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Skew shaped positroids and double Bott–Samelson varieties

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Abstract. Skew shaped positroid varieties are subvarieties of the Grassmannian defined by determinantal equations, and are special cases of open positroid varieties. Double Bott–Samelson varieties are algebraic varieties defined in terms of configurations of flags depending on a positive braid word. We explicitly realize every skew shaped positroid in $\mathrm{Gr}(k,n)$ as a double Bott–Samelson variety, and use this to construct splicing maps for skew shaped positroid varieties, generalizing those constructed by the first and third authors in the case of maximal positroid cells.

Keywords: skew shaped positroids, double Bott–Samelson varieties, cluster algebras, quasi-cluster isomorphism

1 Introduction

In this paper, we study the relation between two classes of varieties admitting a cluster structure: *skew shaped positroids*, which are a special subclass of positroid varieties in the Grassmannian [12] (see also [15]); and *double Bott–Samelson varieties* introduced by Elek–Lu [8, 2]. The cluster structures we consider were constructed by Galashin–Lam [6] for the case of skew shaped positroids, and by Shen–Weng [17] for double Bott–Samelson varieties.

Let us fix positive integers $0 < k < n$, and consider two partitions $\mu \subseteq \lambda$ both fitting inside a $k \times (n-k)$-rectangle. Associated to the pair $(\mu \subseteq \lambda)$, the skew shaped positroid variety $S^\circ_{\lambda/\mu}$ is an affine subvariety of the Grassmannian $\mathrm{Gr}(k,n)$ defined by the vanishing of certain minors, and the non-vanishing of other minors. These minors are determined by the skew diagram λ/μ, and the skew shaped positroid $S^\circ_{\lambda/\mu}$ is in fact

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a positroid variety in the sense of [6]. By the results of that paper, $\mathbb{C}[S^\circ_{\lambda/\mu}]$ admits a cluster algebra structure. In a nutshell, this means that the structure of the coordinate algebra $\mathbb{C}[S^\circ_{\lambda/\mu}]$ can be codified into the combinatorics of a quiver and its mutations, see Section 2 for details.

On the other hand, for each positive braid $\beta \in \mathrm{Br}_k^+$, the double Bott–Samelson variety $\mathrm{BS}(\beta)$ is defined as a space of flag configurations dictated by the braid β, see Definition 4.1 for a precise definition. By the work of several authors, see [2, 8], the coordinate algebra $\mathbb{C}[\mathrm{BS}(\beta)]$ admits a cluster algebra structure, that was made explicit in [17] and later generalized to the setting of *braid varieties* in [1] and [7].

Theorem 1.1. *Let $n > 0$ and $0 < k < n$. For any two partitions $\mu \subseteq \lambda$ fitting inside a $k \times (n-k)$-rectangle, there exist a k-stranded braid $\beta_{\lambda/\mu} \in \mathrm{Br}_k^+$, $s \geq 0$ and an isomorphism:*

$$\mathrm{BS}(\beta_{\lambda/\mu}) \times (\mathbb{C}^\times)^s \cong S^\circ_{\lambda/\mu}.$$

Moreover, the isomorphism can be chosen so that the natural cluster structure in $\mathrm{BS}(\beta_{\lambda/\mu}) \times (\mathbb{C}^\times)^s$ is quasi-isomorphic to the cluster structure in $S^\circ_{\lambda/\mu}$ defined in [6].

We refer the reader to Section 2 for the notion of a cluster quasi-isomorphism. Roughly speaking, a quasi-isomorphism preserves all the geometric information of a cluster algebra, while possibly changing the combinatorics.

As an application of Theorem 1.1, we provide a splicing map for skew shaped positroids, generalizing that constructed in [11] for the maximal positroid cell. More precisely, we show that double Bott–Samelson varieties admit splicing maps, which are predicted by results in link homology and Fock–Goncharov amalgamation [3, 17].

Theorem 1.2 ([9])**.** *Let $\beta = \beta_1\beta_2 \in \mathrm{Br}_n^+$ be a positive braid. Then, there exists an open set $U(\beta_1, \beta_2) \subseteq \mathrm{BS}(\beta)$ admitting a cluster structure induced from that on $\mathrm{BS}(\beta)$ and a cluster quasi-isomorphism*

$$U(\beta_1, \beta_2) \cong \mathrm{BS}(\beta_1) \times \mathrm{BS}(\beta_2).$$

We combine Theorems 1.1 and 1.2 as follows. Let $\mu \subseteq \lambda$ be partitions fitting inside a $k \times (n-k)$ rectangle, and choose $1 \leq j \leq n-k$. Let $\lambda^{1,j}$ be the partition obtained by considering only the first j columns of λ, and define $\mu^{1,j}$ similarly. Note that $\mu^{1,j} \subseteq \lambda^{1,j}$ are partitions fitting inside a $k \times j$-rectangle, so we can consider the skew shaped positroid $S_{\lambda^{1,j}/\mu^{1,j}} \subseteq \mathrm{Gr}(k, k+j)$. Now let $\lambda^{2,j}$ be the partition obtained by considering the $j+1, \dots, n-k$-columns of λ, and similarly for $\mu^{2,j}$. Note that $\mu^{2,j} \subseteq \lambda^{2,j}$ fit inside a $k \times (n-k-j)$-rectangle, so we can consider the skew shaped positroid $S^\circ_{\lambda^{2,j}/\mu^{2,j}} \subseteq \mathrm{Gr}(k, n-j)$.

