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Molecules of affine fixed-point-free W-graphs

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Abstract. RSK correspondence gives a bijection from a permutation to a pair of semistandard Young tableaux with same shape. For the symmetric groups, Kazhdan and Lusztig use the RSK correspondence to classify the molecules and cells of the S_n-graphs. To generalize this to the affine case, Chmutov, Pylyavskyy and Yudovina set up an affine RSK correspondence, giving a bijection from an affine permutation to a triple (P, Q, ρ) where P and Q are tabloids of same shape and ρ is a dominant weight. Moreover, Chmutov, Lewis and Pylyavskyy use it to classify the affine molecules. Marberg generalizes Kazhdan and Lusztig's W-graphs, using affine fixed-point-free involutions as their indices. In this paper, we use affine RSK correspondence to classify the molecules of Marberg's $\tilde{S}_n$-graphs.

Keywords: Affine permutation, Kazhdan–Lusztig cells, Affine matrix-ball construction, molecules, W-graphs

1 Introduction

In an important paper [3], Kazhdan and Lusztig laid a basis for a new approach to representation theory of Hecke algebras. Since then, this approach, called *Kazhdan–Lusztig theory*, has been developed significantly. Of particular importance in this theory are the objects called *cells*. Briefly, their definition is as follows. Each Hecke algebra is associated with a Coxeter group W. Kazhdan and Lusztig define a pre-order $\leqslant_L$ on elements of W. Some pairs v, w of elements of W satisfy both $v \leqslant_L w$ and $w \leqslant_L v$, in which case we say that they are left-equivalent, denoted $v \sim_L w$. Similarly one can define right equivalence $\sim_R$. The respective equivalence classes are called the *left cells* and the *right cells*.

Another way to describe cells is via the Kazhdan–Lusztig W-graph; it is a certain directed graph whose vertices are the elements of W. The graph has the property that $v \leqslant_L w$ precisely when there is a directed path from v to w. Thus the cells are the strongly connected components of the W-graph. Some edges of the W-graph are bidirected, i.e., between a pair of vertices v and w there is an edge $v \to w$ and an edge $w \to v$. The connected component of the subgraph of W-graph considering only the bidirected edges are called *molecules*. Of course, v and w belong to the same Kazhdan–Lusztig cell if they belong to the same molecule.

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In type A, cells and molecules can be determined by the Robinson–Schensted correspondence. Two permutations are in the same left cell if and only if they have the same recording tableau while two permutations are in the same right cell if and only if they have the same insertion tableau. The bidirected edges are determined by Knuth moves, and in this type, each left cells have only one molecules.

In affine type A, when W is an affine symmetric group, Chmutov, Pylyavskyy, and Yudovina [2] described, via a combinatorial algorithm called the Affine Matrix Ball Construction (AMBC), a bijection $W \to \Omega_{\text{dom}}$, where Ω_{dom} is the set of triples (P, Q, ρ) such that P and Q are tabloids of the same shape and ρ is an integer vector (called a *dominant weight*) satisfying certain inequalities that depend on P and Q. Relying on the work of Shi on Kazhdan–Lusztig cells in affine type A, they show that this bijection affords a description of cells analogous to the non-affine case: fixing the tabloid Q gives all affine permutations in a left cell while fixing the tabloid P gives all affine permutations in a right cell. The bidirected edges in this case (what Shi called *star operations* [8]) are natural analogues of Knuth moves.

In [6], E. Marberg constructed two $\tilde{S}_n$-graphs with affine fixed-point-free involutions as their vertices. Analogue to the Kazhdan–Lusztig W-graphs, we can define the cells and molecules via the graphs. In this paper, we use AMBC to determined the molecules of one of the two $\tilde{S}_n$-graphs. We have the following theorems.

Theorem 1.1. For two affine involutions w and v, they are in the same molecule in $\Gamma^{\mathbf{m}}$ only if they have the same sign, and corresponding to tabloids of the same shape with same dominant weight applying AMBC.

Note that this is just a necessary condition, which is not sufficient.

Theorem 1.2. The molecules of $\Gamma^{\mathbf{m}}_n$ with same sign, same shape and same dominant weight are isomorphic to each other.

2 Preliminary

2.1 Affine permutations and involutions

Let n be a positive integer. Write $\mathbb{Z}$ for the set of integers and define $[n] = \{1, 2, \dots, n\}$. Let $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{P} = \{1, 2, 3, \dots\}$ be the sets of nonnegative and positive integers.

Definition 2.1. The *affine symmetric group* $\tilde{S}_n$ is the group of bijections $\pi : \mathbb{Z} \to \mathbb{Z}$ satisfying $\pi(i + n) = \pi(i) + n$ for all $i \in \mathbb{Z}$ and $\pi(1) + \pi(2) + \cdots + \pi(n) = 1 + 2 + \cdots + n$.

