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A conjectural basis for the $(1,2)$-bosonic-fermionic coinvariant ring

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Abstract. We give the first conjectural construction of a monomial basis for the coinvariant ring $R_n^{(1,2)}$, for the symmetric group $\mathfrak{S}_n$ acting on one set of bosonic (commuting) and two sets of fermionic (anticommuting) variables. Our construction interpolates between the modified Motzkin path basis for $R_n^{(0,2)}$ of Kim–Rhoades (2022) and the super-Artin basis for $R_n^{(1,1)}$ conjectured by Sagan–Swanson (2024) and proven by Angarone et al. (2024). We prove that our proposed basis has cardinality $2^{n-1}n!$, aligning with a conjecture of Zabrocki (2020) on the dimension of $R_n^{(1,2)}$, and show how it gives a combinatorial expression for the Hilbert series. We also conjecture a Frobenius series for $R_n^{(1,2)}$, including formulas for hook characters and the m_μ coefficients.

Keywords: Coinvariant ring, Hilbert series, Frobenius series, Artin basis

1 Introduction

The classical coinvariant ring $R_n^{(1,0)} = \mathbb{C}[\boldsymbol{x}_n]/\langle \mathbb{C}[\boldsymbol{x}_n]_+^{\mathfrak{S}_n} \rangle$ is the quotient of a polynomial ring in n variables $\boldsymbol{x}_n = \{x_1, \ldots, x_n\}$ by $\mathfrak{S}_n$-invariant polynomials with no constant term. It is well-known (see for example [11, Section 1.5]) to have dimension $n!$, Hilbert series $[n]_q!$, and Frobenius series

$$\sum_{\lambda \vdash n} \sum_{T \in \mathrm{SYT}(\lambda)} q^{\mathrm{maj}(T)} s_\lambda.$$

An important basis of the classical coinvariant ring is the Artin basis.

In 1994, Haiman [11] introduced the diagonal coinvariant ring $R_n^{(2,0)} = \mathbb{C}[\boldsymbol{x}_n, \boldsymbol{y}_n]/\langle \mathbb{C}[\boldsymbol{x}_n, \boldsymbol{y}_n]_+^{\mathfrak{S}_n} \rangle$, which extends the classical coinvariant ring to two sets of n variables. Here and throughout, $\mathfrak{S}_n$ acts diagonally by permuting the indices of the variables. In 2002, Haiman [12] proved that $R_n^{(2,0)}$ has dimension $(n+1)^{n-1}$, Hilbert series $\langle \nabla_{q,t}(e_n), h_1^n \rangle$, and Frobenius series $\nabla_{q,t}(e_n)$, using several deep results in algebraic geometry. A combinatorial formula for its Hilbert series was conjectured by Haglund and Loehr [10], and eventually was proven as a consequence of the more general shuffle theorem, conjectured by Haglund, Haiman, Loehr, Remmel, and Ulyanov [9] and proven by Carlsson

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and Mellit [5], which gives a combinatorial formula for $\nabla_{q,t}(e_n)$. A monomial basis was given by Carlsson and Oblomkov [6].

Recently, there has been interest (see [3, 2, 4, 21]) in extending the setting to include coinvariant rings with k sets of n commuting variables $\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots$ and j sets of n anti-commuting variables $\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n, \ldots$ (which commute with $\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots$). We denote this by

$$R_n^{(k,j)} = \mathbb{C}[\underbrace{\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots}_{k}, \underbrace{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n, \ldots}_{j}] / \langle \mathbb{C}[\underbrace{\boldsymbol{x}_n, \boldsymbol{y}_n, \boldsymbol{z}_n, \ldots}_{k}, \underbrace{\boldsymbol{\theta}_n, \boldsymbol{\xi}_n, \boldsymbol{\rho}_n, \ldots}_{j}]_+^{\mathfrak{S}_n} \rangle.$$

We briefly overview recent results on some special cases of $R_n^{(k,j)}$ for which there has been significant progress. Kim and Rhoades [14] showed that the fermionic diagonal coinvariant ring $R_n^{(0,2)} = \mathbb{C}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n] / \langle \mathbb{C}[\boldsymbol{\theta}_n, \boldsymbol{\xi}_n]_+^{\mathfrak{S}_n} \rangle$ has dimension $\binom{2n-1}{n}$, confirming a conjecture of Zabrocki [21]. They gave a combinatorial formula for its Hilbert series:

$$\mathrm{Hilb}(R_n^{(0,2)}; u, v) = \sum_{\pi \in \Pi(n)_{>0}} u^{\deg_\theta(\pi)} v^{\deg_\xi(\pi)}, \tag{1.1}$$

where $\Pi(n)_{>0}$ denotes the set of modified Motzkin paths of length n. They also gave a monomial basis for $R_n^{(0,2)}$ and found its Frobenius series [14, Theorem 6.1].