Theorem 1.3. *Let $n > 0$ and $0 < k < n$. Let $\mu \subseteq \lambda$ be partitions fitting inside a $k \times (n-k)$-rectangle. Choose $1 \leq j \leq n-k$, and let $U_j(S^\circ_{\lambda/\mu})$ be the principal open set in $S^\circ_{\lambda/\mu}$ defined by the non-vanishing of the cluster variables in the j-th column of λ/μ. Then,*

$$U_j(S^\circ_{\lambda/\mu}) \cong S^\circ_{\lambda^{1,j}/\mu^{1,j}} \times S^\circ_{\lambda^{2,j}/\mu^{2,j}}.$$

Notation. If n is a positive integer, we denote by $[n]$ the set $\{1, \ldots, n\}$.

2 Cluster algebras

A ***quiver*** is a directed graph Q on a finite edge set that permits multiple edges but does not allow directed cycles of length one or two. An ***ice quiver*** is obtained by specifying each vertex of Q as either ***mutable*** or ***frozen***. A ***seed*** Σ is a quiver where all vertices are labeled with algebraically independent variables called ***cluster variables***, which are deemed mutable or frozen according to the vertex they label. A ***mutation*** is a local operation that can be performed at a mutable vertex of Σ that takes Σ to another seed Σ'. After mutating at the jth vertex of Σ, the associated cluster variable x_j changes to x'_j where x_j and x'_j are related by $x_j x'_j = A_j + B_j$, where $A = \prod_{i \to j} x_i^{\#\{i \to j\}}$ and $B = \prod_{j \to k} x_k^{\#\{j \to k\}}$. We define the ***exchange ratio*** $\widehat{y}_j$ as the ratio $\frac{A_j}{B_j}$. We denote $\Sigma \sim \Sigma'$ if Σ' can be obtained from Σ after performing a finite sequence of mutations.

The ***cluster algebra*** $\mathcal{A}(\Sigma)$ is defined to be the algebra generated by all cluster variables in all possible seeds obtained from mutations on Σ. If $\Sigma \sim \Sigma'$ then $\mathcal{A}(\Sigma)$ and $\mathcal{A}(\Sigma')$ are isomorphic. For more details on cluster algebras, see [4].

We say that an affine algebraic variety X is a ***cluster variety*** if there exists some seed Σ such that X is isomorphic to $Spec(\mathcal{A}(\Sigma))$. In our setting, skew shaped positroids and double Bott–Samelson varieties are all cluster varieties, [6, 17].

A commutative algebra $\mathcal{A}$ may admit more than one cluster structure, that is, there may exist two seeds Σ, Σ' which are not mutation equivalent but such that $\mathcal{A}(\Sigma) \cong \mathcal{A} \cong \mathcal{A}(\Sigma')$. Following Fraser [5], see also [13, Section 5.2], we define a class of morphisms between cluster algebras of perhaps non-mutation-equivalent seeds. We say that two cluster structures $\mathcal{A}(\Sigma) \cong \mathcal{A} \cong \mathcal{A}(\Sigma')$ on the commutative algebra $\mathcal{A}$ are ***quasi-cluster equivalent*** if:

1. Both Σ and Σ' have the same number of mutable as well as of frozen variables.

2. Each frozen variable in Σ' is a Laurent monomial in the frozen variables of Σ, and vice versa.

3. Mutable variables coincide up to multiplication by frozen variables.

4. The exchange ratios coincide, i.e., $\widehat{y}_i = \widehat{y}'_i$ for $i \in [r]$.

Additionally, we say that two cluster varieties $Spec(\mathcal{A}(\Sigma))$ and $Spec(\mathcal{A}(\Sigma'))$ are ***quasi-isomorphic*** if there is an isomorphism between $Spec(\mathcal{A}(\Sigma))$ and $Spec(\mathcal{A}(\Sigma'))$ and also Σ is quasi-cluster equivalent to Σ'.

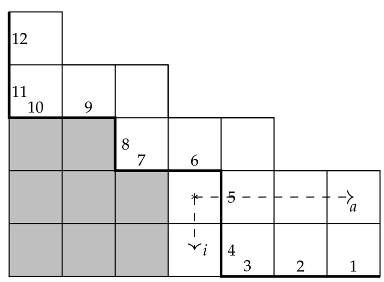
3 Skew shaped positroid varieties

We fix $0 < k < n$ and consider a partition λ whose Young diagram fits inside an $(n-k) \times k$-rectangle, that is, $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \leq n-k$. We draw partitions using the French convention, i.e., the boxes are lower left justified. Given partitions λ and μ with $\mu_i \leq \lambda_i$ for all $1 \leq i \leq k$, the ***skew diagram*** λ/μ denotes the set-theoretic difference of the Young diagrams of λ and μ.

