{sgn})$ as it does not depend on the choice of Ψ, is spanned by what we call *positive n-roots*. As we explain in Section 2.2, the positive n-roots can be identified in a natural and precise way with the subsets of Φ consisting of n pairwise orthogonal positive roots.

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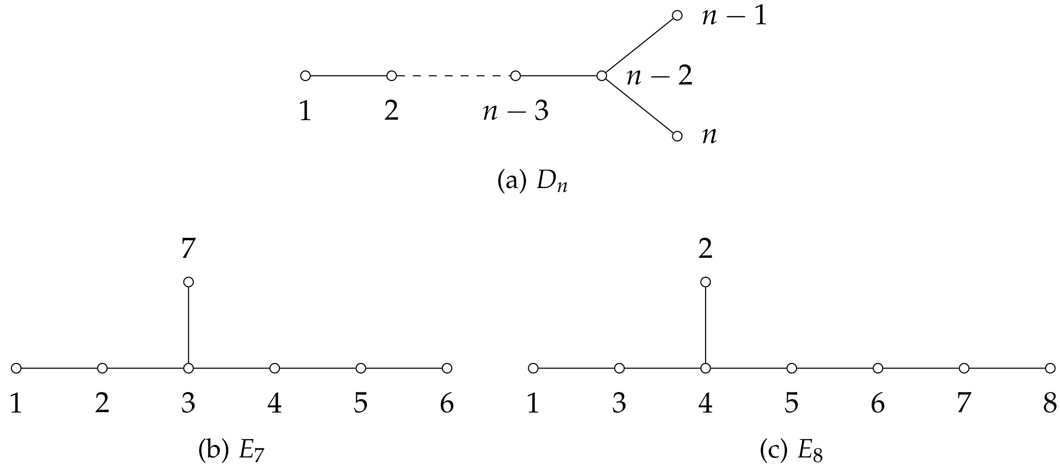


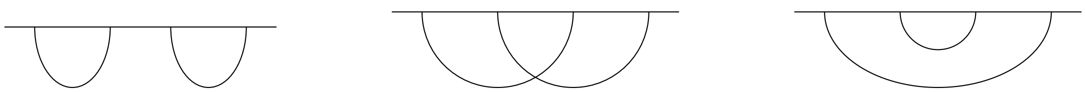
Figure 1: Dynkin diagrams for root systems of type D_n, E_7 and E_8.

The goal of this paper is to explore the rich combinatorial properties of positive n-roots and use them to elucidate the structure of the representation $j_{nA_1}^{\Phi}(\mathrm{sgn})$. Our results have natural connections to many previous works. In type D_n for $n = 2k$ even, the positive n-roots can be identified with the perfect matchings of the set $[n] = \{1, 2, \ldots, n\}$ and with the fixed-point-free involutions of the symmetric group S_n. The Macdonald representation $j_{nA_1}^{\Phi}(\mathrm{sgn})$ in this case is a lift of the Specht module $S^{(k,k)}$ of S_n indexed by the two-row partition $(k, k) \vdash n$ studied recently in [12, 13, 8], and our results have implications for $S^{(k,k)}$ (Remark 4.2). In type E_7, $j_{nA_1}^{\Phi}(\mathrm{sgn})$ is known as the Coble representation [4, Proposition 4.12], which has a long history going back to the work of Coble in 1916 on the Göpel variety [3, (65)]. In type E_8, we use 8-roots to give a convenient construction of a certain strongly regular graph recently studied by Schmidt [14] in connection to quantum isomorphisms of graphs in the sense of Atserias et al [1] (Proposition 4.3.3).

Let W be a Weyl group of type E_7, E_8, or D_n for n even, and let Φ be the root system of W. The positive n-roots of W can be identified with the subsets of Φ consisting of n pairwise orthogonal positive roots, and the collection X of all such subsets admits a W-action induced by the action of W on Φ. The set X turns out to have a rich structure as a W-set: Theorem 3.5, our first main theorem, shows that X is a quasiparabolic W-set in the sense of Rains–Vazirani [11]. A quasiparabolic W-set for a Weyl group W is a W-set equipped with an integer-valued level function satisfying certain axioms that specify how the action of a reflection changes the level (Definition 3.1). These axioms generalize properties satisfied by the natural action of W on any quotient W/W_I of W by a parabolic subgroup W_I, and they enable the deformation of the W-action on X to create a module for the Iwahori–Hecke algebra of W [11, Section 7]. In our setting, the W-action on X in type D_n for n even induces an action of S_n on X, and Theorem 3.5 implies that the fixed-point-free involutions in S_n form a quasiparabolic S_n-set, which is the one of the original motivating examples for the definition of quasiparabolic W-sets

[11, Section 4].

