Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #132, 12 pp.

Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

Enumerating 1324-avoiders with few inversions

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Abstract. The problem of determining the number of 1324-avoiding permutations of length n has received much attention. We work towards this goal by enumerating $\mathrm{av}_n^k(1324)$, the number of 1324-avoiding n-permutations with exactly k inversions, for all k and $n \geq (k+7)/2$. This is achieved with a new structural characterization of such permutations in terms of a new notion of almost-decomposability. In particular, our enumeration verifies half of a conjecture of Claesson, Jelínek and Steingrímsson, according to which $\mathrm{av}_n^k(1324) \leq \mathrm{av}_{n+1}^k(1324)$ for all n and k. Proving the full conjecture would improve the best known upper bound for the exponential growth rate of the number of 1324-avoiders from 13.5 to approximately 13.002.

Keywords: 1324, enumeration, inversions, pattern avoidance, permutations

1 Introduction

A permutation $\pi \in \mathfrak{S}_n$ *contains* a *pattern* $\tau \in \mathfrak{S}_m$ if there exist indices $i_1 < \ldots < i_m$ such that $\pi(i_a) < \pi(i_b)$ if and only if $\tau(a) < \tau(b)$ for all $a, b \in [m]$. Otherwise, π *avoids* τ. An *inversion* in π is a pair of indices (i, j) such that $i < j$ and $\pi_i > \pi_j$. We denote by $\mathrm{Av}_n(\tau)$ the set of all permutations of length n avoiding τ, and by $\mathrm{Av}_n^k(\tau) \subseteq \mathrm{Av}_n(\tau)$ those with exactly k inversions. Furthermore, we set $\mathrm{av}_n(\tau) = |\mathrm{Av}_n(\tau)|$ and $\mathrm{av}_n^k(\tau) = |\mathrm{Av}_n^k(\tau)|$. Two patterns σ and τ are called *Wilf equivalent* if $\mathrm{av}_n(\sigma) = \mathrm{av}_n(\tau)$ for all n.

1.1 Avoiding 1324

Patterns of length four or lower are generally well understood, except for a single case: the pattern 1324. The numbers $\mathrm{av}_n(1324)$ have been determined computationally up to $n = 50$ (sequence A061552 in the OEIS [16]), but in general not even the asymptotics are well-understood. The *Stanley–Wilf limit*

$$L(\tau) = \lim_{n\to\infty} \mathrm{av}_n(\tau)^{1/n}$$

exists for all patterns τ due to the Marcus–Tardos theorem [2, 15], but when $\tau = 1324$, only loose bounds are know. Table 1 shows the timeline of the evolution of these bounds;

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currently they are $10.27 < L(1324) < 13.5$ [4]. Conway, Guttmann and Zinn-Justin have convincingly estimated that $L(1324) \approx 11.600 \pm 0.003$ [12].

	Lower	Upper
2004. Bóna [7]		288
2005. Bóna [8]	9	
2006. Albert et al. [1]	9.47	
2012. Claesson, Jelínek and Steingrímsson [11]		16
2014. Bóna [9]		13.93
2015. Bóna [10]		13.74
2015. Bevan [3]	9.81	
2020. Bevan et al. [4]	10.27	13.5

Table 1: Best known upper and lower bounds for $L(1324)$ throughout history.

One possible avenue towards improvement is suggested by a conjecture of Claesson, Jelínek and Steingrímsson.

Conjecture 1.1 ([11, Conjecture 13])**.** *For all nonnegative integers n and k,*

$$\mathrm{av}_n^k(1324) \leq \mathrm{av}_{n+1}^k(1324).$$

As was demonstrated in [11], the conjecture implies a new upper bound $L(1324) \leq \exp\left(\pi\sqrt{2/3}\right) < 13.002$, using the fact that $\mathrm{av}_n^k(1324)$ is constant when the number k of inversions is fixed and $n \geq k+2$. Our main result proves half of the conjecture.