Given $1 \leq a \leq n-k, 1 \leq i \leq k$ we define the square $\square_{a,i}$ as the i-th box from the bottom in the a-th column from the right. Let $R(\lambda/\mu)$ denote the ribbon shaped boundary of λ/μ: a box $\square \in \lambda/\mu$ belongs to $R(\lambda/\mu)$ if the box northeast of $\square$ does *not* belong to λ/μ. For simplicity, we will assume that $R(\lambda/\mu)$ is connected. For discussions regarding the general case, see [10].

Given a square $\square_{a,i} \in (n-k)^k$, we define $\mu_{a,i}$ as the minimal Young diagram such that $\mu \subseteq \mu_{a,i}$ and $\square_{a,i} \in \mu_{a,i}$. Note that $\mu_{a,i} = \mu$ if and only if $\square_{a,i} \in \mu$, and if $\square_{a,i} \in \lambda/\mu$ then $\mu_{a,i} \subseteq \lambda$. We will identify $\mu_{a,i}$ with its ***boundary path*** $P_{a,i}$ from the southeast to the northwest corner of the $(n-k) \times k$-rectangle. We label all the steps of $P_{a,i}$ with numbers from 1 to n, starting at the southeast corner and increasing as we go west and north. The ***label*** of $\square_{a,i}$ is the k-element subset $I'(\square_{a,i}) \subseteq [n]$ consisting of the labels of the vertical steps of $P_{a,i}$. We denote by $I_\mu := I'(\square_{a,i})$ for any square $\square_{a,i} \in \mu$.

Example 3.1. Our running example will be $n = 12, k = 5$, $\lambda = (7,7,5,3,1)$ and $\mu = (3,3,2) \subseteq \lambda$. The next figure shows that $I'(\square_{4,2}) = \{4,5,8,11,12\}$.



Definition 3.2. Given $\mu \subseteq \lambda$ as above, the *skew shaped positroid variety* $S^\circ_{\lambda/\mu} \subseteq \mathrm{Gr}(k,n)$ consists of those subspaces V admitting a matrix representative $M_V \in \mathrm{Mat}_{k\times n}(\mathbb{C})$ satisfying the following conditions:

1. $\Delta_{I'(a,i)}(M_V) = 0$ for $\square_{a,i} \in (n-k)^k/\lambda$.
2. $\Delta_{I'(a,i)}(M_V) \neq 0$ for $\square_{a,i} \in R(\lambda/\mu)$.
3. $\Delta_{I_\mu}(M_V) \neq 0$.

By Condition (3), **we can and will restrict to the affine chart of** $\mathrm{Gr}(k,n)$ **where** $\Delta_{I_\mu} = 1$**.**

By definition, $S^\circ_{\lambda/\mu}$ is a locally closed subset of $\mathrm{Gr}(k,n)$. In fact, any skew shaped positroid is in fact a *positroid variety* inside the Grassmannian. Introduced by Knutson–Lam–Speyer [12], positroid varieties provide a stratification of the Grassmannian. By the work of Galashin–Lam [6], the coordinate ring of open positroid variety is a cluster algebra, and hence coordinate ring of a skew shaped posiroid admits a cluster algebra structure. We simplify their initial seed for the case of skew shaped positroids as follows.

Theorem 3.3. *Consider a skew diagram λ/μ. An initial seed for the cluster structure on $S^\circ_{\lambda/\mu}$ can be described as follows:*

Cluster variables: *For each box $\square_{a,i} \in \lambda/\mu$, we have a cluster variable $x_{a,i} := \Delta_{I'(a,i)}$.*

Frozen variables: *A variable $x_{a,i}$ is frozen if the box $\square_{a,i}$ belongs to the ribbon shaped boundary $R(\lambda/\mu)$.*

Quiver: *The vertices of the quiver $Q_{\lambda/\mu}$ are in bijection with the boxes $\square_{a,i} \in \lambda/\mu$. There are arrows:*

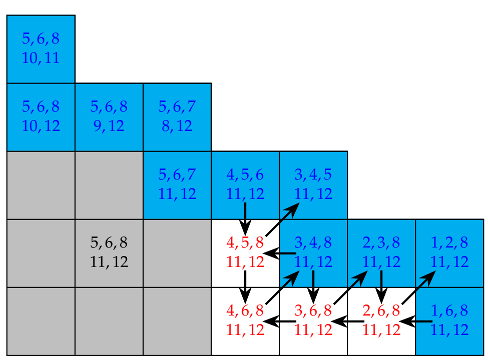
$$\square_{a,i} \to \square_{a+1,i}, \qquad \square_{a,i} \to \square_{a,i-1}, \qquad \square_{a,i} \to \square_{a-1,i+1}$$

provided at least one of these boxes corresponds to a mutable vertex.

Remark 3.4. In fact, the sets $I'(a,i)$ for $\square_{a,i} \in R(\lambda/\mu)$, together with I_μ form a *source Grassmann necklace*. Consequently, we also get a bounded affine permutation. See [12] for details.

