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Modified Macdonald polynomials and Mahonian statistics

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Abstract. We establish an equidistribution between the pairs of statistics $(\mathsf{inv}, \mathsf{maj})$ and $(\mathsf{quinv}, \mathsf{maj})$ on any row-equivalency class $[\tau]$ where τ is a filling of a given Young diagram. In particular if τ is a filling of a rectangular diagram, the triples $(\mathsf{inv}, \mathsf{quinv}, \mathsf{maj})$ and $(\mathsf{quinv}, \mathsf{inv}, \mathsf{maj})$ have the same distribution over $[\tau]$. Our main result affirms a conjecture proposed by Ayyer, Mandelshtam and Martin, thus presenting the equivalence between two refined formulas for the modified Macdonald polynomials.

Keywords: modified Macdonald polynomials, bijections, inversions, queue inversions, the major index, equidistribution

1 Introduction and main results

Macdonald polynomials $P_\lambda(X; q, t)$ indexed by partitions are polynomials in infinitely many variables $X = \{x_1, x_2, \ldots\}$ with coefficients in the field $\mathbb{Q}(q, t)$ of rational functions of two variables q and t. Several important classes of symmetric polynomials are well–studied specializations of Macdonald polynomials such as Schur polynomials (when $q = t$), Hall–Littlewood polynomials (when $q = 0$) and Jack polynomials (when $q = t^\alpha$ and let $t \to 1$).

Macdonald polynomials $P_\lambda(X; q, t)$ are defined as the unique basis for the ring of symmetric functions over the field $\mathbb{Q}(q, t)$ with orthogonal property and lower triangular property. The former is defined through the Hall scalar product and the latter by an expansion of $P_\lambda(X; q, t)$ with respect to monomial symmetric functions $m_\lambda(X)$. Since the coefficients in this expansion have nontrivial denominators, Macdonald introduced the *integral form* of $P_\lambda(X; q, t)$, denoted by $J_\lambda(X; q, t)$, which is defined as

$$J_\lambda(X; q, t) = \sum_\mu K_{\mu\lambda}(q, t) s_\mu[X(1 - t)], \tag{1.1}$$

where $f[X]$ denotes the plethystic substitution of X into the symmetric function f, s_μ is the Schur function and $K_{\mu\lambda}(q, t)$ is the q, t-Kostka numbers.

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Subsequently, another widely studied variant of Macdonald polynomials, called *modified Macdonald polynomials* $\tilde{H}_\lambda(X;q,t)$ was introduced by Garsia and Haiman [5]. Let $\tilde{K}_{\lambda\mu}(q,t) = t^{n(\mu)}K_{\lambda\mu}(q,t^{-1})$ where $n(\lambda) = \sum_i (i-1)\lambda_i$. Then

$$\tilde{H}_\lambda(X;q,t) = \sum_\mu \tilde{K}_{\mu\lambda}(q,t)s_\mu(X).$$

Haiman remarkably found that $\tilde{H}_\lambda(X;q,t)$ equals the Frobenius series of a space as the linear span of certain polynomials and their all partial derivatives [7]. In parallel, the combinatorial investigation of modified Macdonald polynomials has been greatly promoted by the celebrated breakthrough on the surprising connections between $\tilde{H}_\lambda(X;q,t)$ and Mahonian statistics on fillings of Young diagrams due to Haglund, Haiman and Loehr [6].

Before we state the combinatorial formula of $\tilde{H}_\lambda(X;q,t)$, let us review definitions of Mahonian statistics inv, quinv and maj of fillings. We represent a Young diagram in a French manner. A partition $\lambda = (\lambda_1, \cdots, \lambda_k)$ of n is sequence of positive integers such that $\lambda_i \geq \lambda_{i+1}$ for all $1 \leq i < k$ and $|\lambda| = \lambda_1 + \cdots + \lambda_k = n$. Each λ_i is called the ith part of λ and k is the length of λ, denoted by $\ell(\lambda)$. The Young diagram of λ, denoted by $\mathsf{dg}(\lambda)$, is an array of boxes with λ_i boxes in the ith row from bottom to top, with the first box in each row left-justified. A box has coordinate (i,j) if it is in the ith row from bottom to top and jth column from left to right. Let λ' be the transpose of λ, that is, $\mathsf{dg}(\lambda')$ is obtained from $\mathsf{dg}(\lambda)$ by reflecting across the main diagonal (boxes with coordinate (i,i)).