Theorem 1.2. *For all nonnegative integers k and $n \geq \frac{k+7}{2}$,*

$$\mathrm{av}_n^k(1324) = [x^k]\left(P(x)^2 - \frac{R_n(x)}{1-x}\right),$$

where

$$R_n(x) = 2(2+x)x^{n-1}P(x)^2,$$

and $P(x)$ is the generating function for the partition numbers. In particular,

$$\mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324) = [x^k]R_n(x) \geq 0,$$

and this difference has a combinatorial interpretation.

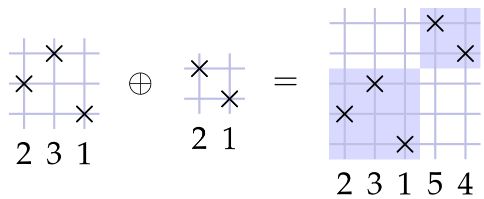
The proof relies on a new notion of *almost decomposable* permutations, and is available in its entirety in our preprint [13]. The following subsection motivates this by defining *decomposable* permutations, and explains the constants $\mathrm{av}_{k+2}^k(1324)$.

1.2 Direct sums and decomposability

For two permutations $\sigma \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_m$, we define the *direct sum* $\sigma \oplus \tau \in \mathfrak{S}_{n+m}$ by

$$(\sigma \oplus \tau)(i) = \begin{cases} \sigma(i) & \text{if } i \leq n, \\ n + \tau(i-n) & \text{if } i > n. \end{cases}$$

For example, $231 \oplus 21$ is obtained in the following way.



If a permutation π is the direct sum of two nonempty permutations, we call π *decomposable*, and otherwise *indecomposable*. We can write π uniquely as a direct sum

$$\pi = \pi^{(1)} \oplus \pi^{(2)} \oplus \ldots \oplus \pi^{(c)},$$

where each *component* $\pi^{(i)}$ is indecomposable (and nonempty). The formula (see [11, Lemma 8])

$$\mathrm{comp}(\pi) + \mathrm{inv}(\pi) \geq |\pi|,$$

where $\mathrm{comp}(\pi)$, $\mathrm{inv}(\pi)$ and $|\pi|$ denote the number of components, the number of inversions and the length of π, respectively, indicates that a permutation with few inversions should have many components. In particular, if $\mathrm{inv}(\pi) \leq |\pi| - 2$, then $\mathrm{comp}(\pi) \geq |\pi| - \mathrm{inv}(\pi) \geq 2$. It is easy to see that a decomposable permutation π avoids 1324 if and only if it is of the form

$$\pi = \pi^{(1)} \oplus 1 \oplus 1 \oplus \ldots \oplus 1 \oplus \pi^{(2)}, \tag{1.1}$$

where $\pi^{(1)}$ avoids 132 and $\pi^{(2)}$ avoids 213. The *inversion table* $b_1 b_2 \ldots b_n$ of a 132-avoider of length n, defined by $b_i = |\{j > i : \pi_j < \pi_i\}|$, is weakly decreasing and therefore – with the exclusion of trailing 0's – an integer partition of $\mathrm{inv}(\pi)$. It follows that

$$\mathrm{av}_n^k(132) = \mathrm{av}_n^k(213) = p(k)$$

for all $n \geq k+1$, where $p(k)$ is the kth partition number. Hence, (1.1) gives

$$\mathrm{av}_n^k(1324) = [x^k]P(x)^2,$$

where $P(x) = \sum_{k \geq 0} p(k) x^k$ and $n \geq k+2$ [11, Proposition 15].

