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Symmetries of periodic and free boundary q-Whittaker measures

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Abstract. We show that the periodic and free boundary q-Whittaker measures, two models of random partitions, exhibit remarkable distributional symmetries. Equivalently, we derive new identities for skew q-Whittaker functions related to bounded Cauchy and Littlewood identities. These extend identities found by Imamura, Mucciconi, and Sasamoto, and in particular give new proofs of these identities.

1 Introduction

Many recent advances in the study of integrable probabilistic systems have been driven by the study of probability measures on partitions defined in terms of symmetric functions. The first such results focused on Schur measures, which are related to zero-temperature growth models, see e.g. [18, 1]. Schur measures have a fermionic (i.e. determinantal/Pfaffian) structure, which aids in asymptotics that are needed for probabilistic applications. More recently, connections were made between generalizations of Schur measures using Macdonald polynomials, and positive temperature models which have no obvious fermionic structure. These were then studied using hard but ad hoc methods, see e.g. [2, 3, 7, 8]. These ideas have also been used to study stationary measures for these systems, see e.g. [9, 10, 4].

Recently, Imamura, Mucciconi, and Sasamoto [15, 17] discovered new identities of symmetric functions relating full- and half-space q-Whittaker measures to periodic and free boundary Schur measures, leading to new asymptotic results for half-space models in the Kardar–Parisi–Zhang universality class [12, 16]. They revealed that q-Whittaker measures, while not fermionic, are intimately related with fermionic measures. Perhaps even more surprising than the identities themselves is that the proof was via an intricate bijection.

Our main results generalize these identities. Moreover, our methods are completely different, and involve symmetric function arguments and the analysis of contour integral

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formulas for Macdonald polynomials. The expressions we work with have interpretations in terms of probability measures on partitions after suitable normalization, which are called periodic and free boundary q-Whittaker measures. The Schur case has been previously studied, see [6, 5, 16], and in the periodic setting, the Macdonald case has also been considered [19]. We also give an interpretation in terms of the six vertex model, a model coming from statistical mechanics, with connections to alternating sign matrices.

This extended abstract summarizes some of the results in [14] as well as in the forthcoming paper [13]. In this paper, we take a symmetric function perspective, and only remark on the interpretations for random partitions. We refer the reader to [14, 13] for full details and proofs.

1.1 Main results

We now state our main results. We let $P_{\lambda/\mu}(x;q,t)$ and $Q_{\lambda/\mu}(y;q,t)$ denote the *Macdonald polynomials* and their duals respectively. We let $(z;q)_n = (1-z)\cdots(1-q^{n-1}z)$ with $n=\infty$ possible. We let

$$h_m(z;q) = \sum_{k=0}^{m} \frac{(q;q)_m}{(q;q)_k (q;q)_{m-k}} z^k \tag{1.1}$$

denote the *Rogers–Szegő polynomials*, and following [22], for a partition λ, we let

$$h_\lambda(a,b;q) = \prod_{i\geq 1} a^{m_{2i-1}} h_{m_{2i-1}}(b/a;q) h_{m_{2i}}(ab;q), \tag{1.2}$$

employing standard multiplicative notation $\lambda = 1^{m_1}2^{m_2}\cdots$ for partitions. In the following, the subscripts P and FB refer to *periodic* and *free boundary*. These terms refer to the boundary conditions on probabilistic systems related to these identities, and have appeared before [6, 5, 19].

Theorem 1.1. *Let u and q be formal variables, and x and y two alphabets. We have that*

$$Z_P(n;x,y;u,q) := \sum_{\lambda,\mu:\lambda_1\leq n} \frac{u^{|\mu|}}{(q;q)_{n-\lambda_1}} \cdot P_{\lambda/\mu}(x;q,0) Q_{\lambda/\mu}(y;q,0) \tag{1.3}$$

is symmetric in the variables u and q.