A filling of $\mathsf{dg}(\lambda)$ is a function $\sigma : \mathsf{dg}(\lambda) \to \mathbb{P}$ ($\mathbb{P}$ is the set of positive integers), which assigns each box u of $\mathsf{dg}(\lambda)$ to a positive integer $\sigma(u)$. We use $\mathsf{South}(u)$ to denote the box right below u. Let $\mathcal{T}(\lambda)$ denote the set of all fillings of $\mathsf{dg}(\lambda)$, and set

$$x^\sigma = \prod_{u \in \mathsf{dg}(\lambda)} x_{\sigma(u)}$$

be the monomial of σ. A *descent* (or *non-descent*) of a filling $\sigma \in \mathcal{T}(\lambda)$ is a pair of entries $(\sigma(x), \sigma(\mathsf{South}(x)))$ such that $\sigma(x) > \sigma(\mathsf{South}(x))$ (or $\sigma(x) \leq \sigma(\mathsf{South}(x))$). Define $\mathsf{Des}(\sigma) = \{x \in \mathsf{dg}(\lambda) : (\sigma(x), \sigma(\mathsf{South}(x)) \text{ is a descent}\}$ to be the *descent set* of σ and $\mathsf{des}(\sigma) = |\mathsf{Des}(\sigma)|$. Let $\mathsf{leg}(u)$ be the number of boxes strictly above u in its column, then

$$\mathsf{maj}(\sigma) = \sum_{u \in \mathsf{Des}(\sigma)} (\mathsf{leg}(u) + 1)$$

is called *the major index* of σ.

Given a filling σ, let $\hat{\sigma}$ be the filling obtained by adding a box with entry 0 above the topmost box of each column of σ. A *queue inversion triple* of σ is a triple (a,b,c) of entries in $\hat{\sigma}$ such that (as shown below on the left)

1. b and c are in the same row and $c \in \sigma$ is to the right of b;

2. a and b are in the same column such that b is right below a;
3. one of the conditions $a < b < c$, $b < c < a$, $c < a < b$ and $a = b \neq c$ is true.

$$\begin{array}{lll} \boxed{a} & & \\ \boxed{b} & \dots & \boxed{c} \end{array} \qquad\qquad \begin{array}{lll} \boxed{a} & \dots & \boxed{c} \\ \boxed{b} & & \end{array}$$

Let $\tilde{\sigma}$ be the filling obtained by adding a box with entry ∞ below the bottommost box of each column of σ. An *inversion triple* of σ is a triple (a, b, c) of entries in $\tilde{\sigma}$ satisfying the above (2)–(3) and (4), as shown above on the right.

4. a and c are in the same row and c is to the right of a.

Let $\mathrm{quinv}(\sigma)$ and $\mathrm{inv}(\sigma)$ be the numbers of queue inversion triples and inversion triples of σ, respectively.

We are now ready to present the combinatorial formula of $\tilde{H}_\lambda(X; q, t)$ by Haglund, Haiman and Loehr [6]:

$$\tilde{H}_\lambda(X; q, t) = \sum_{\sigma \in \mathcal{T}(\lambda)} x^\sigma q^{\mathsf{maj}(\sigma)} t^{\mathsf{inv}(\sigma)} \tag{1.2}$$

Recently, Corteel, Haglund, Mandelshtam, Mason and Williams [4, 3] discovered a compact formula for $\tilde{H}_\lambda(X; q, t)$ which is summed over sorted tableaux and made a conjecture on an equivalent form of Equation (1.2):

$$\tilde{H}_\lambda(X; q, t) = \sum_{\sigma \in \mathcal{T}(\lambda)} x^\sigma q^{\mathsf{maj}(\sigma)} t^{\mathsf{quinv}(\sigma)} \tag{1.3}$$

This conjecture was confirmed by Ayyer, Mandelshtam and Martin [1] by proving that the RHS of Equation (1.3) satisfies certain orthogonal and triangular conditions which uniquely determine the modified Macdonald polynomials $\tilde{H}_\lambda(X; q, t)$.