1.3 Interpreting the main result

Observe that due to the preceding discussion, Conjecture 1.1 trivially holds (with equality) for all $n \geq k+2$. Our main result, Theorem 1.2, improves this to $n \geq \frac{k+7}{2}$, and therefore essentially proves half of the conjecture, along with providing an enumeration for those values of $\mathrm{av}_n^k(1324)$. The strategy is to find an injection $\mathrm{Av}_n^k(1324) \to \mathrm{Av}_{n+1}^k(1324)$ and analyze it in order to enumerate the permutations not contained in its image. The injection relies on *almost decomposability*, which is related to normal decomposability, so it is not surprising that the partition numbers show up.

It is useful to keep in mind Table 2, in which the entry on row n and column k equals $\mathrm{av}_n^k(1324)$. Conjecture 1.1 is equivalent to the statement that each column of the diagram is weakly increasing as n increases. The blue cells indicate the constant parts of each column; the sequence $1, 2, 5, 10, 20, \ldots$ comes from the generating function $P(x)^2$. The red cells contain the new numbers enumerated by Theorem 1.2. Specifically, the sequence of numbers in the blue and red cells on row n is given by the first $2n-6$ coefficients of the generating function $P(x)^2 - R_n(x)/(1-x)$.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	
1	1													
2	1	1												
3	1	2	2	1										
4	1	2	5	6	5	3	1							
5	1	2	5	10	16	20	20	15	9	4	1			
6	1	2	5	10	20	32	51	67	79	80	68	49	29	...
7	1	2	5	10	20	36	61	96	148	208	268	321	351	...
8	1	2	5	10	20	36	65	106	171	262	397	568	784	...
9	1	2	5	10	20	36	65	110	181	286	443	664	985	...
10	1	2	5	10	20	36	65	110	185	296	467	714	1077	...
11	1	2	5	10	20	36	65	110	185	300	477	738	1127	...
12	1	2	5	10	20	36	65	110	185	300	481	748	1151	...

Table 2: The numbers $\mathrm{av}_n^k(1324)$.

The differences $\mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324)$ are displayed in Table 3. The blue 0's come from the constant part of each column, and the numbers in the red cells are given by $R_n(x)$. The diagram also shows that $n \geq \frac{k+7}{2}$ is the best possible bound for our method: if $n < \frac{k+7}{2}$ (and $k \geq 7$), then $\mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324)$ no longer equals $[x^k]R_n(x)$. An expanded version of Table 2 is available at https://akc.is/inv-mono/ courtesy of Anders Claesson.

$n\backslash k$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	
1	0	1													
2	0	1	2	1											
3	0	0	3	5	5	3	1								
4	0	0	0	4	11	17	19	15	9	4	1				
5	0	0	0	0	4	12	31	52	70	76	67	49	29	14	...
6	0	0	0	0	0	4	10	29	69	128	200	272	322	333	...
7	0	0	0	0	0	0	4	10	23	54	129	247	433	672	...
8	0	0	0	0	0	0	0	4	10	24	46	96	201	397	...
9	0	0	0	0	0	0	0	0	4	10	24	50	92	166	...
10	0	0	0	0	0	0	0	0	0	4	10	24	50	100	...
11	0	0	0	0	0	0	0	0	0	0	4	10	24	50	...
12	0	0	0	0	0	0	0	0	0	0	0	4	10	24	...

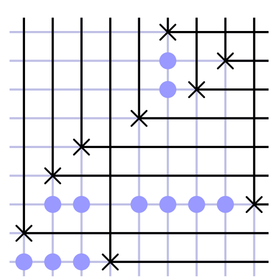
Table 3: The numbers $\mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324)$.

2 Almost decomposable permutations

We will often utilize the *plots* $\{(i, \pi_i) : i \in [n]\}$ (in cartesian coordinates) of permutations $\pi \in \mathfrak{S}_n$. Inverting π corresponds with reflecting its plot across the line $y = x$, and the *reverse-complement* $\mathrm{rc}(\pi)_i = n + 1 - \pi_{n+1-i}$ rotates the plot by 180 degrees. Both π^{-1} and $\mathrm{rc}(\pi)$ preserve 1324-avoidance and the number of inversions of π, so these are useful operations for us.