Theorem 1.2. *Let u,q,a,b,c,d be formal variables, and x an alphabet. We have that*

$$Z_{FB}(n;x;u,q,a,b,c,d) := \sum_{\lambda,\mu:\lambda_1\leq n} \frac{h_{n-\lambda_1}(ab;q) h_{\lambda'}(a,b;q)}{(q;q)_{n-\lambda_1}\prod_{i\geq 1}(q;q)_{\lambda_i-\lambda_{i+1}}} \cdot P_{\lambda/\mu}(x;q,0) \cdot u^{|\mu|/2} h_{\mu'}(c/\sqrt{u}, d/\sqrt{u};q) \tag{1.4}$$

is symmetric in the variables u and q, and separately symmetric under any permutation of the variables a,b,c,d. Here λ' denotes the conjugate partition associated to λ.

Example 1.3. In the case $n = 1$ and where x, y are single variables, one can explicitly evaluate

$$Z_P(1; x, y; u, q) = \frac{(1-uq)(1+xy)}{(1-u)(1-q)}$$
$$Z_{FB}(1; x; u, q, a, b, c, d) = \tag{1.5}$$
$$\frac{(1+a)(1+b)(1+c)(1+d)(1+x) + (1-a)(1-b)(1-c)(1-d)(1-x)}{2(1-u)(1-q)}$$

and see the symmetry properties given by Theorems 1.1 and 1.2. One can also compute the $n \to \infty$ limits, which are important as normalization constants to define probability measures on partitions. For alphabets $x = (x_1, \dots)$ and $y = (y_1, \dots)$, the partition function $Z_P(\infty; x, y; u, q)$ was computed in [19] as

$$\frac{1}{(u;u)_\infty (q;q)_\infty} \prod_{i,j} \frac{1}{(x_i y_j; u, q)_\infty}, \tag{1.6}$$

where $(z; q, t)_\infty = \prod_{k,l \geq 0} (1 - q^k t^l z)$ is a two-parameter version of the q-Pochhammer symbol. We can compute $Z_{FB}(\infty; x; u, q, a, b, c, d)$ to be

$$\frac{1}{(u;u)_\infty (q;q)_\infty (uq; u, q)_\infty} \prod_{s \in \left\{ \substack{ab, ac, ad, \\ bc, bd, cd} \right\}} \frac{1}{(s; u, q)_\infty} \prod_{\substack{i \\ s \in \{a,b,c,d\}}} \frac{1}{(s x_i; u, q)_\infty} \prod_{i<j} \frac{1}{(x_i x_j; u, q)_\infty}. \tag{1.7}$$

As we explain later in the text, these symmetries are actually consequences of explicit contour integral formulas for both expressions when the alphabets are finite.

Remark 1.4. Theorems 1.1 and 1.2 generalize identities in [17]. In particular, the equalities $Z_P(n; x; q, 0) = Z_P(n; x; 0, q)$ and $Z_{FB}(n; x; q, 0, a, 0, \sqrt{q}a, 0) = Z_{FB}(n; x; 0, q, a, \sqrt{q}a, 0)$ recover Theorems 10.11 and 10.12 in [17] respectively.

Remark 1.5. The expressions (1.3) and (1.4) can be viewed as the probability that $\lambda_1 + \chi \leq n$ after suitably normalizing, where λ_1 is the first part of a random partition and χ is an independent random variable. The corresponding measures on partitions are known as *periodic* and *free boundary q-Whittaker measures*.

We next state some related identities involving skew Hall–Littlewood polynomials. The expressions turn out to evaluate to Macdonald and Koornwinder polynomials. The free boundary version is essentially derived in [11], and generalizes Theorem 4.7 of [21]. We first state the periodic version, which to our best knowledge appears to be new.