We refer to elements of $\tilde{S}_n$ as *affine permutations*. A *window* for an affine permutation $\pi \in \tilde{S}_n$ is a sequence of the form $[\pi(i + 1), \pi(i + 2), \dots, \pi(i + n)]$ where $i \in \mathbb{Z}$. An element $\pi \in \tilde{S}_n$ is uniquely determined by any of its windows, and a sequence of n

distinct integers is a window for some $\pi \in \tilde{S}_n$ if and only if the integers represent each congruence class modulo n exactly once.

Let s_i for $i \in \mathbb{Z}$ be the unique element of $\tilde{S}_n$ that interchanges i and $i+1$ while fixing every integer $j \notin \{i, i+1\} + n\mathbb{Z}$. One has $s_i = s_{i+n}$ for all $i \in \mathbb{Z}$, and $\{s_1, s_2, \dots, s_n\}$ generates the group $\tilde{S}_n$. With respect to this generating set, $\tilde{S}_n$ is the affine Coxeter group of type $\tilde{A}_{n-1}$. The parabolic subgroup $S_n = \langle s_1, s_2, \dots, s_{n-1} \rangle \subset \tilde{S}_n$ is the finite Coxeter group of type A_{n-1}; its elements are the permutations $\pi \in \tilde{S}_n$ with $\pi([n]) = [n]$.

A *reduced expression* for $\pi \in \tilde{S}_n$ is a minimal-length factorization $\pi = s_{i_1} s_{i_2} \cdots s_{i_l}$. The *length* of $\pi \in \tilde{S}_n$, denoted $\ell(\pi)$, is the number of factors in any of its reduced expressions. The value of $\ell(\pi)$ is also the number of equivalence classes in the set $\mathrm{Inv}(\pi) = \{(i, j) \in \mathbb{Z} \times \mathbb{Z} : i < j \text{ and } \pi(i) > \pi(j)\}$ under the relation $\sim$ on $\mathbb{Z} \times \mathbb{Z}$ with $(a, b) \sim (a', b')$ if and only if $a - a' = b - b' \in n\mathbb{Z}$.

From here, let n be an even integer. An affine involution is $z \in \tilde{S}_n$ such that $z^2 = 1$. An affine fixed-point-free involution is an affine involution z such that there are no $x \in [n]$ with $z(x) = x$. The set of all involutions is denoted as $\tilde{I}_n$ while the set of all fixed-point-free involutions is denoted as $\mathcal{F}_n$. On $\mathcal{F}_n$, we define $\ell^{\mathsf{FPF}}(z) = \frac{1}{2}(\ell(z) - \frac{n}{2})$.

Definition 2.2. Given $\pi \in \tilde{S}_n$, define $\beta(\pi) = \frac{1}{2n} \sum_{i=1}^{n} |\pi(i) - r_n(\pi(i))|$, where $r_n(i)$ for $i \in \mathbb{Z}$ denotes the unique element of $[n]$ that satisfies $r_n(i) \equiv i \pmod{n}$. For $z \in \mathcal{F}_n$, define $\mathrm{sgn}_{\mathsf{FPF}}(z) = (-1)^{\beta(z)}$.

Let

$$\Theta^+ = s_1 s_3 \cdots s_{n-1} = [2, 1, 4, 3, \dots, n, n-1] \in \tilde{I}_n$$

and

$$\Theta^- = s_2 s_4 \cdots s_n = [1, 0, 3, 2, \dots, n-1, n-2] \in \tilde{I}_n,$$

so that $\mathrm{sgn}_{\mathsf{FPF}}(\Theta^\pm) = \pm 1$. We reserve the symbol Θ to denote one of the two elements of $\{\Theta^\pm\}$. Define $\mathcal{F}_n^+$ as the $\tilde{S}_n$-conjugacy class of Θ^+ and $\mathcal{F}_n^-$ as the $\tilde{S}_n$-conjugacy class of Θ^-. One can show that

$$\mathcal{F}_n^+ = \{z \in \mathcal{F}_n : \mathrm{sgn}_{\mathsf{FPF}}(z) = 1\} \qquad \text{and} \qquad \mathcal{F}_n^- = \{z \in \mathcal{F}_n : \mathrm{sgn}_{\mathsf{FPF}}(z) = -1\} \tag{2.1}$$

and hence that $\mathcal{F}_n = \mathcal{F}_n^+ \sqcup \mathcal{F}_n^-$; see, e.g., [5, Theorem 5.4].

2.2 Tabloids

Let $\lambda = (\lambda_1, \dots, \lambda_\ell)$ be a partition of size $\sum_i \lambda_i \leqslant n$. A *tabloid* P of shape λ is an equivalence class of fillings of the Young diagram of shape λ with elements of $[\overline{n}]$ under identification of fillings that differ by reordering elements within rows. Here $\overline{i}$ denotes the equivalence class of integers $k \equiv i \pmod{n}$. We think of the i-th row of a tabloid P as a set $P_i \subseteq [\overline{n}]$.