Interestingly, a refinement of the equivalence between Equations (1.2) and (1.3) was conjectured by Ayyer, Mandelshtam and Martin ([1, Conjecture 10.3]). Our main result is an affirmation of this conjecture; see Theorem 1.1,Equation (1.4). As a bonus of our approach, we find the equidistribution (1.5) between the triples $(\mathsf{inv}, \mathsf{quinv}, \mathsf{maj})$ and $(\mathsf{quinv}, \mathsf{inv}, \mathsf{maj})$ for all rectangular diagrams. To be precise, two fillings σ, τ of $\mathsf{dg}(\lambda)$ are called *row-equivalent*, denoted by $\sigma \sim \tau$, if the multisets of entries in the ith row of σ and τ are exactly the same for all i. The precise statement of our main result is the following.

Theorem 1.1. *Let $[\sigma]$ denote the row-equivalent class of σ, then*

$$\sum_{\tau \in [\sigma]} q^{\mathsf{maj}(\tau)} t^{\mathsf{inv}(\tau)} = \sum_{\tau \in [\sigma]} q^{\mathsf{maj}(\tau)} t^{\mathsf{quinv}(\tau)}. \tag{1.4}$$

If σ is a filling of a rectangular diagram, then

$$\sum_{\tau \in [\sigma]} q^{\mathsf{maj}(\tau)} t^{\mathsf{inv}(\tau)} u^{\mathsf{quinv}(\tau)} = \sum_{\tau \in [\sigma]} q^{\mathsf{maj}(\tau)} u^{\mathsf{inv}(\tau)} t^{\mathsf{quinv}(\tau)}. \tag{1.5}$$

It is worth pointing out that the symmetric distribution (1.5) is not true for arbitrary filling σ and we provide such an example in Remark 3.6.

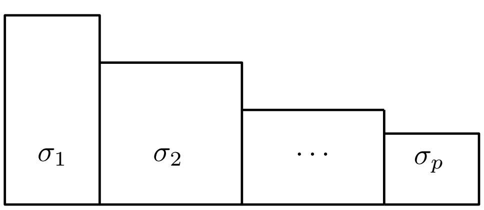
Three subsets of $[\sigma]$, respectively with extreme values of the major index or (queue) inversion numbers are shown to satisfy Equation (1.4) by Bhattacharya, Ratheesh and Viswanath [2, 12]. Their proofs are bijective, which develop novel connections between different combinatorial models, maps and statistics such that Gelfand–Tsetlin patterns, partitions overlaid patterns, box complementation [2] and charge and cocharge on words [12].

We take a different approach, highlighting that the reverse operator and a column switch operator are sufficient to prove Theorem 1.1. The rest of the extended abstract is organized as follows: In Section 2 the reverse operator and flip operator as the starting point of our proof are described. Section 3 is devoted to proving Theorem 1.1, with more details provided in [8].

2 Reverse operator and flip operator

This section is concentrated on two operators, reverse operator and a queue inversion flip operator tailored to the statistic quinv given by Ayyer, Mandelshtam and Martin [1]. The latter was inspired by the column switch operator for the statistic inv by Loehr and Niese [10].

Both operators are related to a decomposition of the Young diagram of λ into rectangles [4]. Each Young diagram $\mathsf{dg}(\lambda)$ is regarded as a concatenation of maximal rectangles in a way that the heights of rectangles are strictly decreasing from left to right. For any $\sigma \in \mathcal{T}(\lambda)$, let σ_i be the filling of the ith rectangle of $\mathsf{dg}(\lambda)$ and $\sigma = \sigma_1 \sqcup \cdots \sqcup \sigma_p$ where p is the number of rectangles of $\mathsf{dg}(\lambda)$; see Figure 2.1 for an example.