Theorem 1.6. *Fix two alphabets $x = (x_1, \dots, x_M)$ and $y = (y_1, \dots, y_N)$, and let (x, y^{-1}) denote the combined alphabet $(x_1, \dots, x_M, y_1^{-1}, \dots, y_N^{-1})$. We have that*

$$\sum_{\lambda, \mu : \lambda_1 \leq n} \frac{(t;t)_{m_n(\mu)}}{(t;t)_{m_n(\lambda)}} u^{|\mu|} \cdot P_{\lambda/\mu}(x; 0, t) Q_{\lambda/\mu}(y; 0, t) = \frac{1}{(u;u)_n} \left(\prod_{j=1}^N y_j^n \right) P_{n^N}(x, y^{-1}; u, t), \tag{1.8}$$

where the left hand side features skew Hall–Littlewood polynomials, and the right hand side is a Macdonald polynomial in parameters (u,t).

Next, we state the free boundary version, which as mentioned can already be found in [11], although written in a very different language. In fact, it could also be proven using the techniques of [21]. We let $K_\lambda(x;q,t,a,b,c,d)$ denote the Koornwinder polynomials.

Theorem 1.7. *Fix an alphabet $x=(x_1,\dots,x_N)$, and let n denote a non-negative integer. We have that*

$$\begin{aligned}\sum_{\lambda,\mu:\lambda_1\leq 2n} \frac{h_\lambda(a,b;t)}{h_{m_{2n}(\lambda)}(ab;t)} \cdot P_{\lambda/\mu}(x;0,t) \cdot \frac{u^{|\mu|/2}}{\prod_i (t;t)_{m_i(\mu)}} h_\mu(c/\sqrt{u}, d/\sqrt{u};t) = \\ C_n(u,t,a,b,c,d)\left(\prod_{i=1}^N x_i^n\right) K_{n^N}(x;u,t,a,b,c,d),\end{aligned} \tag{1.9}$$

where the left hand side features skew Hall–Littlewood polynomials, and the right hand side is a Koornwinder polynomial in parameters (u,t,a,b,c,d), and $C_n(u,t,a,b,c,d)$ is given by

$$\frac{\prod_{i=n-1}^{2n-2}(abcdu^i;t)_\infty}{\prod_{i=1}^n (u^i;t)_\infty \prod_{i=1}^{n-1}(abu^i;t)_\infty \prod_{i=0}^{n-1}\prod_{s\in\{ac,ad,bc,bd,cd\}}(su^i;t)_\infty}. \tag{1.10}$$

Remark 1.8. It would be interesting if our results could be lifted to the Macdonald setting (i.e. allowing both q and t to be non-zero), and indeed the periodic Macdonald measure has already been studied [19]. This is unclear, although we have checked some obvious attempts at generalizing these identities which do not seem to work. Note that the definition of h_λ needs to be changed, see [22] for some discussion of this. Since special cases have a bijective proof [17], it's natural to ask for bijective proofs of Theorems 1.1 and 1.2. This seems to require new ideas, and we leave it as an open problem.

1.2 Organization

The rest of the paper is organized as follows. In Section 2, we give some notation and background on Macdonald polynomials. In Section 3, we give contour integral formulas which imply Theorems 1.1 and 1.2. In Section 4, we give interpretations for Z_P and Z_{FB} in terms of vertex models.

2 Preliminaries

This section reviews background on symmetric functions and Macdonald polynomials, and we refer the reader to [20] for further details. A *partition* is a finite non-increasing sequence of positive integers $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k)$, and we call k the *length*, denoted $l(\lambda)$, and $|\lambda| = \sum_{i\geq 1} \lambda_i$ its *size*. It is useful to view a partition as its *Young diagram*, where we place λ_i boxes in the ith row from top to bottom, so that each row is aligned on the left. We define the *conjugate partition* λ' as the partition obtained from λ by reflecting its Young diagram, switching the rows and columns. We will let $m_i(\lambda)$ denote the number of occurrences of i in λ.