Remark 3.5. In part of the literature, there is an extra frozen variable $x_\mu = \Delta_{I_\mu}$, and consequently the quiver $Q_{\lambda/\mu}$ has an extra frozen vertex. This yields a cluster structure on the affine cone over $S^\circ_{\lambda/\mu}$. Since we work inside $\mathrm{Gr}(k,n)$ by setting $\Delta_{I_\mu} = 1$, we do not see this frozen variable.

Example 3.6. Continuing with n, k, λ and μ as in our running Example 3.1 we obtain the following quiver, where the frozen variables are drawn in blue, and the mutable variables in red.



Remark 3.7. Skew shaped positroids differ from the skew Schubert varieties of [16], and are closely related to Grassmannian Richardson varieties, see [10, Remark 3.2.7].

4 Double Bott–Samelson varieties

We now define double Bott–Samelson varieties. We will work with the positive braid monoid on k strands, i.e. Br_k^+ is the monoid with generators $\sigma_1, \ldots, \sigma_{k-1}$ satisfying the usual braid relations: $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| > 1$ and $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$. We will picture an element $\beta \in \mathrm{Br}_k^+$ via its ***wiring diagram***, obtained by taking k horizontal strands, numbered from bottom to top, representing a generator σ_i by a crossing between the i-th and $(i+1)$-st strands, and concatenating left-to-right as we read β in the same direction.

We will also need the flag variety $\mathcal{F}(k)$ of complete flags in $\mathbb{C}^k$. If $F_1^\bullet = (0 \subseteq F_1^1 \subseteq \cdots \subseteq F_1^k = \mathbb{C}^k)$ and $F_2^\bullet$ are two flags in $\mathbb{C}^k$, and $j \in \{1, \ldots, k-1\}$, then we say that $F_1^\bullet$ is in position s_j with respect to $F_2^\bullet$ if

$$F_1^j \neq F_2^j, \qquad \text{and} \qquad F_1^i = F_2^i \text{ for } i \neq j.$$

we write $F_1^\bullet \xrightarrow{j} F_2^\bullet$ to express that $F_1^\bullet$ is in position s_j with respect to $F_2^\bullet$. On the other hand, we say that the flags $F_1^\bullet$ and $F_2^\bullet$ are transverse if, for each $j = 1, \ldots, k-1$

$$F_1^j \cap F_2^{k-j} = \{0\}, \qquad \text{equivalently} \qquad F_1^j + F_2^{k-j} = \mathbb{C}^k.$$

We write $F_1^\bullet \pitchfork F_2^\bullet$ to express the fact that $F_1^\bullet$ and $F_2^\bullet$ are transverse. Finally, we will need two special flags: the standard and antistandard flags, defined by

$$\begin{aligned} F_{\mathrm{std}}^\bullet &= \{0\} \subseteq \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \cdots \subseteq \langle e_1, e_2, \ldots, e_k \rangle = \mathbb{C}^k, \\ F_{\mathrm{ant}}^\bullet &= \{0\} \subseteq \langle e_k \rangle \subseteq \langle e_k, e_{k-1} \rangle \subseteq \cdots \subseteq \langle e_k, e_{k-1}, \ldots, e_1 \rangle = \mathbb{C}^k. \end{aligned}$$

Definition 4.1. Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}_+^k$. We define the *double Bott–Samelson variety* $\mathrm{BS}(\beta)$ as follows:

$$\mathrm{BS}(\beta) = \{(F_0^\bullet, \ldots, F_\ell^\bullet) \in \mathcal{F}(k)^{\ell+1} \mid F_0^\bullet = F_{\mathrm{std}}^\bullet \xrightarrow{i_1} F_1^\bullet \xrightarrow{i_2} \cdots \xrightarrow{i_\ell} F_\ell^\bullet \pitchfork F_{\mathrm{ant}}^\bullet\}.$$

In fact, for any braid $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell}$, the double Bott–Samelson variety $\mathrm{BS}(\beta)$ is a smooth, affine algebraic variety, that is a principal open set in the affine space $\mathbb{C}^\ell$. To see this, let us give coordinates to the double Bott–Samelson variety $\mathrm{BS}(\beta)$. We identify the flag variety $\mathcal{F}(k)$ with the quotient $\mathrm{GL}(k)/B(k)$, where $B(k) \subseteq \mathrm{GL}(k)$ is the group of upper triangular matrices, as follows. If $M = (m_1|m_2|\cdots|m_k) \in \mathrm{GL}(k)$ has columns $m_1, \ldots, m_k$, we define the flag

$$F_M^\bullet = \{0\} \subseteq \langle m_1 \rangle \subseteq \langle m_1, m_2 \rangle \subseteq \cdots \subseteq \langle m_1, m_2, \ldots, m_k \rangle = \mathbb{C}^k.$$

Note that $F_M^\bullet = F_N^\bullet$ if and only if there exists an upper-triangular matrix U such that $M = NU$, and this provides the identification $\mathcal{F}(k) \cong \mathrm{GL}(k)/B(k)$. Now we need the following results. Recall that if $I, J \subseteq [k]$ are sets of the same size and $M \in \mathrm{GL}(n)$, then the minor $\Delta_{I,J}(M)$ is the determinant of the $|I| \times |J|$-submatrix of M obtained by deleting all rows (resp. columns) that do not belong to I (resp. J).