3	3	2
1	5	1
2	4	3
1	3	3

2	3	3
1	5	1
3	4	2
3	3	1

Figure 2.1: A decomposition of $\sigma = \sigma_1 \sqcup \sigma_2 \sqcup \cdots \sqcup \sigma_p$ (left), a filling σ of a rectangle diagram (middle) and its reverse (right).

Definition 2.1 (reverse operator)**.** For a partition λ and $\sigma = \sigma_1 \sqcup \cdots \sqcup \sigma_p \in \mathcal{T}(\lambda)$, define $\sigma^r = \sigma_1^r \sqcup \cdots \sqcup \sigma_p^r$ as the reverse of σ where the filling σ^r is obtained by reversing the sequence of entries of each row; see Figure 2.1.

We adopt some notations from [1, 11] to describe the flip operator. Define $\mathcal{Q}(a, b, c) = 1$ if (a, b, c) is a queue inversion triple or an inversion triple; otherwise $\mathcal{Q}(a, b, c) = 0$. We say that i is λ*-compatible* if $\lambda_i' = \lambda_{i+1}' \geq 1$.

Definition 2.2 (flip operator)**.** For $\sigma \in \mathcal{T}(\lambda)$ and λ-compatible i, let $t_i^{(r)}$ be the operator that acts on σ by interchanging the entries $\sigma(r,i)$ and $\sigma(r,i+1)$. For $1 \leq r,s \leq \lambda'_i$, let

$$t_i^{[r,s]} := t_i^{(r)} \circ t_i^{(r+1)} \cdots \circ t_i^{(s)}$$

denote the *flip operator* that swaps entries of boxes (x,i) and $(x,i+1)$ for all x with $r \leq x \leq s$. The *flip operator* ρ_i^r is defined as follows: if columns i and $i+1$ are identical in σ, then $\rho_i^r(\sigma) = \sigma$; otherwise, let k be the maximal integer such that $\sigma(k,i) \neq \sigma(k,i+1)$ and $k \leq r$. Let h be maximal such that $h \leq k$, $\sigma(h,i) \neq \sigma(h,i+1)$ and

$$\mathcal{Q}(\sigma(h,i),\sigma(h-1,i),\sigma(h-1,i+1)) = \mathcal{Q}(\sigma(h,i+1),\sigma(h-1,i),\sigma(h-1,i+1)),$$

where $\sigma(0,i) = \infty$ for all i. Define $\rho_i^r(\sigma) = t_i^{[h,k]}$, that is, reverse the pair of entries in every row between rows h and k, columns i and $i+1$. We call row k (or h) *the starting row* (or *the ending row*) of ρ_i^r. For simplicity, we denote

$$\rho_i = \rho_i^{\lambda'_i} \quad \text{and} \quad t_i = t_i^{(\lambda'_i)}.$$

By definition $\rho_i^r \circ \rho_i^r(\sigma) = \sigma$, that is, ρ_i^r is an involution on $\mathcal{T}(\lambda)$.

Remark 2.3. We point out two differences between flip operator in Definition 2.2 and the queue inversion flip operator introduced in [1, 11]. First the latter always starts from the topmost row, that is ρ_i, while we allow the flip operator to begin from any row. Second, we add the condition $\sigma(h,i) \neq \sigma(h,i+1)$ to the terminating row h. These two modifications are intended to develop a precise relation between the change of quinv, inv and $\mathcal{N}\mathsf{des}$ (see Theorem 3.4), which contributes to the proof of Theorem 1.1.