To understand the quasiparabolic structure of X, it is useful to consider certain special quadruples of orthogonal positive roots that we call *alignments*, *crossings*, and *nestings*. These terms are motivated by the perfect matching diagrams corresponding to the positive 4-roots in type D_4 (Figure 2), but they are defined in a type-free way and prove to be useful for understanding X even in types E_7 and E_8, where no diagrammatic interpretation of n-roots seems to be available. The level function associated to the quasiparabolic set X can be obtained by counting crossings and nestings, and levels of n-roots in type D_n have interesting connections to combinatorial game distributions of Steiner systems, Gaussian q-distributions, Laguerre polynomials, and q-Bessel numbers; see [6, Section 6.1].



(a) γ_A (b) γ_C (c) γ_N

Figure 2: Perfect matchings corresponding to the positive 4-roots in type D_4, which include exactly one alignment, γ_A, one crossing, γ_C, and one nesting, γ_N. These 4-roots satisfy the three-term relation $\gamma_C = \gamma_N + \gamma_A$, which can be interpreted as a skein relation among the matchings.

We define a positive n-root to be *noncrossing*, *nonnesting*, or *alignment-free* if its set of components contains no crossing, nesting, or alignment as a subset, respectively. These "feature-avoiding" elements have many remarkable properties, and we summarize some of them in Theorem 4.1, our second main result. In particular, we prove that the noncrossing elements of X form a canonical basis of $j^{\Phi}_{nA_1}(\mathrm{sgn})$ that behaves in $j^{\Phi}_{nA_1}(\mathrm{sgn})$ somewhat like a simple system in the reflection representation of W. The set of nonnesting elements not only forms another basis of $j^{\Phi}_{nA_1}(\mathrm{sgn})$ dual to the noncrossing basis, but also has the structure of a distributive lattice induced by the weak Bruhat order on W. The alignment-free elements form a quasiparabolic W_I-set, X_I, for a suitable parabolic subgroup W_I of W. Moreover, the three types of feature-avoiding n-roots also fit together in satisfying ways. We define a equivalence relation on X, called *σ-equivalence*, whose equivalence classes induce a canonical bijection between the noncrossing and nonnesting elements of X. Any set of σ-equivalence class representatives forms a $\mathbb{Q}$-basis for $j^{\Phi}_{nA_1}(\mathrm{sgn})$, and any two such bases, which include the nonnesting and noncrossing bases of $j^{\Phi}_{nA_1}(\mathrm{sgn})$, admit a unitriangular change of basis matrix with integer entries.

The rest of this abstract is organized as follows. Section 2 reviews preliminary materials leading to the notion of positive n-roots. Section 3 introduces crossings, nestings, and alignments, and then explains how the positive n-roots form a quasiparabolic W-set. Section 4 discusses feature-avoiding n-roots, including both type-independent results and some type-specific properties of the alignment-free positive n-roots. This abstract is based on the full paper [6], where all proofs omitted in the abstract can be found.

2 Preliminaries

We recall the construction of Macdonald representations and explain their connections to positive n-roots in this section. Throughout the abstract, we work over the rational field $\mathbb{Q}$ and assume basic familiarity with root systems. For notation and terminology related to root systems, we follow [7, Chapter 1].

2.1 Macdonald representations

Let W be a finite Weyl group with a root system Φ of rank n. Let Γ be the Dynkin diagram of Φ shown in Figure 1, and let $\Delta = \{\alpha_i : i \in \Gamma\}$ be a simple system of Φ. Let V be the reflection representation of W defined over $\mathbb{Q}$. We recall that Δ is a basis of V, and that V admits a positive-definite bilinear form B defined by

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{if } \ i = j; \\ -1 & \text{if } \ i \text{ and } j \text{ are adjacent in } \Gamma; \\ 0 & \text{otherwise.} \end{cases}$$

Each root $\alpha \in \Phi$ gives rise to a reflection, s_α, which acts on Φ by the formula

$$s_\alpha(\beta) = \beta - B(\alpha, \beta)\alpha$$

and extends linearly to an action on V. For each simple root α_i, we write s_i for s_{α_i}. The bilinear form B is W-invariant in the sense that $B(\alpha, \beta) = B(w.\alpha, w.\beta)$ for all $w \in W$ and all $\alpha, \beta \in \Phi$. We say two roots are $\alpha, \beta \in \Phi$ are *orthogonal* if $B(\alpha, \beta) = 0$.