Example 2.3.

$$\begin{array}{|c|c|c|}\hline \bar{1} & \bar{2} & \bar{3} \\ \hline \bar{4} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline \bar{3} & \bar{1} & \bar{2} \\ \hline \bar{4} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline \bar{4} & \bar{2} & \bar{3} \\ \hline \bar{1} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline \bar{4} & \bar{1} & \bar{2} \\ \hline \bar{3} \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|c|}\hline \bar{1} & \bar{4} & \bar{3} \\ \hline \bar{2} \\ \cline{1-1}\end{array}$$

These are several tabloids of shape $(3,1)$. The first two tabloids are equal, since they differ only by permuting elements within rows.

The order of elements in each row does not matter, but it is convenient to regard elements of T_i as being ordered with respect to the linear ordering $\bar{1} < \cdots < \bar{n}$. In this case we say that T is presented *row-standard*. In the remaining sections, we assume all tabloids are presented row-standard. For a partition $\lambda \vdash n$, define $\mathrm{RSYT}(\lambda)$ to be the set of row-standard Young tabloid of shape λ.

We call the analogue of the descent set for tabloids the *τ-invariant*, in reference to Vogan's (generalized) τ-invariant [11].

Definition 2.4. For a tabloid T filled with all the elements of $[\bar{n}]$, define the *τ-invariant* by

$$\tau(T) := \{\bar{i} \in [\bar{n}] : \bar{i} \text{ lies in a strictly higher row of } T \text{ than } \overline{i+1}\}.$$

Example 2.5. The following tabloid T has $\tau(T) = \{\bar{2}, \bar{5}, \bar{8}\}$.

$$T := \begin{array}{|c|c|c|}\hline \bar{2} & \bar{4} & \bar{5} \\ \hline \bar{6} & \bar{7} & \bar{8} \\ \hline \bar{1} & \bar{3} \\ \cline{1-2}\end{array}$$

2.3 AMBC

Here we briefly recall some notations from [2] and [1] and discuss the relations between the affine matrix-ball construction (abbreviated AMBC) and Kazhdan–Lusztig cells.

For $T = (T_1, \ldots, T_l) \in \mathrm{RSYT}(\lambda)$ and $i \in [1, l-1]$ we define $\mathrm{lch}_i(T)$, called the local charge in row i of T, as follows. Suppose that $T_i = (a_1, \ldots, a_s)$ and $T_{i+1} = (b_1, \ldots, b_t)$. Then $\mathrm{lch}_i(T)$ is the smallest $d \in \mathbb{N}$ such that $a_{l-d} < b_l$ for $l \in [d+1, t]$. Pictorially, this measures necessarily shift of T_i to the right so that (T_i, T_{i+1}) becomes a standard Young tableau (of skew shape). For example, if $T_i = (3,5,7,8)$ and $T_{i+1} = (1,2,4,6)$ then we have $\mathrm{lch}_i(T) = 2$ which can be obtained from the following picture.

$$\begin{array}{|c|c|c|c|}\hline 3 & 5 & 7 & 8 \\ \hline 1 & 2 & 4 & 6 \\ \hline\end{array} \quad \Rightarrow \quad \begin{array}{cc|c|c|c|c|}\cline{3-6} & & 3 & 5 & 7 & 8 \\ \hline \multicolumn{1}{|c|}{1} & 2 & 4 & 6 & & \\ \cline{1-4}\end{array}$$

For $P, Q \in \mathrm{RSYT}(\lambda)$ where $\lambda = (\lambda_1, \ldots, \lambda_l)$, we define the symmetrized offset constants with respect to (P, Q), denoted by $\mathbf{s}(P,Q) = (s_1, \ldots, s_l) \in \mathbb{Z}^l$, as follows.

$$s_i = \begin{cases} 0 & \text{if } i = 1 \text{ or } \lambda_i > \lambda_{i+1}, \\ s_{i-1} + \mathrm{lch}_{i-1}(P) - \mathrm{lch}_{i-1}(Q) & \text{otherwise.} \end{cases}$$

In other words, we have $s_i - s_{i-1} = \mathrm{lch}_{i-1}(P) - \mathrm{lch}_{i-1}(Q)$ whenever $\lambda_{i-1} = \lambda_i$. (See [2, Definition 5.8] and [1, Theorem 5.10] for the equivalent definitions.) It is easy to show that for tabloids P, Q, R of the same shape, we have $\mathbf{s}(P,Q) + \mathbf{s}(Q,R) = \mathbf{s}(P,R)$, thus in particular $\mathbf{s}(P,Q) + \mathbf{s}(Q,P) = \mathbf{s}(P,P) = 0$.

Example 2.6. Let $P = \begin{array}{|c|c|c|}\hline \overline{2} & \overline{4} & \overline{6} \\ \hline \overline{3} & \overline{7} & \overline{8} \\ \hline \overline{1} & \overline{5} & \overline{9} \\ \hline \end{array}$ and $Q = \begin{array}{|c|c|c|}\hline \overline{3} & \overline{5} & \overline{7} \\ \hline \overline{1} & \overline{2} & \overline{8} \\ \hline \overline{4} & \overline{6} & \overline{9} \\ \hline \end{array}$. Then $\mathrm{lch}_1(P) = 0, \mathrm{lch}_2(P) = 1, \mathrm{lch}_1(Q) = 2$, and $\mathrm{lch}_2(Q) = 0$. Thus it follows that $\mathbf{s}(P,Q) = (0, -2, -1)$.