The superspace coinvariant ring is $R_n^{(1,1)} = \mathbb{C}[\boldsymbol{x}_n, \boldsymbol{\theta}_n] / \langle \mathbb{C}[\boldsymbol{x}_n, \boldsymbol{\theta}_n]_+^{\mathfrak{S}_n} \rangle$. Sagan and Swanson [18] conjectured, and Rhoades and Wilson [17] proved, that its Hilbert series is

$$\mathrm{Hilb}(R_n^{(1,1)}; q; u) = \sum_{k=1}^{n} u^{n-k} [k]_q! \, \mathrm{Stir}_q(n,k), \tag{1.2}$$

where $\mathrm{Stir}_q(n,k)$ is a q-analogue of the Stirling numbers. The dimension of $R_n^{(1,1)}$ is the ordered Bell number, which counts the number of ordered set partitions of $\{1, \ldots, n\}$. Sagan and Swanson [18] conjectured and Angarone, Commins, Karn, Murai, and Rhoades [1] proved that a certain super-Artin set is a basis for $R_n^{(1,1)}$. A Frobenius series has been conjectured for $R_n^{(1,1)}$ [4, Equation 5.2].

Zabrocki [20] conjectured a Frobenius series for $R_n^{(2,1)}$. Then D'Adderio, Iraci, and Vanden Wyngaerd [8] introduced certain symmetric function operators Θ_f, called Theta operators, where f is any symmetric function, to extend Zabrocki's conjecture to a conjectural Frobenius series for $R_n^{(2,2)}$. They conjectured that

$$\mathrm{Frob}(R_n^{(2,2)}; q, t; u, v) = \sum_{\substack{k, \ell \geq 0, \\ k+\ell < n}} u^k v^\ell \Theta_{e_k} \Theta_{e_\ell} \nabla_{q,t}(e_{n-k-\ell}), \tag{1.3}$$

which is known as the Theta conjecture.

The ring $R_n^{(1,2)} = \mathbb{C}[\boldsymbol{x}_n, \boldsymbol{\theta}_n, \boldsymbol{\xi}_n]/\langle \mathbb{C}[\boldsymbol{x}_n, \boldsymbol{\theta}_n, \boldsymbol{\xi}_n]_+^{\mathfrak{S}_n}\rangle$ is the main object of study in this paper. Zabrocki [21] conjectured that the dimension of $R_n^{(1,2)}$ is $2^{n-1}n!$, and an ungraded Frobenius series was conjectured by Bergeron [2]. The Theta conjecture specialized at $t = 0$ immediately gives a conjectural Frobenius series for $R_n^{(1,2)}$, however, the specialization can only be done after applying both of the Theta operators, so the resulting formula does not easily simplify. Recently, Iraci, Nadeau, and Vanden Wyngaerd [13] have given a conjectural Hilbert series and conjectural Frobenius series using the combinatorics of segmented permutations. However, they did not propose a monomial basis.

Our first main contribution is a combinatorial construction of a set of monomials, denoted by $B_n^{(1,2)}$ (Definition 3.1), which we conjecture to give a basis of $R_n^{(1,2)}$ (Conjecture 3.2). If the conjecture holds, it implies the following combinatorial Hilbert series (Proposition 3.5):

$$\operatorname{Hilb}(R_n^{(1,2)}; q; u, v) = \sum_{\pi \in \Pi(n)_{>0}} u^{\deg_\theta(\pi)} v^{\deg_\xi(\pi)} \operatorname{stair}_q(\pi). \tag{1.4}$$

In support of this conjecture, we show that the cardinality of $B_n^{(1,2)}$ is $2^{n-1}n!$ (Theorem 3.6). By establishing a weight-preserving bijection between $B_n^{(1,2)}$ and the set of segmented permutations $\mathrm{SW}(1^n)$ (Theorem 5.2), it follows that our conjectural Hilbert series is equivalent to a conjectural Hilbert series of Iraci, Nadeau, and Vanden Wyngaerd [13].