Let Ψ be a *subsystem* of Φ, i.e., a nonempty subset of Ψ which is itself a root system. Let Φ^+ be the set of positive roots in Φ, and let $\Psi^+ = \Psi \cap \Phi^+$. The positive-definite form B allows us to identify V with its dual, V^*, and the symmetric algebra $\mathrm{Sym}(V^*)$ is a W-module under the contragredient action $(w \cdot \phi)(x) = \phi(w^{-1}(x))$. The *Macdonald representation of W associated to* Ψ, denoted $j_\Psi^\Phi(\mathrm{sgn})$, is the cyclic W-submodule of $\mathrm{Sym}(V^*)$ generated by the element $\pi_\Psi = \prod_{\alpha \in \Psi^+} \alpha$. The representation $j_\Psi^\Phi(\mathrm{sgn})$ is irreducible, and all irreducible representations of W can be constructed as a Macdonald representation for a suitable choice of Ψ in types A_n and B_n [9].

2.2 Positive n-roots

When W has type E_7, E_8 or D_n for n even, the root system Φ contains proper subsystems Ψ of type nA_1, and we denote the corresponding Macdonald representation $j_\Psi^\Phi(\mathrm{sgn})$ as

$j_{nA_1}^{\Phi}(\text{sgn})$. The set Ψ^+ consists of n orthogonal positive roots, and we call each element of the form $w.\pi_\Psi \in j_{nA_1}^{\Phi}(\text{sgn})$ an *n-root*. Since the bilinear form B is W-invariant, every n-root has the form $\alpha = \prod_{i=1}^{n} \beta_i$ where the elements β_i are orthogonal roots. Conversely, every product of n orthogonal roots is an n-root, because W acts transitively on the set of maximal sets of orthogonal roots ([6, Lemma 3.1.2]).

We say an n-root α is *positive* if it can be written in the form $\alpha = \prod_{i=1}^{n} \beta_i$ where all the factors β_i are positive, and we call the roots β_i the *components* of α in this case. The components are well defined because they are the irreducible factors of α in the unique factorization domain $\mathbb{Q}[\alpha_1, \alpha_2, \ldots, \alpha_n]$. It follows from the definitions that if α is an n-root then $-\alpha$ is also an n-root, and that precisely one of α and $-\alpha$ is positive. Whenever α is either a root or an n-root, we define the *absolute value* of α to be the positive element in the pair $\{\alpha, -\alpha\}$ and denote it by $|\alpha|$.

The set Φ_n^+ of all positive n-roots admits a natural W-action given by $w(\alpha) = |w.\alpha|$. Similarly, the set X of sets of n orthogonal positive roots admits a W-action given by $w(\{\beta_1, \cdots, \beta_n\}) = \{|w(\beta_1)|, \cdots, |w(\beta_n)|\}$. The map sending each set $\{\beta_1, \cdots, \beta_n\} \in X$ to the product $\prod_{i=1}^{n} \beta_i \in \Phi_n^+$ respects these two W-actions, and we will henceforth use it to identify X with Φ_n^+. This identification provides a convenient passage between the set X and the representation $j_{nA_1}^{\Phi}(\text{sgn})$.

2.3 Root systems of types A_n and D_n

We now review some well-known facts about root systems of types A_n and D_n, including explicit constructions of the root systems. Details about similar constructions for types E_7 and E_8 are also relevant to our treatment of n-roots, and they are described in [6, Section 2.3], but we omit them here because they will not be directly used in this abstract.

Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be an orthonormal basis for $\mathbb{R}^n$. The vectors $\{\varepsilon_i - \varepsilon_j : 1 \leq i \neq j \leq n\}$ form a root system of type A_{n-1}. The simple roots $\{\alpha_1, \alpha_2, \ldots, \alpha_{n-1}\}$ are given by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. The bilinear form B is the usual dot product. A root $\varepsilon_i - \varepsilon_j$ is positive if $i < j$ and negative if $i > j$. The Weyl group is isomorphic to the symmetric group S_n and acts on Φ by permuting the basis $\varepsilon_1, .., \varepsilon_n$.

The vectors $\{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq n\}$ form a root system of type D_n. The simple roots $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ are given by $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i < n$, and $\alpha_n = \varepsilon_i + \varepsilon_{i+1}$. If $i < j$, then the roots $\varepsilon_i \pm \varepsilon_j$ are positive, and the roots $-\varepsilon_i \pm \varepsilon_j$ are negative. The bilinear form B is the usual dot product. The Weyl group acts on Φ by signed permutations of the orthonormal basis, with the restriction that each element effects an even number of sign changes [7, Section 2.10]. There is a well-known homomorphism ϕ from W to S_n resulting from forgetting the signs in a signed permutation, given as follows:

$$\phi(s_i) = \begin{cases} (i, i+1) & \text{if } i < n \\ (n-1, n) & \text{otherwise.} \end{cases} \tag{2.1}$$

2.4 A key example

Suppose that Φ is a root system of type D_n for an even integer $n = 2k \geq 4$. It follows from Section 2.3 that two roots $\alpha, \beta \in \Phi$ are orthogonal if and only if either (a) α and β have disjoint support or (b) α and β have the same support, but $\alpha \neq \pm\beta$. Using this fact, it is straightforward to show that every orthogonal set of positive roots in Φ that is maximal with respect to set containment has the form

$$R = \{\varepsilon_{i_1} - \varepsilon_{j_1},\ \varepsilon_{i_1} + \varepsilon_{j_1},\ \varepsilon_{i_2} - \varepsilon_{j_2},\ \varepsilon_{i_2} + \varepsilon_{j_2},\ \ldots, \varepsilon_{i_k} - \varepsilon_{j_k},\ \varepsilon_{i_k} + \varepsilon_{j_k}\}$$

for numbers $i_1, j_1, \ldots, i_k, j_k$ such that $i_1 < j_1, \ldots, i_k < j_k$, and $\{i_1, j_1, \ldots, i_k, j_k\} = [n]$. Identifying the set R with the perfect matching $\{\{i_1, j_1\},\ \{i_2, j_2\},\ \ldots, \{i_k, j_k\}\}$, we obtain a bijection between the positive n-roots and the perfect matchings of $[n]$. Here, by a perfect matching of $[n]$ we mean a collection of pairwise disjoint size-2 subsets of $[n]$ whose union is the whole of $[n]$.

The case where Φ has type D_4 turns out to be particularly instructive for understanding the general theory of n-roots in all types. In type D_4, there are exactly three n-roots, namely:

$$\gamma_A = (\varepsilon_1 - \varepsilon_2)(\varepsilon_1 + \varepsilon_2)(\varepsilon_3 - \varepsilon_4)(\varepsilon_3 + \varepsilon_4) = (\varepsilon_1^2 - \varepsilon_2^2)(\varepsilon_3^2 - \varepsilon_4^2),$$
$$\gamma_C = (\varepsilon_1 - \varepsilon_3)(\varepsilon_1 + \varepsilon_3)(\varepsilon_2 - \varepsilon_4)(\varepsilon_2 + \varepsilon_4) = (\varepsilon_1^2 - \varepsilon_3^2)(\varepsilon_2^2 - \varepsilon_4^2),$$

and

$$\gamma_N = (\varepsilon_1 - \varepsilon_4)(\varepsilon_1 + \varepsilon_4)(\varepsilon_2 - \varepsilon_3)(\varepsilon_2 + \varepsilon_3) = (\varepsilon_1^2 - \varepsilon_4^2)(\varepsilon_2^2 - \varepsilon_3^2).$$

Viewed as maximal orthogonal sets, these 4-roots form a partition of the 12 positive roots of Φ, and they correspond to the three perfect matchings $\{\{1,2\}\ \{3,4\}\}$, $\{\{1,3\}\ \{2,4\}\}$, and $\{\{1,4\}\ \{2,3\}\}$ depicted in Figure 2, respectively. Furthermore, these 4-roots satisfy the relation

$$\gamma_C = \gamma_N + \gamma_A$$

by a straightforward application of the Ptolemy relation

$$(A - C)(B - D) = (A - D)(B - C) + (A - B)(C - D),$$

and they have the property that for any component α in one of them, the reflection s_α interchanges the other two 4-roots in the Macdonald representation.