For $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_r)$, we define $\mathrm{rev}_\lambda(\boldsymbol{\rho})$ to be the integer vector obtained from $\boldsymbol{\rho}$ by reversing the order of elements corresponding to the parts of the same length in λ. For example, if $\lambda = (2,2,1,1,1)$ and $\boldsymbol{\rho} = (3,1,5,2,4)$ then we have $\mathrm{rev}_\lambda(\boldsymbol{\rho}) = (1,3,4,2,5)$. We say that $\boldsymbol{\rho} \in \mathbb{Z}^{l(\lambda)}$ is *dominant* with respect to (P,Q) if $-(\boldsymbol{\rho} - \mathbf{s}(P,Q))$ is non-increasing, or equivalently $\mathrm{rev}_\lambda(\boldsymbol{\rho} - \mathbf{s}(P,Q))$ is non-decreasing.

Example 2.7. In the Example 2.6 the vector $\boldsymbol{\rho} = (2,0,2) \in \mathbb{Z}^3$ is dominant with respect to (P,Q) because $\boldsymbol{\rho} - \mathbf{s}(P,Q) = (2,2,3)$ is a nondecreasing sequence.

We set

$$\Omega := \bigsqcup_{\lambda \vdash n} \mathrm{RSYT}(\lambda) \times \mathrm{RSYT}(\lambda) \times \mathbb{Z}^{l(\lambda)},$$

$$\Omega_{\mathrm{dom}} := \Big\{ (P,Q,\boldsymbol{\rho}) \in \Omega \mid \boldsymbol{\rho} \text{ is dominant with respect to } (P,Q) \text{ with } \sum_i \rho_i = 0 \text{ for } \boldsymbol{\rho} = (\rho_1 \cdots, \rho_d) \Big\}.$$

We define $\Phi \colon \tilde{S}_n \to \Omega_{\mathrm{dom}}$ by $w \mapsto (P(w), Q(w), \boldsymbol{\rho}(w))$ to be the bijection defined in [2] using the affine matrix-ball construction. Also, let $\Psi \colon \Omega \to \tilde{S}_n$ be the surjection defined by the backward AMBC. Then by [2, Theorem 5.12] we have $\Psi|_{\Omega_{\mathrm{dom}}} = \Phi^{-1}$. Both AMBC and backward AMBC were explained in great detail in both [2] and [1].

For the left and right descent set $L(w)$ and $R(w)$ for w, we have

Proposition 2.8. For any permutation w, $L(w) = \tau(P(w))$ and $R(w) = \tau(Q(w))$.

2.4 Affine FPF graphs and molecules

The following definitions are from Stembridge's papers [9, 10].

Definition 2.9. For an algebra $\mathcal{A}$, an *I-labeled graph* for a finite set I is a triple $\Gamma = (V, \omega, \nu)$ where

(i) V is a finite vertex set;

(ii) $\omega : V \times V \to \mathcal{A}$ is a map;

(iii) $\nu : V \to \mathcal{P}(I)$ is a map assigning a subset of I to each vertex.

We view Γ as a weighted directed graph on the vertex set V with an edge $x \xrightarrow{\omega(x,y)} y$ if $\omega(x, y) \neq 0$.

Now assume that (W, S) is a Coxeter system and $\mathcal{H}$ is the corresponding Iwahori–Hecke algebra.

Definition 2.10. An S-labeled graph $\Gamma = (V, \omega, \nu)$ is a *W-graph* if the free $\mathcal{A}$-module generated by V can be given an $\mathcal{H}$-module structure with

$$H_s x = \begin{cases} vx & s \notin \nu(x) \\ -v^{-1}x + \sum_{y \in V; s \notin \nu(y)} \omega(x, y) y & s \in \nu(x) \end{cases} \qquad \text{for } s \in S \text{ and } x \in V.$$

Example 2.11. There exist a unique ring homomorphism $\mathcal{H} \to \mathcal{H}$ with $v \mapsto v^{-1}$ and $H_s \mapsto H_s^{-1}$; we denote this map by $H \mapsto \overline{H}$, and refer to it as the *bar operator* of $\mathcal{H}$. Write $<$ for the Bruhat order on W. By well-known results of Kazhdan and Lusztig [3], for each $w \in W$ there is a unique $\underline{H}_w \in \mathcal{H}$ with

$$\underline{H}_w = \overline{\underline{H}_w} \in H_w + \sum_{y < w} v^{-1}\mathbb{Z}[v^{-1}] H_y.$$

The elements $\{\underline{H}_w\}_{w \in W}$ form an $\mathcal{A}$-basis for $\mathcal{H}$, called the *Kazhdan–Lusztig basis*. Define $h_{x,y} \in \mathbb{Z}[v^{-1}]$ for $x, y \in W$ such that $\underline{H}_y = \sum_{x \in W} h_{x,y} H_x$, and let $\mu(x, y)$ be the coefficient of v^{-1} in $h_{x,y}$. Finally, let

$$\nu(x) = \{s \in S : \ell(sx) < \ell(x)\} \qquad \text{and} \qquad \omega(x, y) = \begin{cases} \mu(x, y) & \text{if } \nu(x) \not\subset \nu(y) \\ 0, & \text{otherwise} \end{cases}$$

for $x, y \in W$. Then $\Gamma = (W, \omega, \nu)$ is a W-graph [3].