Our second main contribution is a simple combinatorial formula for the conjectural Frobenius series of $R_n^{(1,2)}$ (Conjecture 4.2):

$$\operatorname{Frob}(R_n^{(1,2)}; q; u, v) = \sum_{b \in B_n^{(1,2)}} u^{\deg_\theta(b)} v^{\deg_\xi(b)} q^{\deg_x(b)} Q_{\operatorname{Asc}(b), n}, \tag{1.5}$$

where $Q_{S,n}$ denotes the fundamental quasisymmetric function. We show that this is equivalent to the conjectural Frobenius series of Iraci, Nadeau, and Vanden Wyngaerd (Theorem 5.3). A benefit of using $B_n^{(1,2)}$ instead of segmented permutations is that determining $\operatorname{Asc}(b)$ and $\deg_x(b)$ for $b \in B_n^{(1,2)}$ is typically more direct than determining $\operatorname{Split}(\sigma)$ and $\operatorname{sminv}(\sigma)$ for $\sigma \in \mathrm{SW}(1^n)$, which fulfill analogous roles.

Section 5 contains some applications demonstrating the utility of our constructions. We give a formula for the m_μ coefficient of our conjectural Frobenius series, which can also be interpreted as a symmetric function identity. We give a formula for the coefficient of a hook Schur function $s_{(d+1,1^{n-d-1})}$ in the symmetric function in Equation (1.5):

$$\sum_{k+\ell<n} u^k v^\ell q^{\binom{n-d-k-\ell}{2}} \begin{bmatrix} n-1-d \\ \ell \end{bmatrix}_q \begin{bmatrix} n-1-k \\ d \end{bmatrix}_q \begin{bmatrix} n-1-\ell \\ k \end{bmatrix}_q,$$

extending a result of Iraci, Nadeau, and Vanden Wyngaerd on column Schur functions $s_{(1^n)}$.

Full results with proofs advertised in this extended abstract, along with some extensions to type B, can be found in [15].

2 Background

In this section, we describe the Kim–Rhoades basis for $R_n^{(0,2)}$ [14] and the super-Artin basis for $R_n^{(1,1)}$ conjectured by Sagan–Swanson [18] and proven by Angarone, Commins, Karn, Murai, and Rhoades [1].

Define the set of *modified Motzkin paths* of length n, $\Pi(n)_{>0}$, to be the set of all paths $\pi = (p_1, \dots, p_n)$ in $\mathbb{Z}^2$ such that each step p_i is one of:

(a) an up-step $(1, 1)$,

(b) a horizontal step $(1, 0)$ with decoration θ_i,

(c) a horizontal step $(1, 0)$ with decoration ξ_i,

(d) or a down-step $(1, -1)$ with decoration $\theta_i \xi_i$,

where the first step must be an up-step, and subsequently, the path never goes below the horizontal line $y = 1$.[1]

Define the *weight* $\mathrm{wt}(p_i)$ of a step p_i of a modified Motzkin path to be its decoration, or 1 if it does not have a decoration. Then define the *weight* $\mathrm{wt}(\pi)$ of a modified Motzkin path $\pi \in \Pi(n)_{>0}$ to be the product of the weights of each step p_i, that is,

$$\mathrm{wt}(\pi) := \prod_{p_i \in (p_1, \dots, p_n) = \pi} \mathrm{wt}(p_i).$$

Definition 2.1 ([14])**.** The *Kim–Rhoades basis* $B_n^{(0,2)}$ is the set of all weights of the modified Motzkin paths $\pi \in \Pi(n)_{>0}$, that is,

$$B_n^{(0,2)} := \{\mathrm{wt}(\pi) \mid \pi \in \Pi(n)_{>0}\}.$$

For an example, see Figure 1.

Theorem 2.2 ([14])**.** *The Kim–Rhoades basis $B_n^{(0,2)}$ is a basis for $R_n^{(0,2)}$.*

[1] Kim and Rhoades defined the modified Motzkin paths in a slightly different manner, where there are no decorations on the down-steps, but they still contribute $\theta_i \xi_i$ to the weight. Furthermore, they defined the decorations to be just θ or ξ instead of θ_i and ξ_i. Converting between these conventions is straightforward.

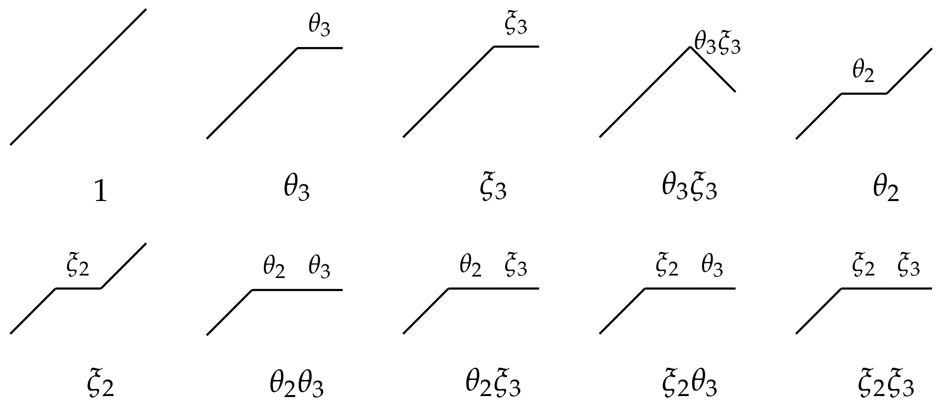


Figure 1: The basis $B_3^{(0,2)}$. Each modified Motzkin path is labeled with its corresponding monomial.

