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Multiplicative permanental inequalities for totally positive matrices

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Abstract. We characterize ratios of permanents of submatrices which are bounded on the set of totally positive matrices. This provides a permanental analog of results of Fallat, Gekhtman, and Johnson [*Adv. Appl. Math.*, **30** no. 3, (2003)] concerning ratios of matrix minors. We also extend work of Drake, Gerrish, and the first author [*Electron. J. Combin.*, **11** no. 1, (2004)] by characterizing differences of monomials in $\mathbb{Z}[x_{1,1}, x_{1,2}, \ldots, x_{n,n}]$ which evaluate positively on all totally positive $n \times n$ matrices.

Keywords: total positivity, permanent, inequalities

1 Introduction

Given an $n \times n$ matrix $A = (a_{i,j})$ and subsets $I, J \subseteq [n] := \{1, \ldots, n\}$, let $A_{I,J} = (a_{i,j})_{i \in I, j \in J}$ denote the (I, J)-*submatrix* of A. For $|I| = |J|$, call $\det(A_{I,J})$ the (I, J)-*minor* of A. A real $n \times n$ matrix A is called *totally positive* (*totally nonnegative*) if every minor of A is positive (nonnegative). Let $\mathcal{M}_n^{\mathrm{TP}} \subset \mathcal{M}_n^{\mathrm{TNN}}$ denote these sets of matrices.

These and the set $\mathcal{M}_n^{\mathrm{HPS}}$ of $n \times n$ Hermitian positive semidefinite matrices arise in many areas of mathematics, and for more than a century mathematicians have been studying inequalities satisfied by their matrix entries. (See, e.g., [7].) Many such inequalities involve minors and permanents. For instance inequalities of Fischer [8], Fan [3], and Lieb [12] state that for all matrices $A \in \mathcal{M}_n^{\mathrm{TNN}} \cup \mathcal{M}_n^{\mathrm{HPS}}$, and for all $I \subseteq [n]$ and $I^c := [n] \smallsetminus I$, we have

$$\begin{aligned} \det(A) &\leq \det(A_{I,I}) \det(A_{I^c,I^c}), \\ \mathrm{per}(A) &\geq \mathrm{per}(A_{I,I}) \,\mathrm{per}(A_{I^c,I^c}). \end{aligned} \tag{1.1}$$

Koteljanskii's inequality [11] states that for $A \in \mathcal{M}_n^{\mathrm{TNN}} \cup \mathcal{M}_n^{\mathrm{HPS}}$ and $I, J \subseteq [n]$ we have

$$\det(A_{I \cup J, I \cup J}) \det(A_{I \cap J, I \cap J}) \leq \det(A_{I,I}) \det(A_{J,J}). \tag{1.2}$$

Many open questions about inequalities exist and seem difficult. For instance, it is known which 8-tuples $(I, J, K, L, I', J', K', L')$ of subsets satisfy

$$\det(A_{I,I'}) \det(A_{J,J'}) \leq \det(A_{K,K'}) \det(A_{L,L'}) \tag{1.3}$$

for all $A \in \mathcal{M}_n^{\mathrm{TNN}}$ [6], [13], but few permanental analogs of such inequalities are known. While some of these 8-tuples also satisfiy

$$\mathrm{per}(A_{I,I'})\,\mathrm{per}(A_{J,J'}) \geq \mathrm{per}(A_{K,K'})\,\mathrm{per}(A_{L,L'}), \tag{1.4}$$

this second inequality is not true in general: the natural permanental analog

$$\mathrm{per}(A_{I\cup J,I\cup J})\,\mathrm{per}(A_{I\cap J,I\cap J}) \geq \mathrm{per}(A_{I,I})\,\mathrm{per}(A_{J,J}) \tag{1.5}$$

of (1.2) holds neither for all $A \in \mathcal{M}_n^{\mathrm{HPS}}$ nor for all $A \in \mathcal{M}_n^{\mathrm{TNN}}$. (See [14, Section 6] for a counterexample with $n = 3$.)

Let us put aside $\mathcal{M}_n^{\mathrm{HPS}}$ and consider conjectured inequalities of the form

$$\mathrm{product}_1 \leq \mathrm{product}_2 \tag{1.6}$$

involving minors and permanents of matrices in $\mathcal{M}_n^{\mathrm{TNN}}$ and $\mathcal{M}_n^{\mathrm{TP}}$. One strategy for studying (1.6) is to view the difference $\mathrm{product}_2 - \mathrm{product}_1$ as a polynomial

$$f(x) := f(x_{1,1}, x_{1,2}, \ldots, x_{n,n}) \in \mathbb{Z}[x] := \mathbb{Z}[x_{1,1}, x_{1,2}, \ldots, x_{n,n}] \tag{1.7}$$