We turn back to the case $W = \tilde{S}_n$. Let $\mathcal{M} = \mathcal{A}\text{-span}\{M_z : z \in \mathcal{F}_n\}$ and $\mathcal{N} = \mathcal{A}\text{-span}\{N_z : z \in \mathcal{F}_n\}$ denote the free $\mathcal{A}$-modules with bases given by the symbols M_z and N_z for $z \in \mathcal{F}_n$. We call $\{M_z\}_{z \in \mathcal{F}_n}$ and $\{N_z\}_{z \in \mathcal{F}_n}$ the standard bases of $\mathcal{M}$ and $\mathcal{N}$, respectively. Let $S = \{s_1, s_2, \ldots, s_n\}$.

Proposition 2.12 ([12, Corollary 4.15]). Both $\mathcal{M}$ and $\mathcal{N}$ have unique $\mathcal{H}$-module structures such that if $s \in S$ and $z \in \mathcal{F}_n$ then we have

$$H_s M_z = \begin{cases} M_{szs} & \ell^{\mathsf{FPF}}(szs) > \ell^{\mathsf{FPF}}(z) \\ v M_z & \ell^{\mathsf{FPF}}(szs) = \ell^{\mathsf{FPF}}(z) \\ M_{szs} + (v - v^{-1}) M_z & \ell^{\mathsf{FPF}}(szs) < \ell^{\mathsf{FPF}}(z) \end{cases}$$

and

$$H_s N_z = \begin{cases} N_{szs} & \ell^{\mathsf{FPF}}(szs) > \ell^{\mathsf{FPF}}(z) \\ -v^{-1} N_z & \ell^{\mathsf{FPF}}(szs) = \ell^{\mathsf{FPF}}(z) \\ N_{szs} + (v - v^{-1}) N_z & \ell^{\mathsf{FPF}}(szs) < \ell^{\mathsf{FPF}}(z). \end{cases}$$

Rains and Vazirani's general theory of quasiparabolic sets gives us for free the $\mathcal{H}$-module structures described in the previous result. These modules are potentially interesting to study on their own. We note a few special properties which follow from results in [6].

We write $f \mapsto \overline{f}$ for the automorphism of $\mathcal{A}$ interchanging v and v^{-1}. A map $U \to V$ of $\mathcal{A}$-modules is *$\mathcal{A}$-antilinear* if $x \mapsto y$ implies $ax \mapsto \overline{a}y$ for all $a \in \mathcal{A}$.

Corollary 2.13 ([12, Corollary 4.16])**.** The $\mathcal{H}$-modules $\mathcal{M}$ and $\mathcal{N}$ have the following properties:

(a) There are unique $\mathcal{A}$-antilinear maps $\mathcal{M} \to \mathcal{M}$ and $\mathcal{N} \to \mathcal{N}$, which we write as $X \mapsto \overline{X}$, with

$$\overline{HM} = \overline{H} \cdot \overline{M} \text{ and } \overline{M_\Theta} = M_\Theta \qquad \text{and} \qquad \overline{HN} = \overline{H} \cdot \overline{N} \text{ and } \overline{N_\Theta} = N_\Theta$$

for all $M \in \mathcal{M}$, $N \in \mathcal{N}$, and $\Theta \in \{\Theta^\pm\}$. Moreover, both of these maps are involutions.

(b) The $\mathcal{H}$-modules $\mathcal{M}$ and $\mathcal{N}$ have unique $\mathcal{A}$-bases $\{\underline{M}_x\}_{x \in \mathcal{F}_n}$ and $\{\underline{N}_x\}_{x \in \mathcal{F}_n}$ satisfying

$$\underline{M}_x = \overline{\underline{M}_x} \in M_x + \sum_{w <_F x} v^{-1}\mathbb{Z}[v^{-1}] M_w \quad \text{and} \quad \underline{N}_x = \overline{\underline{N}_x} \in N_x + \sum_{w <_F x} v^{-1}\mathbb{Z}[v^{-1}] N_w$$

where both sums are over $w \in \mathcal{F}_n$. We refer to these as the *canonical bases* of $\mathcal{M}$ and $\mathcal{N}$.