Next, we recall the super-Artin basis, defined by Sagan and Swanson. Let $\chi(P)$ be 1 if the proposition P is true, and 0 if the proposition P is false. Let θ_T denote the ordered product $\theta_{t_1} \cdots \theta_{t_k}$ for any subset $T = \{t_1 < \cdots < t_k\} \subseteq \{1, \ldots, n\}$. For any $T \subseteq \{2, \ldots, n\}$, define the *α-sequence* $\alpha(T) = (\alpha_1(T), \ldots, \alpha_n(T))$ recursively by the initial condition $\alpha_1(T) = 0$ and for $2 \leq i \leq n$,

$$\alpha_i(T) = \alpha_{i-1}(T) + \chi(i \notin T).$$

Definition 2.3 ([18])**.** The super-Artin set is

$$B_n^{(1,1)} := \{x^\alpha \theta_T \mid T \subseteq \{2, \ldots, n\} \text{ and } \alpha \leq \alpha(T) \text{ componentwise}\}.$$

See Figure 2 for an example. The following result was conjectured by Sagan and Swanson, and was proven by Angarone, Commins, Karn, Murai, and Rhoades.

Theorem 2.4 ([1])**.** *The super-Artin set* $B_n^{(1,1)}$ *is a basis for* $R_n^{(1,1)}$*.*

We assume familiarity with the basics of symmetric function theory (see for example [16, Chapter I] or [19, Chapter 7]). Let m_λ, e_λ, h_λ, p_λ, and s_λ denote respectively the monomial, elementary, complete homogeneous, power-sum, and Schur symmetric functions in infinitely many variables.

Finally, we record the definitions of the multigraded Hilbert and Frobenius series of $R_n^{(2,2)}$ (see for example [2]). For $R_n^{(k,j)}$ with $k, j \leq 2$, we can further specialize the following. Setting $t = 0$ gives us the series for $(1,2)$, etc. $R_n^{(2,2)}$ decomposes as a direct sum of multihomogenous components, which are $\mathfrak{S}_n$-modules:

$$R_n^{(2,2)} = \bigoplus_{r_1, r_2, s_1, s_2 \geq 0} (R_n^{(2,2)})_{r_1, r_2, s_1, s_2}.$$

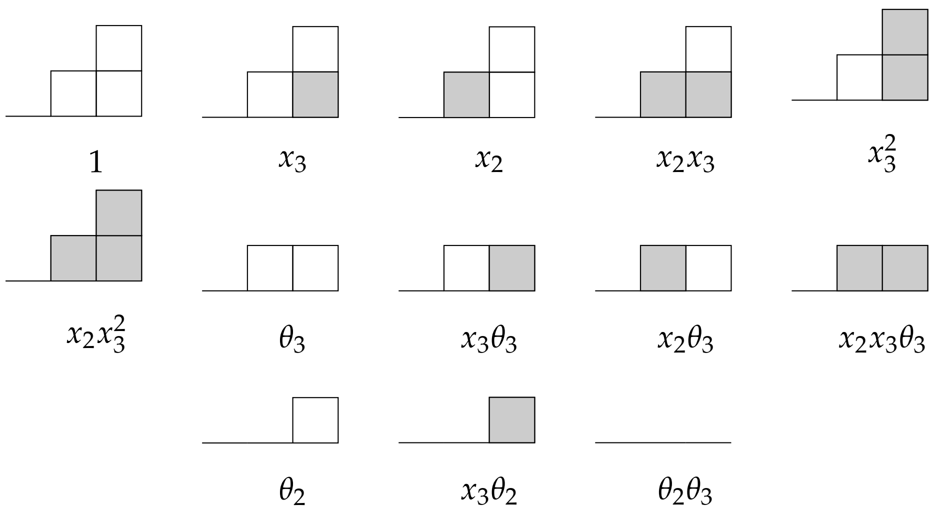


Figure 2: The basis $B_3^{(1,1)}$. The α-sequence is shown as the outline of all boxes, and those x_i used for a particular basis element are shaded in gray.