in matrix entries. Then the validity of the inequality (1.6) is equivalent to the statement that for all $A = (a_{i,j}) \in \mathcal{M}_n^{\mathrm{TNN}}$, we have

$$f(A) := f(a_{1,1}, a_{1,2}, \ldots, a_{n,n}) \geq 0. \tag{1.8}$$

We call a polynomial (1.7) with this property a *totally nonnegative polynomial*. Since $\mathcal{M}_n^{\mathrm{TP}}$ is dense in $\mathcal{M}_n^{\mathrm{TNN}}$, the inequality (1.8) holds for all $A \in \mathcal{M}_n^{\mathrm{TP}}$ if and only if it holds for all $A \in \mathcal{M}_n^{\mathrm{TNN}}$.

A second strategy for studying (variations of) a potential inequality (1.6) is to ask for which positive constants k_1, k_2 the modified inequalities

$$k_1 \cdot \mathrm{product}_1 \leq \mathrm{product}_2 \leq k_2 \cdot \mathrm{product}_1 \tag{1.9}$$

hold for all $A \in \mathcal{M}_n^{\mathrm{TNN}}$. Bounds of $k_1 = 1$ or $k_2 = 1$ imply the inequality (1.6) or its reverse to hold; other bounds give information not apparent in the proof or disproof of (1.6). Equivalently, we may view the ratio of $\mathrm{product}_2$ to $\mathrm{product}_1$ as a rational function

$$R(x) := R(x_{1,1}, x_{1,2}, \ldots, x_{n,n}) \in \mathbb{Q}(x) := \mathbb{Q}(x_{1,1}, x_{1,2}, \ldots, x_{n,n}) \tag{1.10}$$

in matrix entries, and we may ask for upper and lower bounds as x varies over $\mathcal{M}_n^{\mathrm{TP}}$. While a ratio (1.10) is not defined everywhere on $\mathcal{M}_n^{\mathrm{TNN}}$, the density of $\mathcal{M}_n^{\mathrm{TP}}$ in $\mathcal{M}_n^{\mathrm{TNN}}$ allows us to restrict our attention to $\mathcal{M}_n^{\mathrm{TP}}$: we have

$$k_1 \leq R(x) \leq k_2 \tag{1.11}$$

for all $x \in \mathcal{M}_n^{\mathrm{TP}}$ if and only if the same inequalities hold for all $x \in \mathcal{M}_n^{\mathrm{TNN}}$ such that $R(x)$ is defined. Clearly the lower bound k_1 is interesting only when positive, since minors and permanents of totally nonnegative matrices are trivially bounded below by 0.

A characterization of all ratios of the form

$$\frac{\det(x_{I,I'})\det(x_{J,J'})}{\det(x_{K,K'})\det(x_{L,L'})}, \qquad I, I', \ldots, L, L' \subset [n], \tag{1.12}$$

which are bounded above and/or nontrivially bounded below on $\mathcal{M}_n^{\mathrm{TP}}$ follows from work in [6], [13]. Each ratio (1.12) is bounded above and/or below by 1, and for each n, it factors as a product of elements of a finite set of indecomposable ratios. This result was extended in [15] to include ratios of products of arbitrarily many minors

$$\frac{\det(x_{I_1,I_1'})\cdots\det(x_{I_p,I_p'})}{\det(x_{J_1,J_1'})\cdots\det(x_{J_p,J_p'})}. \tag{1.13}$$

Again, each such ratio factors as a product of elements belonging to a finite set of indecomposable ratios. For $n = 3$, each ratio (1.13) is bounded above and/or below by 1; for $n \geq 4$, such bounds are conjectured [2].

While the permanental version (1.5) of Koteljanskii's inequality is false, we will show in Section 3 that the corresponding ratio is bounded above and nontrivially below. Specifically,

$$\frac{1}{|I \cup J|!\,|I \cap J|!} \leq \frac{\mathrm{per}(x_{I,I})\,\mathrm{per}(x_{J,J})}{\mathrm{per}(x_{I\cup J,I\cup J})\,\mathrm{per}(x_{I\cap J,I\cap J})} \leq |I|!\,|J|! \tag{1.14}$$

for all $I, J \subseteq [n]$ and $x \in \mathcal{M}_n^{\mathrm{TP}}$. The failure of (1.5), combined with (1.14), exposes a difference between ratios of minors and of permanents: unlike the bounded ratios in (1.12), *not* all bounded ratios of permanents are bounded by 1. Thus it is natural to ask which ratios

$$R(x) = \frac{\mathrm{per}(x_{I_1,I_1'})\mathrm{per}(x_{I_2,I_2'})\cdots\mathrm{per}(x_{I_r,I_r'})}{\mathrm{per}(x_{J_1,J_1'})\mathrm{per}(x_{J_2,J_2'})\cdots\mathrm{per}(x_{J_q,J_q'})} \tag{1.15}$$

are bounded above and/or nontrivially below as real-valued functions on $\mathcal{M}_n^{\mathrm{TP}}$, and to look for bounds.