Lemma 4.2. *Let $M \in \mathrm{GL}(k)$. Then $F^\bullet_M \pitchfork F^\bullet_{ant}$ if and only if for each $i = 1, \dots, k$, the minor $\Delta_{[i],[i]}(M)$ is nonzero.*

Lemma 4.3. *Let $M \in \mathrm{GL}(k)$ and let $j \in [k-1]$. The space of all flags $F^\bullet \in \mathcal{F}(k)$ such that $F^\bullet_M \xrightarrow{j} F^\bullet$ forms an affine line $\mathbb{C}$, and consists of all the flags of the form $F^\bullet_{MB_j(z)}$, where $z \in \mathbb{C}$ and $B_j(z)$ is the matrix*

$$B_j(z) = I_k + (z-1)E_{i,i} - E_{i+1,i+1} - E_{i,i+1} + E_{i+1,1}$$

where $E_{r,s}$ is the matrix with a 1 on the (r,s)-position and zeroes everywhere else.

If $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}^+_k$ is a positive braid and $(z_1, \dots, z_\ell) \in \mathbb{C}^\ell$, define $B_\beta(z_1, \dots, z_\ell) := B_{i_1}(z_1) \cdots B_{i_r}(z_r) \in \mathrm{GL}(n)$. Lemmas 4.2 and 4.3 have the following consequence.

Theorem 4.4. *Let $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}^+_k$. Then,*

$$\mathrm{BS}(\beta) \cong \{(z_1, \dots, z_\ell) \in \mathbb{C}^\ell \mid \Delta_{[i],[i]}(B_\beta(z_1, \dots, z_\ell)) \neq 0 \textit{ for every } i \in [k]\}.$$

Let us give another equivalent characterization of the variety $\mathrm{BS}(\beta)$ that will be useful later. The wiring diagram of β separates the horizontal strip where the wiring diagram is drawn in several connected components, which moreover can be arranged by "levels," i.e., by horizontal sections of the diagram. Then, an element in $\mathrm{BS}(\beta)$ is an assignment of a subspace V_i of dimension i to each region of level i satisfying the following ***incidence conditions***:

(I1) To the unique left-unbounded region of level i, we assign the subspace $\langle e_1, \dots, e_i \rangle$.

(I2) If V_i is the subspace assigned to the unique right-unbounded region of level i, then $V_i \cap \langle e_{i+1}, \dots, e_k \rangle = \{0\}$.

(I3) If V_i and V_{i+1} are assigned to regions with intersecting boundaries, then $V_i \subseteq V_{i+1}$.

(I4) If V_i, V'_i are assigned to regions separated by a single crossing, then $V_i \neq V'_i$.

To finish this section, we describe the initial seed in the cluster structure on $\mathrm{BS}(\beta)$ constructed by Shen and Weng in [17]. We set $\beta = \sigma_{i_1} \cdots \sigma_{i_\ell} \in \mathrm{Br}^+_k$ and use the parametrization of $\mathrm{BS}(\beta)$ given by Theorem 4.4.

Cluster variables. For each $e \in [\ell]$, set $x_e := \Delta_{[i_e],[i_e]}(B_{\sigma_{i_1} \cdots \sigma_{i_e}}(z_1, \dots, z_e))$. The cluster variables in Σ are $x_1, \dots, x_\ell$. We picture these variables in the wiring diagram of β, by putting the variable x_j on the region to the right of the crossing σ_{i_j}.

Frozen variables. A variable x_e is frozen if and only if there is no $d > e$ with $i_e = i_d$. Note that these correspond to the rightmost regions in the wiring diagram of β.

Quiver. The quiver Q_β is read from the wiring diagram of β, see [17] and Example 5.1 below.

5 Skew shaped positroids and double Bott–Samelson varieties

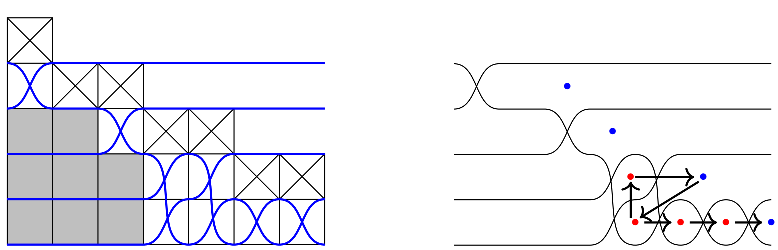
Let $0 < k < n$ and take partitions $\mu \subseteq \lambda \subseteq (n-k)^k$. We define a k-stranded braid $\beta_{\lambda/\mu}$ via the following procedure.