We denote the multigraded Hilbert series by

$$\mathrm{Hilb}(R_n^{(2,2)}; q,t;u,v) := \sum_{r_1,r_2,s_1,s_2 \geq 0} \dim \left((R_n^{(2,2)})_{r_1,r_2,s_1,s_2} \right) q^{r_1} t^{r_2} u^{s_1} v^{s_2},$$

and the multigraded Frobenius series by

$$\mathrm{Frob}(R_n^{(2,2)}; q,t;u,v) := \sum_{r_1,r_2,s_1,s_2 \geq 0} F\,\mathrm{char} \left((R_n^{(2,2)})_{r_1,r_2,s_1,s_2} \right) q^{r_1} t^{r_2} u^{s_1} v^{s_2},$$

where F denotes the Frobenius characteristic map and char denotes the character. Recall that $\langle \mathrm{Frob}(R_n^{(2,2)}; q,t;u,v), h_1^n \rangle = \mathrm{Hilb}(R_n^{(2,2)}; q,t;u,v)$.

3 The conjectural monomial basis

The goal of this section is to interpolate between the Kim–Rhoades basis and the super-Artin basis to construct a new set $B_n^{(1,2)}$, which we conjecture to be a basis for $R_n^{(1,2)}$.

We generalize the α-sequence as follows. Let ξ_S denote the ordered product $\xi_{s_1} \cdots \xi_{s_k}$ for any subset $S = \{s_1 < \cdots < s_k\} \subseteq \{1,\ldots,n\}$. For any $T, S \subseteq \{2,\ldots,n\}$, define the *generalized α-sequence* $\alpha(T,S) = (\alpha_1(T,S),\ldots,\alpha_n(T,S))$ recursively by the initial condition $\alpha_1(T,S) = 0$ and for $2 \leq i \leq n$,

$$\alpha_i(T,S) = \alpha_{i-1}(T,S) - 1 + \chi(i \notin T) + \chi(i \notin S).$$

Definition 3.1. We let

$$B_n^{(1,2)} := \{x^\alpha \theta_T \xi_S \mid \theta_T \xi_S \in B_n^{(0,2)} \text{ and } 0 \le \alpha_i \le \alpha_i(T,S) \text{ for all } i \in \{1,\ldots,n\}\}.$$

See Figure 3 for an example of $B_n^{(1,2)}$ at $n = 3$.