Example 2.4. Given a filling σ as below, $\rho_1(\sigma) = t_1^{[3,5]}(\sigma)$ is generated as follows. Since $\begin{array}{|c|c|}\hline 9 & 3 \\ \hline\end{array}$ is the topmost row with different entries, the flipping process ρ_1 starts from this row, i.e., row 5 and continues until row 3, where $\mathcal{Q}(3,5,8) = \mathcal{Q}(9,5,8) = 1$. Consequently $\rho_1 = t_1^{[3,5]}$.

$$\begin{array}{|c|c|}\hline 3 & 3 \\ \hline 9 & 3 \\ \hline 2 & 5 \\ \hline 3 & 9 \\ \hline 5 & 8 \\ \hline 9 & 3 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 3 & 9 \\ \hline 2 & 5 \\ \hline 3 & 9 \\ \hline 5 & 8 \\ \hline 9 & 3 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 3 & 9 \\ \hline 5 & 2 \\ \hline 3 & 9 \\ \hline 5 & 8 \\ \hline 9 & 3 \\ \hline\end{array} \to \begin{array}{|c|c|}\hline 3 & 3 \\ \hline 3 & 9 \\ \hline 5 & 2 \\ \hline 9 & 3 \\ \hline 5 & 8 \\ \hline 9 & 3 \\ \hline\end{array}$$

3 Our proof strategy

The proof of Theorem 1.1 is bijective, namely, for a partition λ, we will construct a bijection $\varphi : \mathcal{T}(\lambda) \to \mathcal{T}(\lambda)$ satisfying

$$(\mathsf{quinv},\mathsf{maj})(\varphi(\sigma)) = (\mathsf{inv},\mathsf{maj})(\sigma). \tag{3.1}$$

In particular, if $\mathsf{dg}(\lambda)$ is a rectangle, we prove that

$$(\mathsf{inv}, \mathsf{quinv}, \mathsf{maj})(\varphi(\sigma)) = (\mathsf{quinv}, \mathsf{inv}, \mathsf{maj})(\sigma);$$

otherwise, we find a filling σ such that Equation (1.5) is no longer true (see Remark 3.6).

The bijection φ is a composition of two bijections associated with the flip operator ρ_i^k. The first one γ is described in Theorem 3.1 (see below), which is reduced to the reverse operator if $\mathsf{dg}(\lambda)$ is a rectangle. Let $\sigma = \sigma_1 \sqcup \cdots \sqcup \sigma_p$, define

$$\kappa(\sigma) := \sum_{i=1}^{p} (\mathsf{quinv}(\sigma_i) - \mathsf{inv}(\sigma_i^r)). \tag{3.2}$$

For any filling τ, τ_1 represents the leftmost rectangle in the decomposition of τ. Define $\mathcal{N}\mathsf{des}(\tau) = (a_1, \ldots, a_k)$ where a_i counts the number of non-descents in column i of τ and set $\mathsf{ndes}(\tau) = a_1 + \cdots + a_k$ be the number of non-descents of τ.

Theorem 3.1. *There is a bijection $\gamma : \mathcal{T}(\lambda) \to \mathcal{T}(\lambda)$ satisfying $\gamma(\sigma) \sim \sigma$,*

$$\mathsf{quinv}(\gamma(\sigma)) = \mathsf{inv}(\sigma) + \kappa(\gamma(\sigma)), \tag{3.3}$$
$$\mathsf{maj}(\gamma(\sigma)) = \mathsf{maj}(\sigma), \tag{3.4}$$
$$\mathcal{N}\mathsf{des}(\sigma_1) = \mathcal{N}\mathsf{des}((\gamma(\sigma)_1)^r) \tag{3.5}$$

and the topmost rows of σ and $\gamma(\sigma)$ are reverse of each other.

The second bijection $\theta : \mathcal{T}(\lambda) \to \mathcal{T}(\lambda)$ acts on each rectangle of the fillings independently and decreases the number of queue inversions by $\kappa(\gamma(\sigma))$ but preserves the major index, by which we find the desired bijection φ with property (3.1).

3.1 The proof of Theorem 1.1

In this first step, we discuss the change of statistics quinv, inv and maj by the reverse operator and the flip operator in Lemmas 3.2 and 3.3.