The above facts about the 4-roots in type D_4 are important because in all of the types E_7, E_8, and D_n for n even, the action of each reflection in the Weyl group on a positive n-root is always controlled by a suitable type-D_4 subsystem of Φ (Proposition 3.3, 2.). The three-term relation $\gamma_C = \gamma_N + \gamma_A$ provides an important connection between the combinatorial structure of the set X as a W-set and the linear structure of the Macdonald representation $j_{nA_1}^{\Phi}(\mathrm{sgn})$ as a W-module. In particular, the three-term relation can help establish the noncrossing and nonnesting bases of $j_{nA_1}^{\Phi}(\mathrm{sgn})$.

3 Quasiparabolic Structure

In this section, we recall the definition of quasiparabolic W-sets and explain how the positive n-roots form a quasiparabolic W-set in types E_7, E_8, and D_n for n even.

3.1 Quasiparabolic W-sets

Rains and Vazirani introduced quasiparabolic W-sets for a general Coxeter system (W, S) and associated a partial order to them as follows:

Definition 3.1. [11, Definitions 2.1, 2.3, and 5.1] Let W be a Weyl group with generating set S and set of reflections T. A *scaled W-set* is a pair (X, λ), where X is a W-set and $\lambda : X \to \mathbb{Z}$ is a function, called the *level* function, such that $|\lambda(sx) - \lambda(x)| \leq 1$ for all $s \in S$. A *quasiparabolic set* for W is a scaled W-set X satisfying the following two properties:

1. for any $r \in T$ and $x \in X$, if $\lambda(rx) = \lambda(x)$, then $rx = x$;
2. for any $r \in T$, $x \in X$, and $s \in S$, if $\lambda(rx) > \lambda(x)$ and $\lambda(srx) < \lambda(sx)$, then $rx = sx$.

For each quasiparabolic W-set (X, λ), the *quasiparabolic order* on a X is the weakest partial order $\leq_Q$ such that $x \leq_Q rx$ whenever we have $x \in X$, $r \in T$, and $\lambda(x) \leq \lambda(rx)$.

We note that Rains and Vazirani call λ the *height* function, and $\leq_Q$ the *Bruhat order*, but we have chosen to call them the *level* function and the *quasiparabolic order* because of the potential for confusion in the context of this abstract.

3.2 Crossing, Nestings, and Alignments

To explain how positive n-roots form a quasiparabolic W-set, we need to specify the level function λ. We do so by examining certain special quadruples of orthogonal roots:

Definition 3.2. Let Φ be a root system of type E_7, E_8 or D_n for n even. We say a quadruple $Q = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ of four mutually orthogonal positive roots in Φ is a *coplanar quadruple* if the element $\gamma = (\beta_1 + \beta_2 + \beta_3 + \beta_4)/2$ is a root in Φ.

The following proposition reveals intimate connections between coplanar quadruples, subsystems of type D_4 in Φ, and the W-action on positive n-roots. Note that if α is a root in a maximal orthogonal set R then the reflection s_α fixes R in the W-action on X, hence the more interesting actions arise in the case where $\alpha \notin R$, as treated in the second part of the proposition below.

Proposition 3.3. *Let Φ be a root system of type E_7, E_8 or D_n for n even. Let $Q = \{\beta_1, \beta_2, \beta_3, \beta_3\}$ be a set of four mutually orthogonal positive roots in Φ, and let $\gamma = (\beta_1 + \beta_2 + \beta_3 + \beta_4)/2$.*