1. Cross out the top box of every column of λ.
2. For each column of $(n-k)^k$, put strands starting on the lower left corner of each box in this column, so that there are precisely k strands. We join the strands to corners in the right border of this column, as follows:
   (a) Join the strand starting in the lower left corner of the crossed box belonging to λ/μ (from Step 1) in this column to the lower right corner of the lowest box in the same column belonging to λ/μ. If the crossed box also belongs to μ, then simply join the strand to the lower right corner of the same box.
   (b) If a strand starts in the lower left corner of a non-crossed box in λ/μ, join it to the upper right corner of the same box.
   (c) If a strand starts on the lower left corner of any box which is not of the above type, join it (in a straight line) to the lower righg corner of the same box.
3. The braid $\beta_{\lambda/\mu}$ is defined to be the concatenation of the $(n-k)$-braids from Step 2. In formulas,

$$\beta_{\lambda/\mu} = C_1 C_2 \cdots C_{n-k}, \qquad C_j = \sigma_{\lambda_j^t - 1}\sigma_{\lambda_j^t - 2} \cdots \sigma_{\mu_j^t + 1} \tag{5.1}$$

   where C_j is the empty braid if $\mu_j^t + 1 > \lambda_j^t - 1$.

Example 5.1. In our running Example 3.1, we draw the braid $\beta_{\lambda/\mu}$ as well as the quiver $Q_{\beta_{\lambda/\mu}}$. Note that $\beta_{\lambda/\mu} = (\sigma_4)()(\sigma_3)(\sigma_2\sigma_1)(\sigma_2\sigma_1)(\sigma_1)(\sigma_1)$, and that the quiver Q_β is the opposite of the quiver obtained from $Q_{\lambda/\mu}$ by deleting the vertices corresponding to crossed out boxes.



Theorem 5.2. *There exist $s \geq 0$ and an isomorphism of algebraic varieties*

$$\mathrm{BS}(\beta_{\lambda/\mu}) \times (\mathbb{C}^\times)^s \cong S^\circ_{\lambda/\mu}$$

that is a quasi-isomorphism of cluster varieties.

The proof of this theorem makes use of the following key concept. The ***short labeling*** $J(\square_{a,i}) \subseteq I'(\square_{a,i})$ of a box $\square_{a,i} \in \lambda/\mu$, consists of the first i (with respect to the usual order of $[n]$) elements of $I'(\square_{a,i})$. The short labels provide a map $\Phi : S^\circ_{\lambda/\mu} \to \mathrm{BS}(\beta_{\lambda/\mu})$. More precisely, given an element $V \in S^\circ_{\lambda/\mu}$, pick a matrix $M_V = (v_1|\ldots|v_n)$ representing it. The matrix M_V is unique if we require that $v_{b_1} = e_1, \ldots, v_{b_k} = e_k$, where $I_\mu = \{b_1 < \cdots < b_k\}$ and $e_1, \ldots, e_k$ is the standard basis of $\mathbb{C}^k$.

The element $\Phi(V)$ is specified by labeling the region to the right of a crossing of $\beta_{k,n}$, which in turn corresponds to a non-crossed box $\square_{i,a}$ of λ/μ, by the subspace spanned by the elements v_j where j belongs to the short labeling $J(\square_{i,a})$.

The inverse map $\Psi : \mathrm{BS}(\beta_{\lambda/\mu}) \to S^\circ_{\lambda/\mu}$ is obtained by reverse-engineering the construction above, using the fact that $\beta_{\lambda/\mu}$ is the product of interval braids, and the following key identity:

$$(v_1|\ldots|v_k)B_{\sigma_{j-1}\cdots\sigma_i}(z_1,\ldots,z_{j-i}) = (v_1|\cdots|v_{i-1}|v_i'|-v_i|\cdots|-v_{j-1}|v_{j+1}|\cdots|v_k) \quad (5.2)$$

where $v_i' = v_j + \sum_{s=1}^{j-i} z_s v_{i+s-1}$. As we will see below in an example, this will allow us to construct the matrix $\Psi(z_1,\ldots,z_\ell) = (v_1|\ldots|v_n)$ for $(z_1,\ldots,z_\ell) \in \mathrm{BS}(\beta_{\lambda/\mu})$. We will get, however, that for every crossed box $\boxtimes_{a,i} \in \lambda/\mu$, $\Delta_{I'(\boxtimes_{a,i})}(\Psi(z_1,\ldots,z_\ell)) = \pm 1$. This accounts for the $(\mathbb{C}^\times)^s$-factor in Theorem 5.2.