Lemma 3.2. *For $\lambda = (n^m)$ and $\sigma \in \mathcal{T}(\lambda)$, let x_i be the number of non-descents in column i of σ. Then, we have*

$$\mathsf{quinv}(\sigma) - \mathsf{inv}(\sigma^r) = \mathsf{inv}(\sigma) - \mathsf{quinv}(\sigma^r) = \sum_{i=1}^{n} x_i(n - 2i + 1). \tag{3.6}$$

Lemma 3.3. *For a partition λ and a λ-compatible i, let $\sigma \in \mathcal{T}(\lambda)$ and suppose that $\rho_i^r = t_i^{[\kappa_1,\kappa_2]}$. Let $\begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline s & t \\ \hline u & v \\ \hline\end{array}$ be parts of σ such that $\begin{array}{|c|c|}\hline c & d \\ \hline\end{array}$ is the starting row κ_2 and $\begin{array}{|c|c|}\hline s & t \\ \hline\end{array}$ is the ending row κ_1. Set $\sigma(0, i) = \infty$ and $\sigma(\lambda_i' + 1, i) = 0$ for all i. If $\mathcal{Q}(a, c, d) = \mathcal{Q}(b, c, d) = 0$, then*

$$\mathsf{quinv}(\sigma) + 1 = \mathsf{quinv}(\rho_i^r(\sigma)).$$

Equivalently, if $\mathcal{Q}(a,c,d) = \mathcal{Q}(b,c,d) = 1$, *then*

$$\mathsf{quinv}(\sigma) - 1 = \mathsf{quinv}(\rho_i^r(\sigma)).$$

If $\mathcal{Q}(s,u,t) = \mathcal{Q}(s,v,t) = 0$, *then*

$$\mathsf{inv}(\sigma) + 1 = \mathsf{inv}(\rho_i^r(\sigma)).$$

Equivalently, if $\mathcal{Q}(s,u,t) = \mathcal{Q}(s,v,t) = 1$, *then*

$$\mathsf{inv}(\sigma) - 1 = \mathsf{inv}(\rho_i^r(\sigma)).$$

For all cases, i.e., $\mathcal{Q}(a,c,d) = \mathcal{Q}(b,c,d)$ *or* $\mathcal{Q}(s,u,t) = \mathcal{Q}(s,v,t)$, *we have*

$$\mathsf{maj}(\sigma) = \mathsf{maj}(\rho_i^r(\sigma)).$$

In the second step, with the help of Lemmas 3.2 and 3.3, we establish the involution ϕ_i in Theorem 3.4, which relates the change of inv and quinv with the number of non-descent pairs.

Theorem 3.4. *For a partition* λ *and a* λ*-compatible* i, *let* $\sigma \in \mathcal{T}(\lambda)$ *and* x_i *be the number of non-descents in the ith column of* σ. *Then there is an involution* $\phi_i : \mathcal{T}(\lambda) \to \mathcal{T}(\lambda)$ *such that* $\phi_i(\sigma) \sim \sigma$, *and for* $\nu \in \{\mathsf{inv}, \mathsf{quinv}\}$,

$$\mathsf{maj}(\phi_i(\sigma)) = \mathsf{maj}(\sigma), \tag{3.7}$$

$$\nu(\phi_i(\sigma)) = \nu(\sigma) + x_{i+1} - x_i, \tag{3.8}$$

$$\mathcal{N}\mathsf{des}(\phi_i(\sigma)) = (i, i+1) \circ \mathcal{N}\mathsf{des}(\sigma), \tag{3.9}$$

where $(i, i+1) \circ (\ldots x_i, x_{i+1} \ldots) = (\ldots x_{i+1}, x_i \ldots)$.

In the last step, building upon the properties Equation (3.7)–(3.9) of ϕ_i, we construct the bijections θ and γ, thereby completing the proof of Theorem 1.1.

Theorem 3.5 (second part of Theorem 1.1)**.** *If the diagram of* λ *is a rectangle, then there is a bijection* $\theta : \mathcal{T}(\lambda) \to \mathcal{T}(\lambda)$ *such that* $(\mathsf{inv}, \mathsf{quinv}, \mathsf{maj})\sigma = (\mathsf{quinv}, \mathsf{inv}, \mathsf{maj})\theta(\sigma)$.