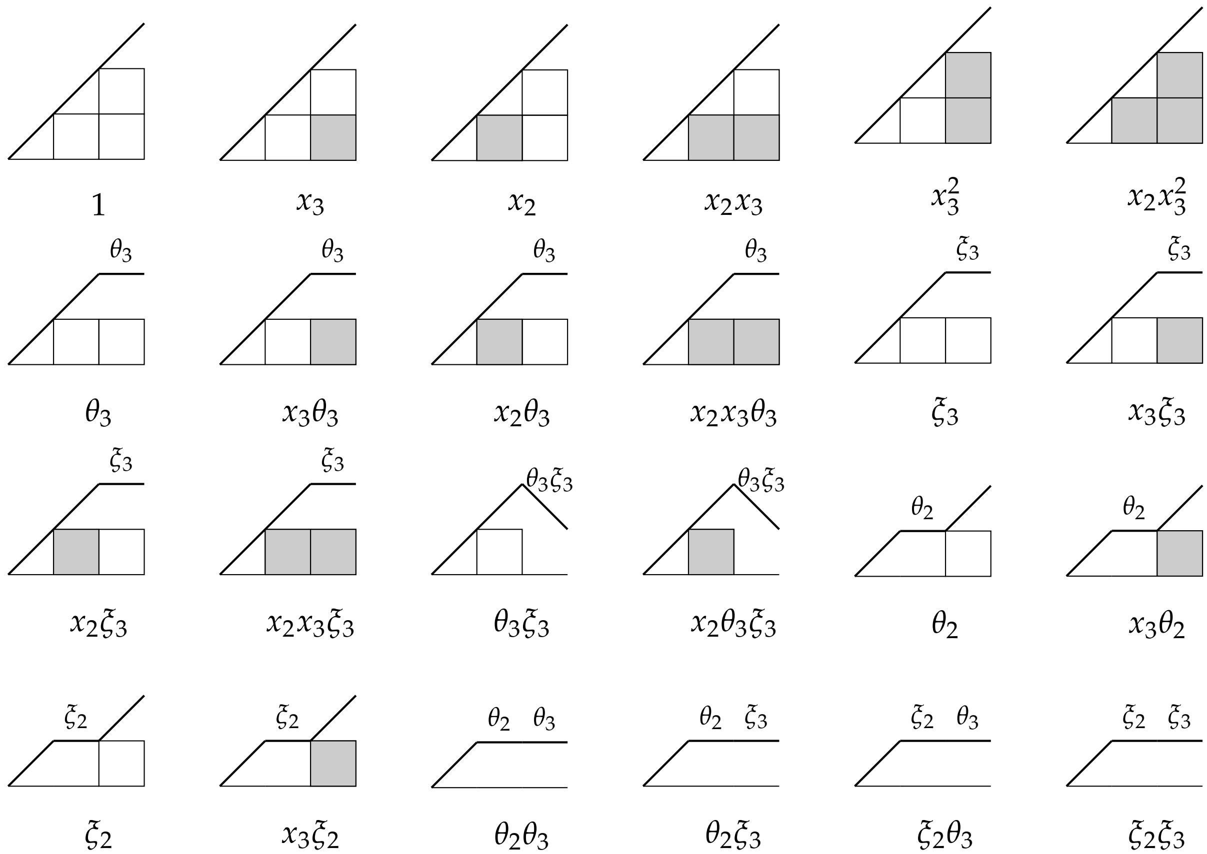


Figure 3: The basis $B_3^{(1,2)}$. Below each modified Motzkin path is the outline of the generalized α-sequence, and those x_i used for a particular basis element are shaded in gray. Note that the outline of the generalized α-sequence can be determined by the following rule: in each position i, the column of boxes extends up until its right side is one unit below the path above it.

We are ready to present our main conjecture.

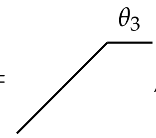
Conjecture 3.2. *The set* $B_n^{(1,2)}$ *is a basis for* $R_n^{(1,2)}$.

Remark 3.3. $B_n^{(1,2)}$ specializes to the super-Artin basis $B_n^{(1,1)}$ by setting all $\xi_i = 0$, and $B_n^{(1,2)}$ specializes to the Kim–Rhoades basis $B_n^{(0,2)}$ by setting all $x_i = 0$.

Next, we consider the implications of Conjecture 3.2 on the Hilbert series of $R_n^{(1,2)}$. Define

$$\operatorname{stair}_q(\pi) := \prod_{k \in \alpha(T(\pi), S(\pi))} [k+1]_q,$$

where $T(\pi)$ and $S(\pi)$ are determined by which elements in $\{2, \ldots, n\}$ appear as indices for θ_i and ξ_i respectively in the weight of the modified Motzkin path π.

Example 3.4. For the modified Motzkin path $\pi =$ 

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we have that $T(\pi) = \{3\}$ and $S(\pi) = \varnothing$, so $\alpha(\{3\}, \varnothing) = (0, 1, 1)$. Thus $\operatorname{stair}_q(\pi) = [1]_q[2]_q[2]_q$. Observe in Figure 3 that there are 4 basis elements corresponding to π.

For a modified Motzkin path π, let $\deg_\theta(\pi) = |T(\pi)|$ and let $\deg_\xi(\pi) = |S(\pi)|$.

Proposition 3.5. *Assuming Conjecture 3.2, it follows that the Hilbert series of $R_n^{(1,2)}$ is*

$$\operatorname{Hilb}(R_n^{(1,2)}; q; u, v) = \sum_{\pi \in \Pi(n)_{>0}} u^{\deg_\theta(\pi)} v^{\deg_\xi(\pi)} \operatorname{stair}_q(\pi).$$

We show that the proposed basis $B_n^{(1,2)}$ has Zabrocki's conjectured dimension, using a similar argument to Corteel–Nunge [7, Lemma 17] which enumerates marked Laguerre histories.

Theorem 3.6. *The cardinality of $B_n^{(1,2)}$ is $2^{n-1}n!$.*

4 A conjectural Frobenius series

The goal of this section is to demonstrate how the conjectural basis can be used to propose a Frobenius series for $R_n^{(1,2)}$.