1. *The following are equivalent:*
   (a) *the set Q is coplanar, i.e., γ is a root;*
   (b) *the set Q is contained in a subsystem $\Psi \subseteq \Phi$ of type D_4;*
   (c) *there is a unique subsystem Ψ of type D_4, Ψ_Q, such that $(Q \cup \{\gamma\}) \subset \Psi_Q \subseteq \Phi$.*
2. *Let $\alpha \in \Phi$ and let R be a set of n orthogonal positive roots such that $\alpha \notin R$.*
   (a) *The root α is orthogonal to all but precisely four elements $Q = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ of R. The elements of Q form a coplanar quadruple.*
   (b) *For the subsystem Ψ_Q as defined in 1.(c), we have $\alpha \in \Psi_Q$, and the set $s_\alpha(Q) = \{s_\alpha(\beta_i) : 1 \leq i \leq 4\}$ is also a coplanar quadruple contained in Ψ_Q.*

To help clarify the relationship between the coplanar quadruples Q and $s_\alpha(Q)$ in the setting of Proposition 3.3.2, we introduce the following notions:

Definition 3.4. Let $Q = \{\beta_1, \beta_2, \beta_3, \beta_4\}$ be a coplanar quadruple, let $\Psi = \Psi_Q$ be the D_4-subsystem associated to Q as in Proposition 3.3, let $\leq$ be the partial order on Ψ relative to the induced simple roots of Ψ, and let $\gamma = (\beta_1 + \beta_2 + \beta_3 + \beta_4)/2$. We say that Q is

1. a *crossing* if $\beta_i \leq \gamma$ for all i and Q contains the unique maximal element of the set $Q \cup (-s_\gamma(Q))$ with respect to $\leq$;
2. a *nesting* if $\beta_i \leq \gamma$ for all i and Q contains the unique minimal element of the set $Q \cup (-s_\gamma(Q))$ with respect to $\leq$;
3. an *alignment* otherwise.

We call each crossing, nesting, or alignment a *feature*. For each maximal orthogonal set $R \in X$, we define $A(R), C(R)$ and $N(R)$ to be the numbers of alignments, crossings, and nestings contained in R, respectively.

It is straightforward to verify that in type D_4, the three 4-roots $\gamma_A, \gamma_C, \gamma_N$ from Section 2.4 form an alignment, a crossing, and a nesting, respectively, which is compatible with the diagrams of their corresponding perfect matchings in the obvious sense. In general types, it can also be shown that a coplanar quadruple is always a crossing, a nesting, or an alignment, but never multiple features at once [6, Theorem 3.2.2.(ii)]. The purpose of Definition 3.4 is to distinguish the three coplanar quadruples in a type-D_4 subsystem using only abstract properties of roots, and doing so allows us to develop the theory of n-roots even in types E_7 and E_8, where no convenient diagrammatic interpretation of n-roots seems to be available.

3.3 The positive n-roots form a quasiparabolic W-set.

We are ready to state our first main theorem precisely. Recall from Section 2.2 that for each root α, we denote by $|\alpha|$ the absolute value of α.

Theorem 3.5. *Let W be a Weyl group of type E_7, E_8, or D_n for n even, and let X be the set of maximal orthogonal sets of positive roots of W, regarded as a W-set under the action*

$$w(\{\beta_1, \cdots, \beta_n\}) = \{|w(\beta_1)|, \cdots, |w(\beta_n)|\}.$$

The pair (X, λ) is a quasiparabolic W-set, where $\lambda : X \to \mathbb{Z}$ is the level function given by $\lambda(x) = C(x) + 2N(x)$.

The proof of the theorem involves a detailed analysis of how a reflection s_α acts on a maximal orthogonal set R. Using Proposition 3.3, it can be shown that s_α either fixes R or changes exactly four roots in R forming a coplanar quadruple Q within a type-D_4 subsystem Ψ_Q of Φ to a different feature, but considerable care is required to deal with other coplanar quadruples in R that overlap with Q, which may undergo changes not local to Ψ_Q under the action of s_α. For example, in type D_6, as the reflection s_α for the root $\alpha = \varepsilon_1 - \varepsilon_3$ takes the maximal orthogonal set $R = \{\varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_5, \varepsilon_3 \pm \varepsilon_6\}$ to the set $\{\varepsilon_1 \pm \varepsilon_6, \varepsilon_2 \pm \varepsilon_5, \varepsilon_3 \pm \varepsilon_4\}$, it not only changes the crossing $Q = \{\varepsilon_1 \pm \varepsilon_4, \varepsilon_3 \pm \varepsilon_6\}$ consisting of the roots not orthogonal to α to a nesting, but also causes the crossings $\{\varepsilon_1 \pm \varepsilon_4, \varepsilon_2 \pm \varepsilon_5\}$ and $\{\varepsilon_2 \pm \varepsilon_5, \varepsilon_3 \pm \varepsilon_6\}$ in R to become nestings.