Example 5.3. Let us carefully construct the maps in our running example. First, we construct the map $\Phi : S^\circ_{\lambda/\mu} \to \mathrm{BS}(\beta_{\lambda/\mu})$. Let $V \in S^\circ_{\lambda/\mu}$ with $M_V = (v_1|\ldots|v_{12})$. Since $I_\mu = \{5,6,8,11,12\}$, we set $v_5 = e_1, v_6 = e_6, v_8 = e_3, v_{11} = e_4, v_{12} = e_5$. Reading the short labels of the boxes corresponding to the crossings of $\beta_{\lambda/\mu}$ (see Example 3.6), we obtain that $\Phi(V)$ is given as in Figure 1. That this labeling satisfies the incidence conditions (I1)–(I4) follows from the equations defining $S^\circ_{\lambda/\mu}$. For example, let us verify that $\langle v_1\rangle \subseteq \langle v_3, v_4\rangle$. From the Grassmann necklace $\mathcal{I}_{\lambda/\mu}$ and Remark 2.4 in [14], we can find the bounded affine permutation $f_{\lambda/\mu}$ which is in bijection with $\mathcal{I}_{\lambda/\mu}$. In this case, $f_{\lambda/\mu}(1) = 3$ and this implies that $\langle v_1, v_2\rangle = \langle v_2, v_3\rangle$. Thus, $\langle v_1\rangle \subseteq \langle v_2, v_3\rangle$. Similarly, we obtain $f_{\lambda/\mu}(2) = 4$, and thus $\langle v_2, v_3\rangle = \langle v_3, v_4\rangle$. Therefore, $\langle v_1\rangle \subseteq \langle v_2, v_3\rangle = \langle v_3, v_4\rangle$. Let us now construct the map $\Psi : \mathrm{BS}(\beta_{\lambda/\mu}) \to S^\circ_{\lambda/\mu}$. Let $(z_1,\ldots,z_8) \in \mathrm{BS}(\beta_{\lambda/\mu})$. We define $V := \Psi(z_1,\ldots,z_8)$ by specifying the columns of the matrix $M_V = (v_1|\ldots|v_{12})$. First, we set $(v_5|v_6|v_8|v_{11}|v_{12}) = I_k$, the $k \times k$-identity matrix. After C_1 in Figure 1 we see the new vector v_{10}, so we define v_{10} so that

$$(v_5|v_6|v_8|v_{11}|v_{12})B_4(z_1) = (v_5|v_6|v_8|v_{10}|-v_{11})$$

that we can do by (5.2). The braid C_2 is empty so we skip it. Looking at C_3 define v_7 so that

$$(v_5|v_6|v_8|v_{10}|-v_{11})B_3(z_2) = (v_5|v_6|v_7|-v_8|-v_{11}).$$

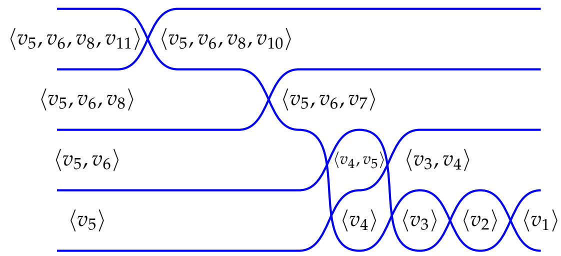


Figure 1: The braid $\beta_{\lambda/\mu} = (\sigma_4)()(\sigma_3)(\sigma_2\sigma_1)(\sigma_2\sigma_1)(\sigma_1)(\sigma_1)$ associated to λ/μ where $\lambda = (7,7,5,3,1)$ and $\mu = (3,3,2)$.

Moving on to C_4, define v_4 so that

$$(v_5|v_6|v_7|-v_8|-v_{11})B_{\sigma_2\sigma_1}(z_3,z_4) = (v_4|-v_5|-v_6|-v_8|-v_{11})$$

where we have once again used (5.2). This allows us to recover every vector $v_1,\ldots,v_{12}$, *except for* v_9, since 9 does not appear in Figure 1. However, the equations of $S^\circ_{\lambda/\mu}$ imply that v_9 is a scalar multiple of v_{10}, and we can recover v_9 by requiring that $\Delta_{5,6,8,9,12} = 1$. Let us discuss the compatibility between cluster structures, by looking at the cluster variables in $\mathrm{BS}(\beta_{\lambda/\mu})$ attached to the crossings of $C_4 = \sigma_2\sigma_1$. These are

$$\begin{aligned}\Delta_{[2],[2]}B_{C_1C_2C_3\sigma_2}(z_1,z_2,z_3) = \quad & \Delta_{[2],[2]}B_{C_1C_2C_3C_4}(z_1,z_2,z_3,z_4)\\ & \text{and } \Delta_{[1],[1]}B_{C_1C_2C_3C_4}(z_1,z_2,z_3,z_4).\end{aligned}$$

By construction,

$$\begin{aligned}\Delta_{[2],[2]}B_{C_1C_2C_3C_4}(z_1,z_2,z_3,z_4) &= \Delta_{[2],[2]}(v_4|-v_5|-v_6|-v_8|-v_{11})\\ &= -\det(v_4|v_5|e_3|e_4|e_5) = -\det(v_4|v_5|v_8|v_{11}|v_{12})\end{aligned}$$

and similarly $\Delta_{[1],[1]}B_{C_1C_2C_3C_4}(z_1,z_2,z_3,z_4) = \det(v_4|v_6|v_8|v_{11}|v_{12})$. Looking at Example 3.6 these are, possibly up to signs, the corresponding cluster variables in $S^\circ_{\lambda/\mu}$.