We also use the *Rothe diagram* of π, which is obtained from the plot of π by drawing lines to north and east from each point (i, π_i), and marking the empty coordinate points – these points are the inversions of π. The following figure shows an example.

245169783 =

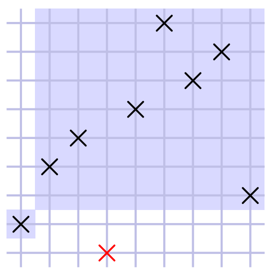


Definition 2.1. For $\pi \in \mathfrak{S}_n$ and $i \in [n]$, we denote by $\pi \setminus \pi_i$ the unique permutation in $\mathfrak{S}_{n-1}$ that is order-isomorphic to $\pi_1 \ldots \pi_{i-1}\pi_{i+1} \ldots \pi_n$. We say that $\pi \setminus \pi_i$ is obtained by *deleting* entry π_i from π.

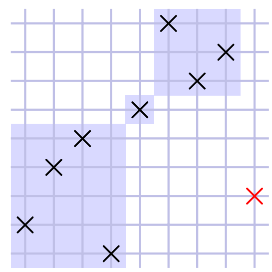
Definition 2.2. A permutation $\pi \in \mathfrak{S}_n$ is called *almost decomposable* if it is indecomposable, but at least one of $\pi \setminus 1$, $\pi \setminus n$, $\pi \setminus \pi_1$, $\pi \setminus \pi_n$ is decomposable.

For example, consider the indecomposable permutation $\pi = 245169783$. Since $\pi \setminus 1 = 13458672 = 1 \oplus 2347561$ is decomposable, π is almost decomposable. Notice that $\pi \setminus 3 = 23415867 = 2341 \oplus 1 \oplus 312$ is also decomposable.

Almost-decomposability means that deleting one of the points from the 'boundary' of the plot of the permutation makes it decomposable. Here are the plots of $\pi \setminus 1$ and $\pi \setminus 3$.



$\pi \setminus 1$



$\pi \setminus 3$

An important detail in Section 3, where the injection $\mathrm{Av}_n^k(1324) \to \mathrm{Av}_{n+1}^k(1324)$ is constructed, is that e.g. *both* $\pi \setminus 1$ and $\pi \setminus n$ can be decomposable when π is almost decomposable. However, not all combinations are possible, and this is critical.

Proposition 2.3. *Let $\pi \in \mathfrak{S}_n$ be indecomposable. If $\pi \setminus 1$ is decomposable then $\pi \setminus \pi_1$ is indecomposable, and similarly if $\pi \setminus n$ is decomposable then $\pi \setminus \pi_n$ is indecomposable.*

The following result is our strucural characterization of $\mathrm{Av}_n^k(1324)$ for $k \leq 2n-7$. Its proof relies on an intricate case-by-case analysis, the details of which are available in [13].

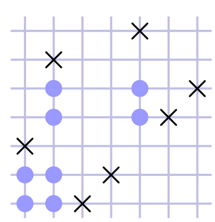
Theorem 2.4. *Each indecomposable permutation in $\mathrm{Av}_n^k(1324)$, where $k \leq 2n-7$, is almost decomposable.*

The bound $k \leq 2n-7$ is tight, since e.g.

$$\pi = [3,6,1,2,\ \ 7,\ldots,n,\ \ 4,5]$$

is 1324-avoiding with $2n-6$ inversions, and neither decomposable nor almost decomposable. Here is the plot of the first such permutation, $\pi = 3612745$.

3612745 =



Interestingly, the same permutation π has been used before to exemplify that two 1324-avoiding permutations can have the same *profile*: π and π^{-1} have the same left-to-right minima and right-to-left maxima in the same positions [5].