Define $\mathbf{m}_{x,y}$ and $\mathbf{n}_{x,y}$ for $x, y \in \mathcal{F}_n$ as the polynomials in $\mathbb{Z}[v^{-1}]$ such that

$$\underline{M}_y = \sum_{x \in \tilde{S}_n} \mathbf{m}_{x,y} M_x \qquad \text{and} \qquad \underline{N}_y = \sum_{x \in \tilde{S}_n} \mathbf{n}_{x,y} N_x.$$

Let $\mu_{\mathbf{m}}(x, y)$ and $\mu_{\mathbf{n}}(x, y)$ denote the coefficients of v^{-1} in $\mathbf{m}_{x,y}$ and $\mathbf{n}_{x,y}$. Define $\nu_{\mathbf{m}}$, $\nu_{\mathbf{n}}$: $\mathcal{F}_n \to \mathcal{P}(S)$ by

$$\nu_{\mathbf{m}}(x) = \{s \in S : sxs \leqslant_F x\} \qquad \text{and} \qquad \nu_{\mathbf{n}}(x) = \{s \in S : x \leqslant_F sxs\}$$

where $S = \{s_1, s_2, \dots, s_n\} \subset \tilde{S}_n$. Finally, let $\omega_{\mathbf{m}}$: $\mathcal{F}_n \times \mathcal{F}_n \to \mathbb{Z}$ be the map with

$$\omega_{\mathbf{m}}(x, y) = \begin{cases} \mu_{\mathbf{m}}(x, y) + \mu_{\mathbf{m}}(y, x) & \nu_{\mathbf{m}}(x) \not\subset \nu_{\mathbf{m}}(y) \\ 0 & \nu_{\mathbf{m}}(x) \subset \nu_{\mathbf{m}}(y). \end{cases}$$

Define $\omega_{\mathbf{n}}$: $\mathcal{F}_n \times \mathcal{F}_n \to \mathbb{Z}$ by the same formula, but with $\mu_{\mathbf{m}}$ and $\nu_{\mathbf{m}}$ replaced by $\mu_{\mathbf{n}}$ and $\nu_{\mathbf{n}}$.

Corollary 2.14 ([6, Theorem 3.26])**.** Both $\Gamma_n^{\mathbf{m}} = (\mathcal{F}_n, \omega_{\mathbf{m}}, \nu_{\mathbf{m}})$ and $\Gamma_n^{\mathbf{n}} = (\mathcal{F}_n, \omega_{\mathbf{n}}, \nu_{\mathbf{n}})$ are $\tilde{S}_n$-graphs.

We call these graphs affine `FPF` graphs. The strongly connected components in a W-graph Γ are called *cells*. The connected components with respect to bidirected edges are called *molecules*. It is generally an interesting problem to determine the cells and molecules in a given W-graph.

If $n = 2$ then one can show that $\Gamma_n^{\mathbf{m}}$ and $\Gamma_n^{\mathbf{n}}$ both decompose into just two cells given by $\mathcal{F}_n^+$ and $\mathcal{F}_n^-$.

3 Affine Knuth moves and involutive transformation

Definition 3.1. Let $i \in \mathbb{Z}$. Two affine permutations w and w' are connected by a Knuth move at position $\bar{i}$ if all of the following hold:

- for all j such that $j \equiv i \pmod{n}$, we have $w'(j) = w(j+1)$ and $w'(j+1) = w(j)$;
- for all j such that $j \not\equiv i \pmod{n}$, $j \not\equiv i+1 \pmod{n}$, we have $w'(j) = w(j)$;
- at least one of $w(i+2)$ and $w(i-1)$ is numerically between $w(i)$ and $w(i+1)$.

We denote it as $w \underset{\mathsf{K}}{\overset{\bar{i}}{\sim}} w'$. Also, we write $w \underset{\mathsf{K}}{\overset{\bar{i}}{\sim}} w$ if $w(i+2)$ and $w(i-1)$ are not numerically between $w(i)$ and $w(i+1)$.

Definition 3.2. Let $n \geqslant 3$. For a tabloid T and each integer $1 \leqslant i \leqslant n$, the *dual equivalence operator* D_i is the map acting on T by

$$D_i(T) := \begin{cases} s_i(T) & \text{if } \bar{i} \in \tau(T) \backslash \tau(s_i(T)), \overline{i \pm 1} \in \tau(s_i(T)) \backslash \tau(T) \\ & \text{or } \overline{i \pm 1} \in \tau(T) \backslash \tau(s_i(T)), \bar{i} \in \tau(s_i(T)) \backslash \tau(T) \\ T & \text{otherwise.} \end{cases}$$

For $x, y \in [n]$, we say $x \prec_T y$ if y appears above x and $x \preceq_T y$ if y appears above x or in the same row as x. Then we have $D_i(T) = s_i(T)$ if and only if one of the following happens: $i+1 \preceq_T i+2 \prec_T i$, $i \prec_T i+2 \preceq_T i+1$, $i+1 \prec_T i-1 \preceq_T i$, $i \preceq_T i-1 \prec_T i+1$.

Definition 3.3. Let $T = (T_1, T_2, \cdots)$ be a tabloid of shape λ and suppose that $i \in T_t$ for some $1 \leqslant i \leqslant n$ and $1 \leqslant t \leqslant l = l(\lambda)$. We define $\delta(T, i) = (\delta_1, \delta_2, \cdots, \delta_l)$ and $\iota(T, i) = (\iota_1, \iota_2, \cdots, \iota_l)$ as follows. Here $[-]$ is the Iverson bracket.