In Section 2 we describe a multigrading of the coordinate ring $\mathbb{Z}[x]$ of $n \times n$ matrices. Extending work in [5], we define a partial order on the monomials in $\mathbb{Z}[x]$ which characterizes the differences $\prod x_{i,j}^{c_{i,j}} - \prod x_{i,j}^{d_{i,j}}$ which are totally nonnegative polynomials. This leads to our main results in Section 3 which characterize ratios (1.15) which are bounded above and nontrivially below as real-valued functions on $\mathcal{M}_n^{\mathrm{TP}}$. We provide some such bounds, which are not necessarily tight, and suggest problems for further investigation.

2 A multigrading of $\mathbb{Z}[x_{1,1}, x_{1,2}, \ldots, x_{n,n}]$ and the total nonnegativity order

We will find it convenient to view degree-r monomials in $\mathbb{Z}[x]$ in terms of permutations in the symmetric group $\mathfrak{S}_r$ and multisets of $[n]$. In particular, given permutations $v, w \in \mathfrak{S}_r$ define the monomial

$$x^{v,w} := x_{v_1,w_1} \cdots x_{v_r,w_r}.$$

Define an *r-element multiset* of $[n]$ to be a nondecreasing r-tuple of elements of $[n]$. In exponential notation, we write i^k to represent k consecutive occurrences of i in such an r-tuple, e.g.,

$$(1,1,2,3) = 1^2 2^1 3^1, \qquad (1,2,2,2) = 1^1 2^3. \tag{2.1}$$

Two r-element multisets

$$M = (m_1, \ldots, m_r) = 1^{\alpha_1} \cdots n^{\alpha_n}, \qquad O = (o_1, \ldots, o_r) = 1^{\beta_1} \cdots n^{\beta_n}, \tag{2.2}$$

determine a *generalized submatrix* $x_{M,O}$ of x by $(x_{M,O})_{i,j} := x_{m_i,o_j}$. For example, when $n = 3$, we have the 4×4 generalized submatrix and monomial

$$x_{1123,1222} = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,2} & x_{1,2} \\ x_{1,1} & x_{1,2} & x_{1,2} & x_{1,2} \\ x_{2,1} & x_{2,2} & x_{2,2} & x_{2,2} \\ x_{3,1} & x_{3,2} & x_{3,2} & x_{3,2} \end{bmatrix}, \qquad (x_{1123,1222})^{1234,4312} = x_{1,2}x_{1,2}x_{2,1}x_{3,2}. \tag{2.3}$$

The ring $\mathbb{Z}[x]$ has a natural multigrading

$$\mathbb{Z}[x] = \bigoplus_{r \geq 0} \bigoplus_{M,O} \mathcal{A}_{M,O}, \tag{2.4}$$

where the second direct sum is over pairs (M, O) of r-element multisets of $[n]$, and

$$\mathcal{A}_{M,O} := \mathrm{span}_{\mathbb{Z}}\{(x_{M,O})^{e,w} \mid w \in \mathfrak{S}_r\}. \tag{2.5}$$

More precisely, for M, O as in (2.2), a basis for $\mathcal{A}_{M,O}$ is given by all monomials

$$\prod_{i,j \in [n]} x_{i,j}^{c_{i,j}} \tag{2.6}$$

with $C = (c_{i,j}) \in \mathrm{Mat}_{n \times n}(\mathbb{N})$ satisfying

$$c_{i,1} + \cdots + c_{i,n} = \alpha_i, \quad c_{1,j} + \cdots + c_{n,j} = \beta_j \quad \text{for } i, j = 1, \ldots, n. \tag{2.7}$$

One may express a monomial (2.6) in the form $(x_{M,O})^{e,w}$ by the following algorithm.

Algorithm 2.1. Given a monomial (2.6) in $\mathcal{A}_{M,O}$ with M, O as in (2.2),

(i) Define the rearrangement $u = u_1 \cdots u_r$ of O by writing (2.6) with variables in lexicographic order as $x_{m_1,u_1} \cdots x_{m_r,u_r}$.

(ii) Let $j_1 < \cdots < j_{\beta_1}$ be the positions of the β_1 ones in u, let $j_{\beta_1+1} < \cdots < j_{\beta_1+\beta_2}$ be the positions of the β_2 twos in u, etc.

(iii) For $i = 1, \ldots, r$, define $w_{j_i} = i$.

(iv) Call the resulting word $w = w(C)$.