Proof. For $\lambda = (n^m)$ and $\sigma \in \mathcal{T}(\lambda)$, let x_i be the number of non-descents in column i. Then $\mathcal{N}\mathsf{des}(\sigma^r) = (x_n, \ldots, x_1)$. Define a bijection $\theta : \mathcal{T}(\lambda) \to \mathcal{T}(\lambda)$ as a product of ϕ_i's as follows. In the first step, apply the bijections ϕ_i for i from $n-1$ to 1 on σ^r, let $\tau_1 = \phi_1 \circ \cdots \circ \phi_{n-1}(\sigma^r)$. Theorem 3.4 ensures that the number of queue inversion triples is increased by

$$\nu(\tau_1) - \nu(\sigma^r) = \sum_{i=2}^{n}(x_1 - x_i) = (n-1)x_1 - \sum_{i=2}^{n} x_i$$

for $\nu \in \{\mathsf{inv}, \mathsf{quinv}\}$ and $\mathcal{N}\mathsf{des}(\tau_1) = (x_1, x_n, \ldots, x_2)$. In the next step, we apply the bijections ϕ_i for i from $n-1$ to 2 on τ_1 and let $\tau_2 = \phi_2 \circ \cdots \circ \phi_{n-1}(\tau_1)$, yielding that the number of queue inversion triples is increased by $(n-2)x_2 - (x_3 + \cdots + x_n)$ and $\mathcal{N}\mathsf{des}(\tau_2) = (x_1, x_2, x_n \ldots, x_3)$. Continue this process until the sequence $\mathcal{N}\mathsf{des}$ of the image becomes $(x_1, x_2, \ldots, x_n)$. Denote the resulting filling by $\theta(\sigma)$ and one sees that $\mathsf{maj}(\theta(\sigma)) = \mathsf{maj}(\sigma)$ by Equation (3.7). Further,

$$\begin{aligned} \nu(\theta(\sigma)) &= \nu(\sigma^r) + \sum_{i=1}^{n-1}(n-i)x_i - \sum_{i=2}^{n}(i-1)x_i \\ &= \nu(\sigma^r) + \sum_{i=1}^{n} x_i(n-2i+1). \end{aligned}$$

Compare with Equation (3.6), we conclude that $(\mathsf{inv}, \mathsf{quinv})(\sigma) = (\mathsf{quinv}, \mathsf{inv})(\theta(\sigma))$, as claimed. □

Remark 3.6. In the table below, we present a row-equivalent filling class $[\sigma]$ of a non-rectangular diagram to disprove Equation (1.5) for arbitrary fillings.

$[\sigma]$	$\begin{array}{ccc} 3 & & \\ 4 & 1 & 2 \\ 3 & 3 & 3 \end{array}$	$\begin{array}{ccc} 3 & & \\ 4 & 2 & 1 \\ 3 & 3 & 3 \end{array}$	$\begin{array}{ccc} 3 & & \\ 1 & 2 & 4 \\ 3 & 3 & 3 \end{array}$	$\begin{array}{ccc} 3 & & \\ 1 & 4 & 2 \\ 3 & 3 & 3 \end{array}$	$\begin{array}{ccc} 3 & & \\ 2 & 4 & 1 \\ 3 & 3 & 3 \end{array}$	$\begin{array}{ccc} 3 & & \\ 2 & 1 & 4 \\ 3 & 3 & 3 \end{array}$
maj	2	2	2	2	2	2
inv	0	1	2	1	2	3
quinv	3	2	2	1	0	1

We proceed by establishing Theorem 1.1 via Theorem 3.1 and Theorem 3.5.