- If $\lambda_t = \lambda_{t-1}$, then we put $\delta(T, i) = 0$. Otherwise, $\delta(T, i) = ([\lambda_j = \lambda_t])_{1 \leqslant j \leqslant l}$.
- $\iota(T, i) = ([j = t])_{1 \leqslant j \leqslant l} = ([i \in T_j])_{1 \leqslant j \leqslant l}$.

Theorem 3.4 ([1, Theorem 3.11])**.** Suppose w is an affine permutation with $\Phi(w) = (P, Q, \rho)$ and $w' \underset{\mathsf{K}}{\overset{\bar{i}}{\sim}} w$. Then $\Phi(w') = (P, D_i(Q), \rho + \iota(Q, 1) - \iota(Q, n))$.

Definition 3.5. Let $i \in \mathbb{Z}$. Two affine permutations w and w' are connected by a dual Knuth move at position $\bar{i}$ if w^{-1} and $(w')^{-1}$ are connected by a Knuth move at position $\bar{i}$.

We denote it as $w \underset{\mathsf{dK}}{\overset{\bar{i}}{\sim}} w'$. We also write $w \underset{\mathsf{dK}}{\overset{\bar{i}}{\sim}} w$ if $w \underset{\mathsf{K}}{\overset{\bar{i}}{\sim}} w$.

Proposition 3.6 ([4, Proposition 5.10])**.** Suppose w is an affine permutation with $\Phi(w) = (P, Q, \rho)$ and $w' \underset{\mathsf{dK}}{\overset{\bar{i}}{\sim}} w$. Then $\Phi(w') = (D_i(P), Q, \rho - \iota(P, 1) + \iota(P, n))$.

Definition 3.7. Let $i \in \mathbb{Z}$. Two affine involutions z and z' are connected by an involutive transformation at position $\bar{i}$ if there exist an affine permutations w such that $z \underset{\mathsf{K}}{\overset{\bar{i}}{\sim}} w \underset{\mathsf{dK}}{\overset{\bar{i}}{\sim}} z'$.

We denote it as $w \underset{\mathsf{It}}{\overset{\bar{i}}{\sim}} w'$.

Corollary 3.8 ([4, Proposition 5.10])**.** Suppose w is an affine involution with $\Phi(w) = (P, P, \rho)$. Then $w' \underset{\mathsf{It}}{\overset{\bar{i}}{\sim}} w$ if and only if $\Phi(w') = (D_i(P), D_i(P), \rho)$.

Theorem 3.9. If $x, y \in \tilde{I}_n$ have $x \underset{\mathsf{It}}{\overset{\bar{i}}{\sim}} y$ and $A := \{i-1, i, i+1\}$ then

$$y = \begin{cases} x & \text{if } x(A) \neq A \text{ and } x(i) \text{ is between } x(i-1) \text{ and } x(i+1) \\ (i-1, i)x(i-1, i) & \text{if } x(A) \neq A \text{ and } x(i+1) \text{ is between } x(i-1) \text{ and } x(i) \\ (i, i+1)x(i, i+1) & \text{if } x(A) \neq A \text{ and } x(i-1) \text{ is between } x(i) \text{ and } x(i+1) \\ (i-1, i+1)x(i-1, i+1) & \text{if } x(A) = A. \end{cases}$$

4 Shift and Knuth moves

Define $\omega = [2, \cdots, n, n+1]$. Then we have the following proposition.

Proposition 4.1 ([4, Proposition 5.5])**.** Suppose w is an affine permutation with $\Phi(w) = (P, Q, \rho)$. Then

$$\Phi(\omega w) = (\omega(P), Q, \rho - \mathbf{s}(P, Q) + \mathbf{s}(\omega(P), Q) + \delta(Q, n)),$$
$$\Phi(w\omega^{-1}) = (P, \omega(Q), \rho - \mathbf{s}(P, Q) + \mathbf{s}(P, \omega(Q)) - \delta(Q, n)).$$

Moreover, if w is an affine involution, then $\Phi(\omega w \omega^{-1}) = (\omega(P), \omega(P), \rho)$.

Proposition 4.2. Suppose x, y are affine involutions, then $x \underset{\mathsf{It}}{\overset{\bar{i}}{\sim}} y$ if and only if $\omega x \omega^{-1} \underset{\mathsf{It}}{\overset{\overline{i+1}}{\sim}} \omega y \omega^{-1}$.

Proposition 4.3. Suppose w is an affine involution, then w and $\omega w \omega^{-1}$ cannot be connected by a sequence of involutive transformation.

Theorem 4.4. Suppose w is an affine involution with $\Phi(w) = (P, P, \rho)$. For each P' with same shape as P, let $w' = \Psi(P', P', \rho)$. Then there exist $i \in [n]$, such that w and $\omega^i w' \omega^{-i}$ are connected by a sequence of involutive transformations.

5 Bidirected edges

In this section, we discuss about the bidirected edges for $\Gamma_n^{\mathbf{m}}$ and $\Gamma_n^{\mathbf{n}}$, which is an analogue of Lemma 3.10 of [7].