For example, it is easy to check that for multisets $(1123, 1222) = (1^2 2^1 3^1, 1^1 2^3 3^0)$ of $\{1, 2, 3\}$, the graded component $\mathcal{A}_{1123,1222}$ of $\mathbb{Z}[x_{1,1}, x_{1,2}, \ldots, x_{3,3}]$ is spanned by monomials (2.6), where $C = (c_{i,j})$ is one of the matrices

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{2.8}$$

having row sums $(2, 1, 1)$ and column sums $(1, 3, 0)$. These monomials are

$$x_{1,1}x_{1,2}x_{2,2}x_{3,2}, \quad x_{1,2}^2 x_{2,1}x_{3,2}, \quad x_{1,2}^2 x_{2,2}x_{3,1}, \tag{2.9}$$

with column index sequences equal to the rearrangements 1222, 2212, 2221 of 1222. Algorithm 2.1 then produces permutations 1234, 2314, 2341 in $\mathfrak{S}_4$, and we may express the monomials (2.9) as

$$(x_{1123,1222})^{1234,1234}, \quad (x_{1123,1222})^{1234,2314}, \quad (x_{1123,1222})^{1234,2341}. \tag{2.10}$$

For r-element multisets M,O of $[n]$, monomials in $\mathcal{A}_{M,O}$ are closely related to parabolic subgroups of $\mathfrak{S}_r$ with standard generators $s_1, \ldots, s_{r-1}$, and double cosets of the form $W_{\iota(M)} w W_{\iota(O)}$ where w belongs to $\mathfrak{S}_r$, W_J is the subgroup of $\mathfrak{S}_r$ generated by J, and

$$\begin{aligned} \iota(M) &:= \{s_1, \ldots, s_{r-1}\} \smallsetminus \{s_{\alpha_1}, s_{\alpha_1+\alpha_2}, \ldots, s_{r-\alpha_n}\} = \{s_j \,|\, m_j = m_{j+1}\}, \\ \iota(O) &:= \{s_1, \ldots, s_{r-1}\} \smallsetminus \{s_{\beta_1}, s_{\beta_1+\beta_2}, \ldots, s_{r-\beta_n}\} = \{s_j \,|\, o_j = o_{j+1}\}. \end{aligned} \tag{2.11}$$

It is easy to see that the map $M \mapsto \iota(M)$ is bijective: one recovers $M = 1^{\alpha_1} \cdots n^{\alpha_n}$ from the generators not in $\iota(M)$ as in (2.11). It is known that each double coset has unique minimal and maximal elements with respect to the *Bruhat order on* $\mathfrak{S}_r$, defined by declaring $v \leq w$ if each reduced expression $s_{i_1} \cdots s_{i_\ell}$ for w contains a subword which is a reduced expression for v. (See, e.g., [1], [4].) Let $W_{\iota(M)} \backslash W / W_{\iota(O)}$ denote the set of all double cosets of $W = \mathfrak{S}_r$ determined by r-element multisets M, O.

Proposition 2.2. *Fix r-element multisets M, O as in (2.2). The double cosets $W_{\iota(M)} \backslash W / W_{\iota(O)}$ satisfy the following.*

(i) Each double coset has a unique Bruhat-minimal element u satisfying $su > u$ for all $s \in \iota(M)$ and $us > u$ for all $s \in \iota(O)$; it has a unique Bruhat-maximal element u' satisfying $su' < u'$ for all $s \in \iota(M)$ and $u's < u'$ for all $s \in \iota(O)$.

(ii) We have $W_{\iota(M)} v W_{\iota(O)} = W_{\iota(M)} w W_{\iota(O)}$ if and only if $(x_{M,O})^{e,v} = (x_{M,O})^{e,w}$.

(iii) The cardinality $|W_{\iota(M)} \backslash W / W_{\iota(O)}|$ is the dimension of $\mathcal{A}_{M,O}$, equivalently, the number of matrices in $\mathrm{Mat}_{n \times n}(\mathbb{N})$ having row sums $(\alpha_1, \ldots, \alpha_n)$ and column sums $(\beta_1, \ldots, \beta_n)$.

(iv) Each permutation w produced by Algorithm 2.1 is the unique Bruhat-minimal element of its coset $W_{\iota(M)} w W_{\iota(O)}$.