For any subset $S \subseteq \{1, \ldots, n-1\}$, the *fundamental quasisymmetric function* $Q_{S,n}$ is defined by

$$Q_{S,n} = \sum_{\substack{a_1 \le a_2 \le \cdots \le a_n, \\ a_i < a_{i+1} \text{ if } i \in S}} z_{a_1} z_{a_2} \cdots z_{a_n}.$$

We need the following definitions, which are motivated by related definitions of Iraci, Nadeau, and Vanden Wyngaerd on segmented permutations. For any $b \in B_n^{(1,2)}$, write

$$b = \pm \prod_{i=1}^{n} x_i^{\alpha_i} \theta_i^{\beta_i} \xi_i^{\gamma_i},$$

for some exponents $\alpha_i \in \mathbb{Z}_{\geq 0}$ and $\beta_i, \gamma_i \in \{0,1\}$. Since this is an ordered product, reordering the factors may change the sign, however, for our purposes the sign does not matter. Then define i to be an *ascent of* b if and only if one of the following occurs:

- $\beta_i < \beta_{i+1}$;
- $\beta_i = \beta_{i+1} = 1$ and $\alpha_i \geq \alpha_{i+1} + \gamma_{i+1}$; or
- $\beta_i = \beta_{i+1} = 0$ and $\alpha_i < \alpha_{i+1} + \gamma_{i+1}$.

For $b \in B_n^{(1,2)}$, we say that $\mathrm{Asc}(b) := \{i \in \{1, \ldots, n-1\} \mid i \text{ is an ascent of } b\}$.

Example 4.1. Consider $b = x_2 x_4^2 \theta_4 \theta_5 \xi_5 \in B_5^{(1,2)}$, which has $\alpha_2 = 1$, $\alpha_4 = 2$, $\beta_4 = 1$, $\beta_5 = 1$, $\gamma_5 = 1$, and all other exponents are 0. This has ascents at $i = 1$ (since $\beta_1 = \beta_2 = 0$ and $\alpha_1 < \alpha_2 + \gamma_2$), at $i = 3$ (since $\beta_3 < \beta_4$), and at $i = 4$ (since $\beta_4 = \beta_5 = 1$ and $\alpha_4 \geq \alpha_5 + \gamma_5$). Thus $\mathrm{Asc}(b) = \{1, 3, 4\}$.

Now we can state a conjectural Frobenius series for $R_n^{(1,2)}$ in terms of the set $B_n^{(1,2)}$.

Conjecture 4.2.

$$\mathrm{Frob}(R_n^{(1,2)}; q; u, v) = \sum_{b \in B_n^{(1,2)}} u^{\deg_\theta(b)} v^{\deg_\xi(b)} q^{\deg_x(b)} Q_{\mathrm{Asc}(b),n}.$$

We will see further evidence for this conjecture with Theorem 5.3.

5 The proposed basis and segmented permutations

In this section, we establish a q, u, v-weight preserving bijection between our proposed basis $B_n^{(1,2)}$ and the set of segmented permutations $\mathrm{SW}(1^n)$. A *segmented permutation* σ is a permutation of $\{1, \ldots, n\}$, where between any letters, there may be a vertical bar $|$. For example, the segmented permutations of length 2 are $12, 1|2, 21, 2|1$. Denote the set of all segmented permutations on $\{1, \ldots, n\}$ by $\mathrm{SW}(1^n)$.[2] The cardinality of $\mathrm{SW}(1^n)$ is $2^{n-1}n!$.

Define a map $\psi : B_n^{(1,2)} \longrightarrow \mathrm{SW}(1^n)$ as follows. For any $b \in B_n^{(1,2)}$, we write it as

$$b = \pm \prod_{i=1}^{n} x_i^{\alpha_i} \theta_i^{\beta_i} \xi_i^{\gamma_i},$$

for some $\alpha_i \in \mathbb{Z}_{\geq 0}$ and $\beta_i, \gamma_i \in \{0,1\}$. We always have $\alpha_1 = \beta_1 = \gamma_1 = 0$; start the corresponding segmented permutation with 1. Then, as i ranges from 2 up through n, do exactly one of the following for each i:

[2] The notation $\mathrm{SW}(1^n)$ comes from the more general class of *segmented Smirnov words*, of which segmented permutations form a proper subset.

(a) if $\beta_i = \gamma_i = 0$: insert "$|i$" or "$i|$" in such a way as to create a new block consisting of only i at position $\alpha_i + 1$ from the rightmost block in the permutation (indexing starting at 1);

(b) if $\beta_i = 1$ and $\gamma_i = 0$: insert i as the last element of an existing block, at position $\alpha_i + 1$ from the rightmost block in the permutation;

(c) if $\beta_i = 0$ and $\gamma_i = 1$: insert i as the first element of an existing block, at position $\alpha_i + 1$ from the rightmost block in the permutation;

(d) if $\beta_i = \gamma_i = 1$: insert i to replace a "$|$" and thus merge two adjacent blocks into one block, which is now at position $\alpha_i + 1$ from the rightmost block in the permutation.

Upon completing this process, the output is some segmented permutation σ in $\mathrm{SW}(1^n)$.