4 Feature-avoiding Elements

We study the noncrossing, nonnesting, and alignment-free positive n-roots in this section. To understand the interactions of these feature-avoiding elements, it is helpful to define two n-roots to be *σ-equivalent* if their components have the same sum. This is an equivalence relation, and we call each equivalence class of the relation a *σ-class*.

4.1 Type-independent properties

The following theorem is a summary of selected results from [6, Sections 5.1–5.5]. We omit the proofs, but note that the three-term relation $\gamma_C = \gamma_N + \gamma_A$ plays a crucial role in the proofs of all the results summarized.

Theorem 4.1. *Let W be a Weyl group of type E_7, E_8, or D_n for n even, and let X be the set of maximal orthogonal sets of positive roots of W.*

1. *The noncrossing n-roots in X form a canonical $\mathbb{Q}$-basis, $\mathcal{B}_{NC}$, for the Macdonald representation $j_{nA_1}^{\Phi}(\mathrm{sgn})$. Moreover, every n-root of W is a $\mathbb{Z}$-linear combination of $\mathcal{B}_{NC}$ with coefficients of like sign. Equivalently, the basis $\mathcal{B}_{NC}$ is a sign-coherent basis for $j_{nA_1}^{\Phi}(\mathrm{sgn})$ in the sense of [5, Definition 6.12].*

2. *The nonnesting n-roots in X form a $\mathbb{Q}$-basis, $\mathcal{B}_{NN}$, for $j_{nA_1}^{\Phi}(\mathrm{sgn})$. Moreover, the set $\mathcal{B}_{NN}$ has the structure of a distributive lattice isomorphic to the lattice $L = \{v \in W : v \leq_L w_N\}$ for a suitable element w_N, where $\leq_L$ is the left weak Bruhat order on W.*

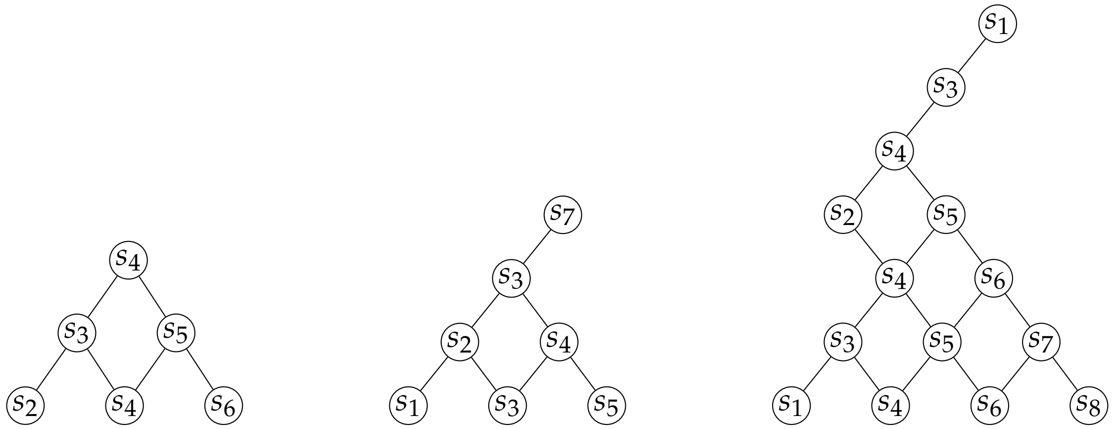
3. *The alignment-free n-roots in X form a quasiparabolic W_I-set, X_I, where W_I is a suitable standard parabolic subgroup of W, under the restriction of the level function λ on X. The set X_I also has a bipartite structure: the n-roots in X_I with even levels and those with odd levels partition X_I into two equal-sized components, and these components are interchanged by every reflection in W_I.*

4. *Each σ-class C of X contains a unique nonnesting n-root, β_1, and a unique noncrossing element, β_2. Moreover, in this case C coincides with the interval*
$$[\beta_1, \beta_2] = \{\gamma \in X : \beta_1 \leq_Q \gamma \leq_Q \beta_2\} \subseteq X$$
with respect to the quasiparabolic order $\leq_Q$. The set X_I coincides with the unique maximal σ-class with respect to a certain natural partial order on the σ-classes.