6 Splicing

Let $\beta = \beta_1\beta_2$ be a decomposition of the braid $\beta \in \mathrm{Br}_k^+$. We let $U(\beta_1,\beta_2) \subseteq \mathrm{BS}(\beta)$ be the open set obtained by the non-vanishing of the cluster variables corresponding to the last appearance of each letter in β_1. Note that $U(\beta_1,\beta_2)$ has a cluster structure, obtained from the cluster structure in $\mathrm{BS}(\beta)$ by freezing the cluster variables just described.

Theorem 6.1. *Let $\beta = \beta_1\beta_2 \in \mathrm{Br}_k^+$ and $U(\beta_1,\beta_2)$ be as above. Then, we have a quasi-cluster isomorphism*

$$U(\beta_1,\beta_2) \cong \mathrm{BS}(\beta_1)\times\mathrm{BS}(\beta_2).$$

The idea behind the previous result is as follows. We let $\beta_1 = \sigma_{i_1} \cdots \sigma_{i_{\ell_1}}$, $\beta_2 = \sigma_{j_1} \cdots \sigma_{j_{\ell_2}}$. Then, $\mathrm{BS}(\beta)$ consists of sequences of flags:

$$F^\bullet_{\mathrm{std}} = F^\bullet_0 \xrightarrow{i_1} F^\bullet_1 \xrightarrow{i_2} \cdots \xrightarrow{i_{\ell_1}} F^\bullet_{\ell_1} \xrightarrow{j_1} F^\bullet_{\ell_1+1} \xrightarrow{j_2} \cdots \xrightarrow{j_{\ell_2}} F^\bullet_{\ell_1+\ell_2} \pitchfork F^\bullet_{\mathrm{ant}}. \tag{6.1}$$

The conditions on $U(\beta_1, \beta_2)$ imply that $F^\bullet_\ell \pitchfork F^\bullet_{\mathrm{ant}}$, so we can try to separate (6.1) as:

$$F^\bullet_0 \xrightarrow{i_1} F^\bullet_1 \xrightarrow{i_2} \cdots \xrightarrow{i_{\ell_1}} F^\bullet_{\ell_1} \qquad \text{and} \qquad F^\bullet_{\ell_1} \xrightarrow{j_1} F^\bullet_{\ell_1+1} \xrightarrow{j_2} \cdots \xrightarrow{j_{\ell_2}} F^\bullet_{\ell_1+\ell_2}. \tag{6.2}$$

The first chain of flags belongs to $\mathrm{BS}(\beta_1)$. The second chain of flags, however, does not belong to $\mathrm{BS}(\beta_2)$, because it would have to start with the standard flag for this to be the case. So the isomorphism in Theorem 6.1 is obtained upon a simultaneous translation of all the flags in the second chain of (6.2).

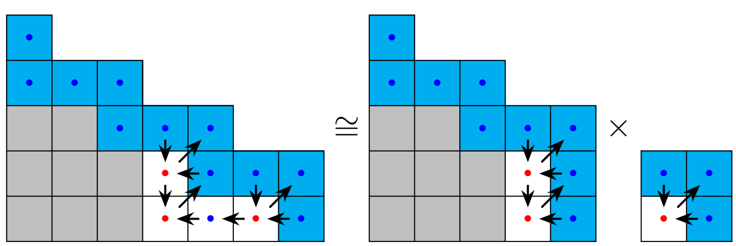
Now let $\mu \subseteq \lambda$ be partitions fitting inside a $k \times (n-k)$-rectangle. By definition, we have a decomposition $\beta_{\lambda/\mu} = C_1 \cdots C_{n-k}$. Pick $1 \le j \le n-k$, and let $\beta_1 = C_1 \cdots C_j$, and $\beta_2 = C_{j+1} \cdots C_{n-k}$, so that we have a decomposition $\beta_{\lambda/\mu} = \beta_1\beta_2$ and we can apply Theorem 6.1. Note that the set $U = U_j$ can be obtained as the non-vanishing locus all the cluster variables corresponding to the j-th column of the $k \times (n-k)$-box which, up to variables associated to crossed boxes, correspond to the letters of C_j. Note also that $\beta_1 = \beta_{\lambda^{1,j}/\mu^{1,j}}$ and $\beta_2 = \beta_{\lambda^{2,j}/\mu^{2,j}}$, where $\lambda^{1,j}, \mu^{1,j}$ and $\lambda^{2,j}, \mu^{2,j}$ are as defined in the introduction. Thus, we obtain the following result:

Theorem 6.2. *With the notation as above, we have a quasi-isomorphism of cluster varieties.*

$$U_j(S^\circ_{\lambda/\mu}) \cong S^\circ_{\lambda^{1,j}/\mu^{1,j}} \times S^\circ_{\lambda^{2,j}/\mu^{2,j}}$$

In [10] we also give a way of understanding this result in a linear algebraic way.

Example 6.3. In our running Example 3.1, take $j = 5$. Then, we need to freeze the variable $x_{7-5,1} = x_{2,1} = \Delta_{3,6,8,11,12}$, which is the only non-frozen variable in the fifth column. We have $\lambda^{1,5} = (5,5,5,3,1)$, $\mu^{1,5} = \mu = (3,3,2)$, while $\lambda^{2,5} = (2,2)$ and $\mu^{2,5} = \emptyset$.



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