Proof. Omitted. □

For any subsets I, J of generators of $\mathfrak{S}_r$, the Bruhat order on $\mathfrak{S}_r$ induces a poset structure on $W_I \backslash W / W_J$ as follows. We declare $W_I v W_J \leq W_I w W_J$ if the minimal element of $W_I v W_J$ is less than or equal to the minimal element of $W_I w W_J$. (Equivalently, we may compare maximal or arbitrary elements of the cosets [4, Lemma 2.2].) We call this poset the *Bruhat order on $W_I \backslash W / W_J$*. Another equivalent inequality can be stated in terms of matrices of exponents defined by monomials in $\mathcal{A}_{M,O}$. (See, e.g., [10].) Given a matrix $C = (c_{i,j}) \in \mathrm{Mat}_{n \times n}(\mathbb{N})$, define the matrix $C^* = (c^*_{i,j}) \in \mathrm{Mat}_{n \times n}(\mathbb{N})$ by

$$c^*_{i,j} = \text{sum of entries of } C_{[i],[j]}. \tag{2.12}$$

Proposition 2.3. *Fix monomials*

$$(x_{M,O})^{e,v} = \prod_{i,j} x_{i,j}^{c_{i,j}}, \quad (x_{M,O})^{e,w} = \prod_{i,j} x_{i,j}^{d_{i,j}},$$

in $\mathcal{A}_{M,O}$ and define matrices C^*, D^* as in (2.12). Then we have $W_{\iota(M)} v W_{\iota(O)} \leq W_{\iota(M)} w W_{\iota(O)}$ in the Bruhat order if and only if $C^* \geq D^*$ in the componentwise order.

The Bruhat order on $W_{\iota(M)} \backslash W / W_{\iota(O)}$ is closely related to certain totally nonnegative polynomials in $\mathcal{A}_{M,O}$. Indeed, when $M = O = 1 \cdots n$, totally nonnegative polynomials of the form $x^{e,v} - x^{e,w}$ are characterized by the Bruhat order on $\mathfrak{S}_n$ [5].

Theorem 2.4. *For $v, w \in \mathfrak{S}_n$, the polynomial $x^{e,v} - x^{e,w}$ is totally nonnegative if and only if $v \leq w$ in the Bruhat order.*

We will now extend this result to all monomials in $\mathbb{Z}[x]$. Let us define a partial order $\leq_T$ on all monomials in $\mathbb{Z}[x]$ by declaring $(x_{M,O})^{e,v} \leq_T (x_{P,Q})^{e,w}$ if $(x_{P,Q})^{e,w} - (x_{M,O})^{e,v}$ is a totally nonnegative polynomial. We call this the *total nonnegativity order* on monomials in $\mathbb{Z}[x]$. It is not hard to show that the total nonnegativity order is a disjoint union of its restrictions to the multigraded components (2.4) of $\mathbb{Z}[x]$.

Lemma 2.5. *Monomials*

$$\prod_{i,j} x_{i,j}^{c_{i,j}}, \quad \prod_{i,j} x_{i,j}^{d_{i,j}} \tag{2.13}$$

are comparable in the total nonnegativity order only if they belong to the same multigraded component of $\mathbb{Z}[x]$.

Proof. Omitted. $\square$

Theorem 2.6. *Fix r-element multisets $M = 1^{\alpha_1} \cdots n^{\alpha_n}$, $O = 1^{\beta_1} \cdots n^{\beta_n}$ as in (2.2), and matrices $C, D \in \mathrm{Mat}_{n\times n}(\mathbb{N})$ with row and column sums $(\alpha_1, \ldots, \alpha_n)$, $(\beta_1, \ldots, \beta_n)$, and define the polynomial*

$$f(x) = \prod_{i,j} x_{i,j}^{c_{i,j}} - \prod_{i,j} x_{i,j}^{d_{i,j}}$$

in $\mathcal{A}_{M,O}$. Then the following are equivalent.

1. *$f(x)$ is totally nonnegative.*
2. *$C^* \geq D^*$ in the componentwise order.*
3. *$w(C) \leq w(D)$ in the Bruhat order on $\mathfrak{S}_r$.*
4. *$f(x)$ is equal to a sum of products of the form $\det(x_{I,J}) x_{u_1,v_1} \cdots x_{u_{r-2},v_{r-2}}$ in $\mathcal{A}_{M,O}$ with $|I| = |J| = 2$.*

Proof. Omitted. $\square$

For example, let us revisit the monomials (2.9) – (2.10) in the graded component $\mathcal{A}_{1123,1222}$ of $\mathbb{Z}[x_{1,1}, x_{1,2}, \ldots, x_{3,3}]$. It is easy to see that $1234 < 2314 < 2341$ in the Bruhat order on $\mathfrak{S}_4$ and that the application of (2.12) to the corresponding matrices in (2.8) yields the componentwise comparisons

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & 3 \\ 1 & 4 & 4 \end{bmatrix} \geq \begin{bmatrix} 0 & 2 & 2 \\ 1 & 3 & 3 \\ 1 & 4 & 4 \end{bmatrix} \geq \begin{bmatrix} 0 & 2 & 2 \\ 0 & 3 & 3 \\ 1 & 4 & 4 \end{bmatrix}. \tag{2.14}$$