Example 5.1. Consider $b = x_2 x_4^2 \theta_4 \theta_5 \xi_5$, which has $\alpha_2 = 1$, $\alpha_4 = 2$, $\beta_4 = 1$, $\beta_5 = 1$, $\gamma_5 = 1$, and all other exponents are 0. Start building a segmented permutation with 1. At $i = 2$, we are in case (a), so we insert $2|$ to get $2|1$, so that 2 is in a new block at position $1 + 1 = 2$ from the right. At $i = 3$, we are in case (a), so we insert $|3$ to get $2|1|3$, so that 3 is in a new block at the right. At $i = 4$, we are in case (b), so we insert 4 as the last element of the block at position $2 + 1 = 3$ from the right, giving $24|1|3$. At $i = 5$, we are in case (d), so we insert 5 to replace a "$|$" and merge two blocks which is now in the rightmost position, giving $24|153$.

In a segmented permutation σ, an *ascent* (resp. *descent*) is an index i such that $\sigma_i < \sigma_{i+1}$ (resp. $\sigma_i > \sigma_{i+1}$) and there is no vertical bar $|$ between σ_i and σ_{i+1}. A consequence of the bijection is the following, where statistics $\mathrm{sminv}(\sigma)$ and $\mathrm{Split}(\sigma)$ on segmented permutations are defined in [13].

Theorem 5.2. *The map $\psi : B_n^{(1,2)} \longrightarrow \mathrm{SW}(1^n)$ is a q, u, v-weight preserving bijection. For any $b \in B_n^{(1,2)}$, we have $\deg_\theta(b) = k$, the number of ascents in $\psi(b)$, $\deg_\xi(b) = \ell$, the number of descents in $\psi(b)$, $\deg_x(b) = \mathrm{sminv}(\psi(b))$, and $\mathrm{Asc}(b) = \mathrm{Split}(\psi(b))$.*

Let $\mathrm{SW}(1^n, k, \ell)$ denote the set of segmented permutation of length n with exactly k ascents and ℓ descents. Now, we are able to establish the equivalence of our conjectural Frobenius series (Conjecture 4.2) with a conjectural Frobenius series of Iraci, Nadeau, and Vanden Wyngaerd. This can also be interpreted as a symmetric function identity.

Theorem 5.3.

$$\sum_{b \in B_n^{(1,2)}} u^{\deg_\theta(b)} v^{\deg_\xi(b)} q^{\deg_x(b)} Q_{\mathrm{Asc}(b),n} = \sum_{k+\ell<n} u^k v^\ell \sum_{\sigma \in \mathrm{SW}(1^n,k,\ell)} q^{\mathrm{sminv}(\sigma)} Q_{\mathrm{Split}(\sigma),n}.$$

Define the symmetric function $\mathrm{SF}(n,k,\ell) = \sum_{\sigma \in \mathrm{SW}(1^n,k,\ell)} q^{\mathrm{sminv}(\sigma)} Q_{\mathrm{Split}(\sigma),n}$ by the inner sum on the right hand side of Theorem 5.3 For a composition $\alpha \vDash n$, let $\mathrm{Set}(\alpha) := (\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \cdots + \alpha_{\ell-1})$. We have a formula for the m_μ coefficient of $\mathrm{SF}(n,k,\ell)$.

Theorem 5.4. *Let $\mu \vdash n$. For any fixed k, ℓ, we have that*

$$\langle \mathrm{SF}(n,k,\ell), h_\mu \rangle = \sum_{\substack{b \in B_n^{(1,2)}, \\ \deg_\theta(b) = k, \\ \deg_\xi(b) = \ell, \\ \mathrm{Asc}(b) \subseteq \mathrm{Set}(\mu)}} q^{\deg_x(b)}.$$

We derive the following formula for hook Schur functions $s_{(d+1,1^{n-d-1})}$, using the combinatorics of our conjectural basis. This generalizes a result on column Schur functions $s_{(1^n)}$ [13, Theorem 5.6], which is recovered at $d = 0$. To get the complete conjectural sign character, multiply by $u^k v^\ell$ and sum over all $k + \ell < n$.

Theorem 5.5. *For $0 \le d \le n-1$,*

$$\langle \mathrm{SF}(n,k,\ell), s_{(d+1,1^{n-d-1})} \rangle = q^{\binom{n-d-k-\ell}{2}} \begin{bmatrix} n-1-d \\ \ell \end{bmatrix}_q \begin{bmatrix} n-1-k \\ d \end{bmatrix}_q \begin{bmatrix} n-1-\ell \\ k \end{bmatrix}_q.$$

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