3 The injection

Denote by $\mathcal{D}_n^k$ and $\mathcal{A}_n^k$ the sets of decomposable and almost decomposable permutations in $\text{Av}_n^k(1324)$, respectively. In this section we will construct injections

$$g : \mathcal{D}_n^k \longrightarrow \text{Av}_{n+1}^k(1324) \qquad \text{and} \qquad f : \mathcal{A}_n^k \longrightarrow \text{Av}_{n+1}^k(1324),$$

with disjoint images, for *all* n and k. If $k \leq 2n-7$ then all permutations in $\text{Av}_n^k(1324)$ are decomposable or almost decomposable by Theorem 2.4, so our mappings combine to an injection

$$\text{Av}_n^k(1324) \longrightarrow \text{Av}_{n+1}^k(1324).$$

In particular, this verifies Conjecture 1.1 for all $k \leq 2n-7$.

First of all, any $\pi \in \mathcal{D}_n^k$ can be written in the form

$$\pi = \pi^{(1)} \oplus \underbrace{1 \oplus \ldots \oplus 1}_{m \text{ times}} \oplus \pi^{(2)}$$

for some $m \geq 0$ by (1.1). This allows us to set

$$g(\pi) = \pi^{(1)} \oplus \underbrace{1 \oplus \ldots \oplus 1}_{m+1 \text{ times}} \oplus \pi^{(2)} \in \mathcal{D}_{n+1}^k.$$

The image $g(\mathcal{D}_n^k)$ is exactly the set of all permutations in $\text{Av}_{n+1}^k(1324)$ with at least three components, and g is clearly injective. Note that when $n \geq k+2$, g is a bijection.

We will define f in a similar way. Let $\pi \in \mathcal{A}_n^k$.

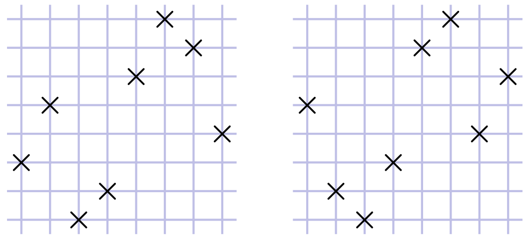
1. If $\pi \setminus \pi_1$ is decomposable, let $f(\pi)$ be the permutation with $f(\pi)_1 = \pi_1$ and $f(\pi) \setminus \pi_1 = g(\pi \setminus \pi_1)$.

2. If $\pi \setminus 1$ is decomposable, let $f(\pi) = f(\pi^{-1})^{-1}$.

3. Otherwise, let $f(\pi) = (\text{rc} \circ f \circ \text{rc})(\pi)$, where $\text{rc}(\pi)$ is the reverse-complement.

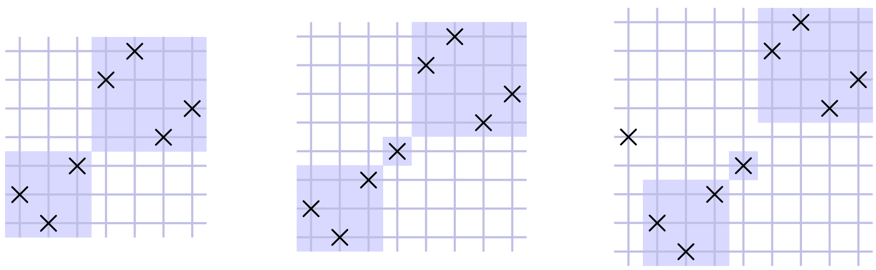
Remark 3.1. The mapping f preserves 1324-avoidance and the number of inversions. Moreover, the permutations in its image have at most two components.

Example 3.2. Consider the permutation $\pi = 35126874 \in \text{Av}_8^8(1324)$. Here are the plots of π and $\text{rc}(\pi)$.