Lemma 5.1. There is a bidirected edge in the $\tilde{S}_n$-graph $\Gamma_n^{\mathbf{m}}$ between $x < y \in \mathcal{F}_n$ if and only if there exist $\{s, t\} = \{s_{i-1}, s_i\}$, such that $txt \leqslant x < sxs = y < tyt$.

In this case, we label this bidirected edge by $\bar{i}$.

Proposition 5.2. There is a bidirected edge in the $\tilde{S}_n$-graph $\Gamma_n^{\mathbf{m}}$ between $x, y \in \mathcal{F}_n$ labeled $\bar{i}$ if and only if $x \underset{\mathsf{It}}{\overset{\bar{i}}{\sim}} y$.

Lemma 5.3. There is a bidirected edge in the $\tilde{S}_n$-graph $\Gamma_n^{\mathbf{n}}$ between $x < y \in \mathcal{F}_n$ if and only if there exist $\{s, t\} = \{s_{i-1}, s_i\}$, such that $txt < x < sxs = y \leqslant tyt$.

In this case, we label this bidirected edge by $\bar{i}$.

Definition 5.4. Suppose $x, y \in \mathcal{F}_n$. For $j \in \{i-1, i, i+1\}$ let

$$v(j) := \begin{cases} x(j) & \text{if } x(j) \notin \{i-1, i, i+1\} \\ j & \text{if } j \neq x(j) \in \{i-1, i, i+1\}. \end{cases} \tag{5.1}$$

Then, we define $x \underset{\mathsf{nIt}}{\overset{i}{\sim}} y$ for some $1 < i < n$ if

$$y = \begin{cases} x & \text{if } v(i) \text{ is between } v(i-1) \text{ and } v(i+1) \\ (i-1, i)x(i-1, i) & \text{if } v(i+1) \text{ is between } v(i-1) \text{ and } v(i) \\ (i, i+1)x(i, i+1) & \text{if } v(i-1) \text{ is between } v(i) \text{ and } v(i+1). \end{cases}$$

If $x \underset{\mathsf{nIt}}{\sim} y$, we say that they are connected by an $\mathbf{n}$-involutive transformation.

Theorem 5.5. There is a bidirected edge in the $\tilde{S}_n$-graph $\Gamma_n^{\mathbf{n}}$ between $x, y \in \mathcal{F}_n$ labeled $\bar{i}$ if and only if $x \underset{\mathsf{nIt}}{\overset{\bar{i}}{\sim}} y$.

6 Molecules of affine fpf graph

Theorem 6.1. For $w, v \in \mathcal{F}_n$, they are in the same molecule in $\Gamma_n^{\mathbf{m}}$ if and only if they are connected by a sequence of involutive transformations.

Corollary 6.2. For two affine involutions w and v, they are in the same molecule in $\Gamma_n^{\mathbf{m}}$ only if they have the same sign, and corresponding to tabloids of the same shape with same dominant weight applying AMBC.

Note that this is just a necessary condition, which is not sufficient, according to Proposition 4.3. Moreover, we have the following example.

For $n = 4$, by definition of molecule, we can find such two molecules:

$$\{[4,3,2,1],[-4,3,2,9],[3,-4,1,10],[-5,4,9,2],[4,-5,10,1],[4,11,-6,1]\}$$

and

$$\{[0,-1,6,5],[0,7,-2,5],[7,0,-3,6],[-1,8,5,-2],[8,-1,6,-3],[-8,-1,6,13]\}.$$

All of them have the same sign $+1$ and are corresponding to tabloids of the same shape $\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array}$ with same dominant weight $\begin{array}{|c|}\hline 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \end{array}$.

Although we cannot distinguish between these molecules with same sign, same shape and same dominant weight, there exist a graph isomorphism between these molecules.

Theorem 6.3. The molecules of $\Gamma_n^{\mathbf{m}}$ with same sign, same shape and same dominant weight are isomorphic to each other.

Proposition 6.4. The number of molecules of $\Gamma_n^{\mathbf{m}}$ with the same shape λ and same dominant weight ρ is the smallest i such that $\omega^i(P(\lambda))$ and $P(\lambda)$ are connected by a sequence of dual equivalence operators. Moreover, this number is independent from the weight ρ.

We denote this number by $o(\lambda)$ and call it the order of λ. Recall that $a|b$ means a divides b for two integers a, b. Moreover, $a|b|c$ means $a|b$ and $b|c$.

Proposition 6.5. We have $2|o(\lambda)|n$. Specifically, $o((1,1,\cdots,1)) = n$ and $o((\frac{n}{2},\frac{n}{2})) = 2$.

Proposition 6.6. Every molecule of $\Gamma_n^{\mathbf{m}}$ has a single minimal element.

Theorem 6.7. For $w, v \in \mathcal{F}_n$, they are in the same molecule in $\Gamma_n^{\mathbf{n}}$ if and only if they are connected by a sequence of $\mathbf{n}$-involutive transformation.

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