Proof of Theorem 1.1. For any $\sigma \in \mathcal{T}(\lambda)$, let $\tau = \gamma(\sigma)$, consider the rectangle decomposition of $\tau = \tau_1 \sqcup \cdots \sqcup \tau_p$, let $\pi = \theta(\tau_1^r) \sqcup \cdots \sqcup \theta(\tau_p^r)$ and we shall see that $\varphi(\sigma) = \pi$ satisfying $\mathsf{quinv}(\pi) = \mathsf{inv}(\sigma)$ and $\mathsf{maj}(\pi) = \mathsf{maj}(\sigma)$. Theorem 3.5 assures that

$$\mathsf{quinv}(\pi) - \mathsf{quinv}(\tau) = \sum_{i=1}^{p}(\mathsf{quinv}(\theta(\tau_i^r)) - \mathsf{quinv}(\tau_i)) = \sum_{i=1}^{p}(\mathsf{inv}(\tau_i^r) - \mathsf{quinv}(\tau_i)) = -\kappa(\tau).$$

In combination of $\mathsf{quinv}(\tau) - \mathsf{inv}(\sigma) = \kappa(\tau)$ by Theorem 3.1, we conclude that $\mathsf{quinv}(\pi)$ equals $\mathsf{inv}(\sigma)$. Furthermore, $\mathsf{maj}(\pi) = \mathsf{maj}(\sigma)$ follows directly by Theorem 3.1 and Theorem 3.5, as wished. □

Example 3.7. Let $\lambda = (7,7,5,5,5,2)$, a filling σ of $\mathsf{dg}(\lambda)$ and the corresponding $\varphi(\sigma)$

are displayed as below left and right, respectively.

5	4					
9	3	6	1	3		
2	5	9	4	8		
3	9	7	3	5		
5	8	4	6	4	8	7
9	3	6	5	2	10	1

4	5					
6	3	1	3	9		
8	4	5	9	2		
3	3	5	7	9		
7	8	4	4	6	8	5
10	1	6	2	5	9	3

One can easily check that $(\mathsf{inv}, \mathsf{maj})(\sigma) = (\mathsf{quinv}, \mathsf{maj})(\varphi(\sigma)) = (40, 33)$, but $\mathsf{quinv}(\sigma) \neq \mathsf{inv}(\varphi(\sigma))$ as $\mathsf{quinv}(\sigma) = 32$ and $\mathsf{inv}(\varphi(\sigma)) = 34$.

4 Discussion

We propose some interesting open questions. A *column strict filling* (CSF) of a diagram is a filling whose entries along each column are strictly decreasing from top to bottom. In other words, the major index of a column strict filling reaches its maximal value.

Bhattacharya, Ratheesh, and Viswanath [2] found the symmetric distribution (1.5) for all the column strict fillings of an arbitrary diagram. That is, let $\mathsf{CSF}(\lambda)$ be the set of CSFs of the diagram of λ, then

$$\sum_{\tau \in [\sigma] \cap \mathsf{CSF}(\lambda)} t^{\mathsf{inv}(\tau)} u^{\mathsf{quinv}(\tau)} = \sum_{\tau \in [\sigma] \cap \mathsf{CSF}(\lambda)} u^{\mathsf{inv}(\tau)} t^{\mathsf{quinv}(\tau)}. \tag{4.1}$$

Open problem 4.1. The equation (4.1) is not directly seen from our bijection φ. It would be interesting to find out whether the bijection φ is applicable to prove Equation (4.1).

Combining Theorem 1.1 and results by Loehr and Niese [10], we are led to

$$\sum_{\tau \in [\sigma]} t^{\mathsf{maj}(\tau')} = \sum_{\tau \in [\sigma]} t^{\mathsf{inv}(\tau)} = \sum_{\tau \in [\sigma]} t^{\mathsf{quinv}(\tau)} = \prod_{i=1}^{\ell(\lambda)} \begin{bmatrix} a_{i,1} + \cdots + a_{i,N} \\ a_{i,1}, \ldots, a_{i,N} \end{bmatrix}_t, \tag{4.2}$$

where τ' is obtained by transposing the filling τ, the ith row of σ consists of $a_{i,1}$ copies of 1, $a_{i,2}$ copies of 2, etc, and the rightmost one is a product of t-multinomial coefficients (see for instance [9, 13]).

Open problem 4.2. Let $G(q, t, u)$ be the LHS of Equation (1.5), can one derive a formula of $G(q, t, u)$ with the symmetric property $G(q, t, u) = G(q, u, t)$? Is there a formula for Equation (1.4) as a natural generalization of Equation (4.2)?

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