π $\quad$ $\mathrm{rc}(\pi)$

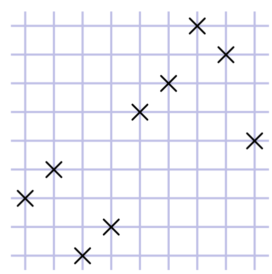
We can see that $\pi \setminus 1$ and $\pi \setminus \pi_1$ are both indecomposable, whereas $\pi \setminus \pi_n$ is decomposable. Therefore $\mathrm{rc}(\pi) \setminus \mathrm{rc}(\pi)_1$ is decomposable. The following figure shows the permutations $\mathrm{rc}(\pi) \setminus \mathrm{rc}(\pi)_1$, $g(\mathrm{rc}(\pi) \setminus \mathrm{rc}(\pi_1))$ and $(f \circ \mathrm{rc})(\pi)$.



$\mathrm{rc}(\pi) \setminus \mathrm{rc}(\pi)_1$ $\quad$ $g(\mathrm{rc}(\pi) \setminus \mathrm{rc}(\pi)_1)$ $\quad$ $(f \circ \mathrm{rc})(\pi)$

Finally, we get the following.

$$f(\pi) = (\mathrm{rc} \circ f \circ \mathrm{rc})(\pi) = 341267985 \quad =$$



Theorem 3.3. *The function* $f : \mathcal{A}_n^k \to \mathrm{Av}_{n+1}^k(1324)$ *is injective for all* n *and* k*. Furthermore,* $\mathrm{av}_n^k(1324) \leq \mathrm{av}_{n+1}^k(1324)$ *whenever* $k \leq 2n-7$*.*

This is not obvious; a complete proof is available in [13].

4 Enumerating the difference

The goal of this section is to describe the set of permutations

$$\mathcal{R}_{n+1}^k := \mathrm{Av}_{n+1}^k(1324) \setminus \left(g(\mathcal{D}_n^k) \cup f(\mathcal{A}_n^k)\right)$$

for all $k \leq 2n-7$. We will assume that $k \leq 2n-7$ throughout.

$\mathcal{R}_{n+1}^k$ consists of the following collections; for details, see [13].

(a) Permutations $\sigma \in \mathrm{Av}_{n+1}^k(1324)$ with $\sigma_1 = n+1$ or $\sigma_{n+1} = 1$. They are enumerated by $[x^k]\big(2x^n P(x)^2\big)$.

(b) Permutations $\sigma \in \mathrm{Av}_{n+1}^k(1324)$ with $\sigma_2 = n+1$ or $\sigma_{n+1} = 2$. These permutations are enumerated by $[x^k]\big(2x^{n-1} P(x)^2\big)$.

(c) Permutations $\sigma \in \mathrm{Av}_{n+1}^k(1324)$ such that $\sigma \setminus 1$, $\sigma \setminus \sigma_1$, $\sigma \setminus (n+1)$ and $\sigma \setminus \sigma_{n+1}$ all have at most two components. However, one can show no such permutations exist.

(d) Let h denote the natural extension of the left-inverse of f to all permutations $\sigma \in \mathrm{Av}_{n+1}^k(1324)$ such that $\mathrm{comp}(\sigma) \le 2$, $\sigma_1 \ne n+1$, $\sigma_{n+1} \ne 1$ and

$$\max\{\mathrm{comp}(\sigma \setminus i) : i = 1, n+1, \sigma_1, \sigma_{n+1}\} \ge 3.$$

This last part of $\mathcal{R}_{n+1}^k$ consists exactly of the permutations $\sigma \in \mathrm{Av}_{n+1}^k(1324)$ with $\sigma_1, \sigma_2 \ne n+1$ and $\sigma_{n+1} \ne 1, 2$, such that $f(h(\sigma)) \ne \sigma$. One can prove that (up to symmetry), all of these permutations have the structure described in Figure 1, leading to the enumeration $[x^k]\big(2x^{n-1} P(x)^2\big)$.