Thus we have $(x_{1123,1222})^{1234,1234} \geq_T (x_{1123,1222})^{1234,2314} \geq_T (x_{1123,1222})^{1234,2341}$, i.e.,

$$x_{1,1}x_{1,2}x_{2,2}x_{3,2} \;\geq_T\; x_{1,2}^2 x_{2,1} x_{3,2} \;\geq_T\; x_{1,2}^2 x_{2,2} x_{3,1}.$$

Furthermore, the chain $1234 < 2134 < 2314 < 2341$ in the Bruhat order on $\mathfrak{S}_4$ with

$$2134 = (1,2)1234, \quad 2314 = (2,3)2134, \quad 2341 = (3,4)2314 \tag{2.15}$$

allows us to write $x_{1,1}x_{1,2}x_{2,2}x_{3,2} - x_{1,2}^2 x_{2,1}x_{3,2}$ as

$$\begin{aligned}\Big((x_{1123,1222})^{1234,1234} - (x_{1123,1222})^{1234,2134}\Big) + \Big((x_{1123,1222})^{1234,2134} - (x_{1123,1222})^{1234,2314}\Big)\\ = \det\begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{1,1} & x_{2,2}\end{bmatrix} x_{2,2}x_{3,2} + x_{1,2}\det\begin{bmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2}\end{bmatrix} x_{3,2},\end{aligned}$$

and to write $x_{1,2}^2 x_{2,1}x_{3,2} - x_{1,2}^2 x_{2,2}x_{3,1}$ as

$$\Big((x_{1123,1222})^{1234,2314} - (x_{1123,1222})^{1234,2341}\Big) = x_{1,2}^2 \det\begin{bmatrix} x_{2,1} & x_{2,2} \\ x_{3,1} & x_{3,2}\end{bmatrix}.$$

3 Main results

Let $\mathcal{M}_n^{\mathrm{TP}}$ be the set of totally positive $n \times n$ matrices. To characterize ratios of products of permanents which are bounded above and/or nontrivially bounded below on the set $\mathcal{M}_n^{\mathrm{TP}}$, we first consider necessary conditions on the multisets of rows and columns appearing in such ratios. Let

$$R(x) = \frac{\mathrm{per}(x_{I_1,I'_1})\mathrm{per}(x_{I_2,I'_2})\cdots\mathrm{per}(x_{I_r,I'_r})}{\mathrm{per}(x_{J_1,J'_1})\mathrm{per}(x_{J_2,J'_2})\cdots\mathrm{per}(x_{J_q,J'_q})}, \tag{3.1}$$

be such a ratio, where

$$(I_1,\ldots,I_r), \quad (I'_1,\ldots,I'_r), \quad (J_1,\ldots,J_q), \quad (J'_1,\ldots,J'_q) \tag{3.2}$$

are multisets of $[n]$ satisfying $|I_k| = |I'_k|$, $|J_k| = |J'_k|$ for all k. In order for $R(x)$ to be bounded above or nontrivially bounded below on $\mathcal{M}_n^{\mathrm{TP}}$ the multisets (3.2) must be related in terms of an operation which we call *multiset union*. Given multisets $M = 1^{\alpha_1}\cdots n^{\alpha_n}$, $O = 1^{\beta_1}\cdots n^{\beta_n}$ of $[n]$, define their multiset union to be

$$M \Cup O := 1^{\alpha_1+\beta_1}\cdots n^{\alpha_n+\beta_n}. \tag{3.3}$$

For example, $1124 \Cup 1233 = 11122334$.

Proposition 3.1. *Given multiset sequences as in* (3.2)*, a ratio* (3.1) *can be bounded above or nontrivially bounded below on $\mathcal{M}_n^{\mathrm{TP}}$ only if we have the multiset equalities*

$$I_1 \Cup \cdots \Cup I_r = J_1 \Cup \cdots \Cup J_q, \qquad I'_1 \Cup \cdots \Cup I'_r = J'_1 \Cup \cdots \Cup J'_q. \tag{3.4}$$

Proof. Omitted. □

To state sufficient conditions for the boundedness of ratios (3.1) we observe that it is possible to bound the permanent above and below as follows.

Proposition 3.2. *For any $n \times n$ totally nonnegative matrix $A = (a_{i,j})$ we have*

$$a_{1,1} \cdots a_{n,n} \leq \mathrm{per}(A) \leq n! \cdot a_{1,1} \cdots a_{n,n}. \tag{3.5}$$

Proof. Omitted. □

Now we state our main result, which characterizes ratios $R(x)$ as in (3.1) which are bounded above for $x \in \mathcal{M}_n^{\mathrm{TP}}$.