5. *Any set of σ-equivalence class representatives forms a $\mathbb{Q}$-basis for $j_{nA_1}^{\Phi}(\mathrm{sgn})$, and any two such bases (when suitably ordered) have a unitriangular change of basis matrix with integer entries. In particular, this is true for the bases $\mathcal{B}_{NC}$ and $\mathcal{B}_{NN}$.*

In Theorem 4.1, the facts that $\mathcal{B}_{NC}$ and $\mathcal{B}_{NN}$ are bases of $j_{nA_1}^{\Phi}(\mathrm{sgn})$ can be established using a version of Bergman's diamond lemma [2]. Simple reflections in W act on $\mathcal{B}_{NC}$ by a simple formula [6, Theorem 5.3.2.(iv)], and the sign-coherence property of $\mathcal{B}_{NC}$ may be amenable to categorification. The element w_N is a fully commutative element (in the sense of [15]) that we can identify explicitly, and the distributive lattice L is isomorphic to the lattice of ideals in the heap poset of w_N (Figure 3). The set X_I has many interesting properties, and we summarize some of them in Section 4.2. The results in (4) give rise to a canonical bijection $\beta_1 \leftrightarrow \beta_2$ between $\mathcal{B}_{NC}$ and $\mathcal{B}_{NN}$, and both (4) and (5) generalize known results for a Specht module, as we explain in the following remark.

Remark 4.2. In type D_n for $n = 2k$ even, the action of W on the representation $j_{nA_1}^{D_n}(\mathrm{sgn})$ factors through the map ϕ from Equation (2.1) to induce an S_n-module structure on $j_{nA_1}^{D_n}(\mathrm{sgn})$. The resulting S_n-module is isomorphic to the Specht module $S^{(k,k)}$, and the nonnesting and noncrossing bases for $j_{nA_1}^{D_n}(\mathrm{sgn}) \cong S^{(k,k)}$ have been studied extensively as the *web basis* and *standard basis* in the works of Rhoades [12], Russell–Tymoczko [13], and Hwang–Jang–Oh [8]. The canonical bijection between $\mathcal{B}_{NC}$ and $\mathcal{B}_{NN}$ generalizes the graph isomorphism ψ from [13], and Theorem 4.1.5 can be used to recover the results on the transition matrix from $\mathcal{B}_{NN}$ to $\mathcal{B}_{NC}$ in [13, Section 5]. If we expand the maximally crossing and maximally nesting positive n-roots (such n-roots exist in all types by [6, Sections 4.3 and 5.2]) as linear combinations of $\mathcal{B}_{NC}$, then the resulting coefficients give rise to *Euler numbers* [10, A000111] and *web permutations* by the results of [8], respectively. For more details, see [6, Remark 6.1.3].



(a) (b) (c)

Figure 3: Hasse diagrams for the heaps of the elements w_N in types D_8, E_7 and E_8.

4.2 Type-specific properties

The alignment-free positive n-roots have the following properties. We refer the interested reader to [6, Sections 6.1–6.3] for more details.

Proposition 4.3. *Let W be a Weyl group of type E_7, E_8, or D_n for n even, and let X_I be the set of alignment-free positive n-roots of W.*

1. *If W has type D_n for $n = 2k$ even, then under the restriction of the quasiparabolic order on positive n-roots, the set X_I is canonically isomorphic to the symmetric group S_k under the strong Bruhat order via the map $\varphi : S_k \to X_I$ sending each element $\tau \in S_n$ to the n-root $\varphi(\tau) = \prod_{i=1}^{k}(\varepsilon_i^2 - \varepsilon_{\tau(i)+k}^2)$.*

2. *If W has type E_7, then the set X_I admits a canonical bijection to the 30 inequivalent labellings of the Fano plane.*

3. *If W has type E_8, then the graphs G_{E_8} and Γ constructed as follows are not isomorphic but quantum isomorphic in the sense of [1]. The vertices of G_{E_8} are the positive roots of type E_8, with two vertices being adjacent if and only if they are orthogonal. The vertices of Γ are the elements of X_I with even levels, with two vertices being adjacent if and only if they share no common components.*

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