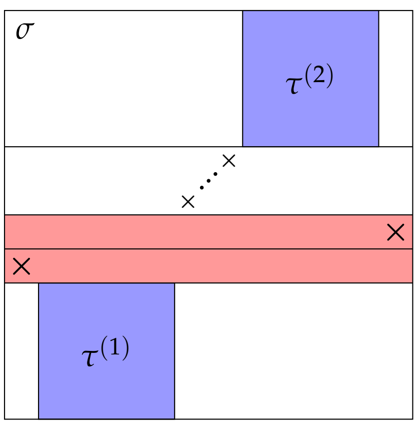


Figure 1: A permutation σ satisfying the assumptions of (d). Note that $\tau := \sigma \setminus \{\sigma_1, \sigma_{n+1}\}$ is decomposable, and that σ_1 is placed just above the first component of τ, with σ_{n+1} one step higher. It is easy to check that $f(h(\sigma)) \ne \sigma$.

The collections described above are disjoint and their union is $\mathcal{R}_{n+1}^k$, so we get

$$\mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324) = |\mathcal{R}_{n+1}^k| = [x^k]\big(2(2+x)x^{n-1} P(x)^2\big).$$

This proves Theorem 1.2.

5 Further directions and conjectures

This section contains discussion of three topics: extending our method; repeated differences of the numbers $\mathrm{av}_n^k(1324)$; and the unimodality of the sequences

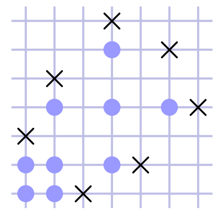
$$\mathrm{av}_n^0(1324), \ \ \mathrm{av}_n^1(1324), \ \ \ldots, \ \ \mathrm{av}_n^{\binom{n}{2}}(1324).$$

Extending almost-decomposability. A natural idea is to delete more than one point from the boundary of the plot of a permutation to make it decomposable. For example, one could imagine handling our earlier counterexample $\pi = 3612745$ by deleting entries 1 and 2. However, some permutations with as few as $2n - 5$ inversions behave poorly in this respect. One example is

$$\pi = [3, 4, \ldots, n-4, n-2, 1, n, 2, n-1, n-3].$$

Consider the case of $n = 7$; it is not clear which points should be deleted to make π decomposable.

3517264 =



Question 5.1. Is there a generalization of almost-decomposability by which the upper bound $2n - 7$ could be improved?

Question 5.2. Does almost-decomposability have other uses?

Repeated differences. We want to understand the numbers $\mathrm{av}_n^k(1324)$ also for $k > 2n - 7$. Studying the numbers when $k \leq 3n - 15$ we have found a tantalising pattern. We here extend the study of differences from Theorem 1.2 to repeated differences. In what follows we will write av_n^k for $\mathrm{av}_n^k(1324)$. Let

$$\begin{aligned} b_{r,n} := \big(\mathrm{av}_{n+3}^{2n+r-3} - \mathrm{av}_{n+2}^{2n+r-3}\big) - \big(\mathrm{av}_{n+2}^{2n+r-4} - \mathrm{av}_{n+1}^{2n+r-4}\big) \\ - \Big(\big(\mathrm{av}_{n+2}^{2n+r-5} - \mathrm{av}_{n+1}^{2n+r-5}\big) - \big(\mathrm{av}_{n+1}^{2n+r-6} - \mathrm{av}_{n}^{2n+r-6}\big)\Big). \end{aligned} \tag{5.1}$$

For example, by inspecting Table 3,

$$b_{0,8} = \underbrace{\big(\mathrm{av}_{11}^{13} - \mathrm{av}_{10}^{13}\big)}_{=100} - \underbrace{\big(\mathrm{av}_{10}^{12} - \mathrm{av}_{9}^{12}\big)}_{=92} - \Big(\underbrace{\big(\mathrm{av}_{10}^{11} - \mathrm{av}_{9}^{11}\big)}_{=50} - \underbrace{\big(\mathrm{av}_{9}^{10} - \mathrm{av}_{8}^{10}\big)}_{=46}\Big) = 4.$$