Theorem 3.3. *Let rational function*

$$R(x) = \frac{\mathrm{per}(x_{I_1,I'_1})\mathrm{per}(x_{I_2,I'_2}) \cdots \mathrm{per}(x_{I_r,I'_r})}{\mathrm{per}(x_{J_1,J'_1})\mathrm{per}(x_{J_2,J'_2}) \cdots \mathrm{per}(x_{J_q,J'_q})} \tag{3.6}$$

have index sets which satisfy (3.4), and define matrices $C = (c_{i,j})$, $C^* = (c^*_{i,j})$, $D = (d_{i,j})$, $D^* = (d^*_{i,j})$ by

$$(x_{I_1,I'_1})^{e,e} \cdots (x_{I_r,I'_r})^{e,e} = \prod x_{i,j}^{c_{i,j}}, \qquad (x_{J_1,J'_1})^{e,e} \cdots (x_{J_q,J'_q})^{e,e} = \prod x_{i,j}^{d_{i,j}}, \tag{3.7}$$

and (2.12). Then $R(x)$ is bounded above on the set of totally positive matrices if and only if $C^* \leq D^*$ in the componentwise order. In this case, it is bounded above by $|I_1|! \cdots |I_r|!$.

Proof. Suppose that $C^* \not\leq D^*$. Then for some indices (k, ℓ) we have $c^*_{k,\ell} > d^*_{k,\ell}$. Define the matrix $B(t) = (b_{i,j}(t))$ by

$$b_{i,j}(t) = \begin{cases} t & \text{if } i \leq k \text{ and } j \leq \ell, \\ 1 & \text{otherwise.} \end{cases}$$

Now, we have $R(B(t)) = \frac{p(t)}{q(t)}$ where $\deg(p(t)) = c^*_{i,j} > d^*_{i,j} = \deg(q(t))$. Thus we have

$$\lim_{t\to\infty} R(B(t)) = t^{c^*_{i,j}-d^*_{i,j}} = \infty.$$

Assume therefore that we have $C^* \leq D^*$ and let A be any $n \times n$ totally positive matrix. Applying the inequalities of Proposition 3.2 to the numerator and denominator of $R(A)$ respectively, we see that $R(A)$ is at most

$$\frac{|I_1|!(A_{I_1,I'_1})^{e,e} \cdots |I_r|!(A_{I_r,I'_r})^{e,e}}{(A_{J_1,J'_1})^{e,e} \cdots (A_{J_q,J'_q})^{e,e}} = \frac{|I_1|! \cdots |I_r|! \prod a_{i,j}^{c_{i,j}}}{\prod a_{i,j}^{d_{i,j}}}. \tag{3.8}$$

By Theorem 2.6, this is at most $|I_1|! \cdots |I_r|!$.

□

Observe that Theorem 3.3 guarantees no nontrivial lower bound for $R(x)$ and gives an upper bound which is sometimes tight. Indeed the ratio $\frac{x_{1,2}x_{2,1}}{x_{1,1}x_{2,2}}$ attains all values in the open interval $(0,1)$ as x varies over matrices in $\mathcal{M}_2^{\mathrm{TP}}$. On the other hand, special cases of the ratios in Theorem 3.3 can be shown to have both upper and nontrivial lower bounds.

Corollary 3.4. *For ratio $R(x)$ and matrices C, D defined as in Theorem 3.3, if $C = D$, then $R(x)$ is bounded above and below by*

$$\frac{1}{|J_1|! \cdots |J_q|!} \leq R(x) \leq |I_1|! \cdots |I_r|! \,, \tag{3.9}$$

for $x \in \mathcal{M}_n^{\mathrm{TP}}$.

For example, consider the ratio

$$R(x) = \frac{\mathrm{per}(x_{12,34})\mathrm{per}(x_{34,12})}{x_{1,3}x_{2,4}x_{3,1}x_{4,2}} \tag{3.10}$$

with $|I_1| = |I_2| = 2$, $|J_1| = |J_2| = |J_3| = |J_4| = 1$, and

$$C = D = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \tag{3.11}$$

By Corollary 3.4, we have $1 \leq R(x) \leq 4$. It is easy to see that $R(x)$ attains values arbitrarily close to 4 as x approaches the matrix of all ones. It is also possible to show that $R(x)$ attains values arbitrarily close to 1. Indeed, consider the matrix $A = A(\epsilon) = (a_{i,j})$ defined by

$$A(\epsilon) = \begin{bmatrix} 1 & 1 & \epsilon & \epsilon^3 \\ 1 & 2 & 1 & \epsilon \\ \epsilon & 1 & 2 & 1 \\ \epsilon^3 & \epsilon & 1 & 1 \end{bmatrix}, \tag{3.12}$$

where ϵ is positive and close to 0. To see that $A(\epsilon)$ is totally positive, it suffices to verify the positivity of the sixteen minors $\det(A_{[a_1,b_1],[a_2,b_2]})$ indexed by pairs of intervals, at least one of which contains 1 [9, Theorem 9]. Observe that we have $a_{1,j} > 0$ and $a_{i,1} > 0$ for all i, j. Also,