Anders Claesson has kindly provided us with data of av_n^k for $k, n \leq 45$, and it appears that for a fixed $r \geq 0$, $b_{r,n}$ is constant for all $n \geq 10 + r$. Up to $r = 9$, these constants are

$$4, 8, 14, 28, 52, 88, 150, 244, 390, 612.$$

Conjecture 5.3. *The numbers $b_{r,n}$ are equal for a fixed r with $n \geq 10 + r$ (call them b_r) and they satisfy*

$$\sum_{r \geq 0} b_r x^r = \frac{2(1+x)(2-x^2)}{1-x} P(x)^2,$$

where $P(x)$ is again the generating function for the partition numbers.

Remark 5.4. To guess the formula in Conjecture 5.3 one really only needs four numbers b_0, b_1, b_2, b_3 since the numerator is a degree 3 polynomial, but it is true for all 10 numbers we have.

Remark 5.5. If Conjecture 5.3 is proven we could still not determine the numbers av_n^k for all $k \leq 3n - 15$ since we also would need starting values when $n = 10 + r$ of $\left(\mathrm{av}_{n+2}^{2n+r-5} - \mathrm{av}_{n+1}^{2n+r-5}\right) - \left(\mathrm{av}_{n+1}^{2n+r-6} - \mathrm{av}_{n}^{2n+r-6}\right)$. That sequence starts

$$12, 24, 41, 120, 274, 553, 1098, 2055.$$

Improved bounds and unimodality. Let $c \leq 1$ be a constant such that the maximal value of each sequence $\left(\mathrm{av}_n^k(1324)\right)_k$ occurs with $k \leq c \cdot \binom{n}{2}$, and assume that Conjecture 1.1 is true. Then, using the same technique as in [11, Theorem 17], we find that

$$L(1324) \leq \exp\left(\pi\sqrt{2c/3}\right).$$

Incidentally, $c = 21/23 \approx 0.913$ gives $L(1324) \leq 11.6004$, which is in the range of the estimation from [12]. Such a large improvement is unrealistic with the current approach. Indeed, large 1324-avoiders generally have many inversions [14]. Note that $c > 0.813$, since $c = 0.813$ gives $L(1324) \leq 10.263$, contradicting the known lower bound 10.27.

Question 5.6. Is there a constant $c < 1$ such that the maximal value of each sequence $\left(\mathrm{av}_n^k(1324)\right)_k$ occurs with $k \leq c \cdot \binom{n}{2}$?

This line of thinking leads to another natural question: is $\mathrm{av}_n^k(1324)$ unimodal in k? It is well-known (see [6] for a nice proof) that $\left(s_n^k\right)_k$ is log-concave, where s_n^k denotes the number of all permutations (not required to avoid any pattern) of length n with k inversions. As far as we know, there are no similar nontrivial results for the pattern avoiding case. The sequences $\left(\mathrm{av}_n^k(1324)\right)_k$ are unimodal in the data we have, but we have not found a proof. Of special interest would be the positions of the 'tops' of the unimodal sequences, due to the discussion above.

Conjecture 5.7. *The sequence* $\left(\mathrm{av}_n^k(1324)\right)_{k=0}^{\binom{n}{2}}$ *is unimodal for each* n*.*

Acknowledgements

We are grateful to Anders Claesson and Bjarki Ágúst Guðmundsson, who provided us with data for the numbers $\mathrm{av}_n^k(1324)$, for $k, n \leq 45$ (see https://akc.is/inv-mono/). Anders Claesson furthermore correctly guessed the generating function $R_n(x) = 2(2 + x)x^{n-1}P(x)^2$ for the differences $\mathrm{av}_{n+1}^k(1324) - \mathrm{av}_n^k(1324)$.

Both authors were funded by the Swedish Research Council, VR, grant 2022-03875.

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