$$\begin{aligned}
&\det(A_{12,12}) = 1, \\
&\det(A_{12,23}) = \det(A_{23,12}) = 1 - 2\epsilon, \\
&\det(A_{12,34}) = \det(A_{34,12}) = \epsilon^2 - \epsilon^3, \\
&\det(A_{123,123}) = 1 + 2\epsilon - 2\epsilon^2, \\
&\det(A_{123,234}) = \det(A_{234,123}) = 1 - 4\epsilon + \epsilon^2 + 3\epsilon^3, \\
&\det(A) = 4\epsilon - 6\epsilon^2 - 2\epsilon^3 + 9\epsilon^4 - 2\epsilon^5 - 3\epsilon^6.
\end{aligned}$$

It follows that we have

$$\lim_{\epsilon\to 0^+} R(A) = \lim_{\epsilon\to 0^+} \frac{(\epsilon^2-\epsilon^3)^2}{\epsilon^4} = \lim_{\epsilon\to 0^+} 1 - 2\epsilon + \epsilon^2 = 1.$$

In the case that all submatrices in (3.6) are principal, the necessary condition (3.4) for boundedness is in fact sufficient to guarantee the existence of upper and nontrivial lower bounds.

Corollary 3.5. *For ratio $R(x)$ as in Theorem 3.3, if only principal submatrices appear ($I_k = I'_k$, $J_k = J'_k$ for all k), then $R(x)$ is bounded above and below as in (3.9).*

Proof. Omitted. $\square$

For example, consider the ratio

$$\frac{\mathrm{per}(x_{I,I})\,\mathrm{per}(x_{J,J})}{\mathrm{per}(x_{I\cup J,I\cup J})\,\mathrm{per}(x_{I\cap J,I\cap J})} \tag{3.13}$$

coming from the (false) permanental version (1.5) of Koteljanskii's inequality (1.2). By Corollary 3.5, the four principal submatrices of x imply that the exponent matrices C and D are equal and diagonal with (i,i) entry equal to the multiplicity of i in $I \uplus J$. Thus Corollary 3.4 gives the lower and upper bounds

$$\frac{1}{|I\cup J|!\,|I\cap J|!}, \qquad |I|!\,|J|! \tag{3.14}$$

as claimed in (1.14). These bounds are not in general tight. Consider the special case

$$\frac{1}{3!1!} \le \frac{\mathrm{per}(x_{12,12})\mathrm{per}(x_{23,23})}{\mathrm{per}(x_{123,123})\mathrm{per}(x_{2,2})} \le (2!)^2, \qquad C = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.15}$$

A more careful analysis leads to improved bounds of $1/2$ and 2, since the difference $2\mathrm{per}(x_{12,12})\mathrm{per}(x_{23,23}) - \mathrm{per}(x_{123,123})\mathrm{per}(x_{2,2})$ equals

$$\det(x_{13,13})x_{22}^2 + \det(x_{23,13})x_{12}x_{22} + \det(x_{12,23})x_{21}x_{32} + x_{11}x_{22}x_{23}x_{32} + x_{12}x_{21}x_{23}x_{32},$$

and the difference $2\mathrm{per}(x_{123,123})\mathrm{per}(x_{2,2}) - \mathrm{per}(x_{12,12})\mathrm{per}(x_{23,23})$ equals

$$x_{11}x_{22}^2x_{33} + \det(x_{23,23})x_{12}x_{21} + 2x_{12}x_{22}x_{23}x_{31} + x_{11}x_{22}x_{23}x_{32} + 2x_{13}x_{21}x_{22}x_{32} + 2x_{13}x_{22}^2x_{31}.$$

The authors believe that even these bounds are not tight. The smallest and greatest values we have found for the ratio in (3.15) are $2/3$ and $121/114$, respectively.

It would be interesting to characterize the ratios (1.15) which are bounded by 1, or to find examples of such ratios.

Problem 3.6. *Characterize the differences* $\mathrm{per}(x_{J_1,J'_1})\cdots\mathrm{per}(x_{J_q,J'_q}) - \mathrm{per}(x_{I_1,I'_1})\cdots\mathrm{per}(x_{I_r,I'_r})$ *which are totally nonnegative polynomials.*

Problem 3.7. *Show that the polynomial* $\mathrm{per}(x_{[n],[n]})\mathrm{per}(x_{[n+1,2n],[n+1,2n]}) - \mathrm{per}(x_{I,I})\mathrm{per}(x_{J,J})$ *is totally nonnegative when $I = [2n] \smallsetminus 2\mathbb{Z}$, $J = [2n] \cap 2\mathbb{Z}$.*

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