Let $x = (x_1, x_2, \dots)$ be a formal alphabet, and $P_\lambda(x; q, t)$ denote the *Macdonald polynomials*, defined as the unique symmetric functions orthogonal with respect to the inner product defined by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda \prod_i \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}}, \tag{2.1}$$

where $p_\lambda = \prod_{i\geq 1} p_{\lambda_i}$, $p_k = \sum_{i\geq 1} x_i^k$, and $z_\lambda = \prod_i m_i(\lambda)! i^{m_i(\lambda)}$, and whose change of basis to the monomial symmetric functions is upper triangular with respect to the dominance ordering on partitions. We let $Q_\lambda(x; q, t)$ denote the dual basis. Macdonald polynomials satisfy a *Cauchy identity*, which states that

$$\sum_\lambda P_\lambda(x; q, t) Q_\lambda(y; q, t) = \prod_{i,j} \frac{(t x_i y_j; q)_\infty}{(x_i y_j; q)_\infty} =: \Pi(x, y; q, t). \tag{2.2}$$

For a skew Young diagram λ/μ, the *skew Macdonald polynomials* $P_{\lambda/\mu}(x; q, t)$ are then defined by

$$\langle P_{\lambda/\mu}, Q_\nu \rangle = \langle P_\lambda, Q_\mu Q_\nu \rangle \tag{2.3}$$

for all Q_ν, and $Q_{\lambda/\mu}$ is defined similarly. They satisfy a *branching rule*, meaning that if we specialize into two sets of variables (x, y), then

$$P_{\lambda/\mu}(x, y; q, t) = \sum_\nu P_{\lambda/\nu}(x; q, t) P_{\nu/\mu}(y; q, t). \tag{2.4}$$

There is an important involution $\omega_{q,t}$ on the ring of symmetric functions $\omega_{q,t}$, which satisfies $\omega_{q,t}(P_{\lambda/\mu}(x; q, t)) = Q_{\lambda'/\mu'}(x; t, q)$, and $\omega_{q,t}(Q_{\lambda/\mu}(x; q, t)) = P_{\lambda'/\mu'}(x; t, q)$. We will be interested in two special cases of the Macdonald polynomials. When $q = 0$, the Macdonald polynomials are called the *Hall–Littlewood polynomials*, and when $t = 0$, they are called the *q-Whittaker polynomials*.

We now give some contour integral formulas for the Hall–Littlewood polynomials which we will need. These could be given for general Macdonald polynomials, but certain constants become much more complicated. Specialize to n variables $z = (z_1, \dots, z_n)$. Following Chapter VI, Section 9 of [20], given two Laurent polynomials f, g in z (or more generally any formal series such that the product below is well-defined), we let $\langle f, g \rangle_n' = \frac{1}{n!}[f(z) g(z^{-1}) \Delta(z; q, t)]_1$, where $[\cdot]_1$ means we take the constant term in z, and

$$\Delta(z; q, t) = \prod_{i \neq j} \frac{(z_i z_j^{-1}; q)_\infty}{(t z_i z_j^{-1}; q)_\infty}. \tag{2.5}$$

Macdonald polynomials are orthogonal with respect to this inner product. If f and g are Laurent polynomials, then

$$\langle f, g\rangle'_n = \frac{1}{n!}\oint_C \frac{dz_1}{2\pi i z_1}\cdots\oint_C \frac{dz_n}{2\pi i z_n} f(z)g(z^{-1})\Delta(z;q,t), \tag{2.6}$$

where C is positively oriented and chosen to include 0 and no other poles.

We now give a well-known contour integral formula for the skew Hall–Littlewood polynomials. A similar formula holds for the q-Whittaker polynomials.

Lemma 2.1. *Let $x = (x_1, \dots)$ be an alphabet of arbitrary size. If $l(\lambda) \le n$, then*

$$P_{\lambda/\mu}(x;0,t) = \frac{(t;t)_{n-l(\lambda)}}{(1-t)^n}\langle P_\lambda(z;0,t), Q_\mu(z;0,t)\Pi(z,x;0,t)\rangle'_n. \tag{2.7}$$

This equality can either be viewed formally, or if $x_i \in \mathbb{C}$, then the contours defining $\langle,\rangle'_n$ should be chosen to be circles centered at 0 and include all x_i.

3 Contour integral formulas

At the heart of our approach are exact contour integral evaluations for Z_P and Z_{FB}. In this section, we state the contour integral formulas and as well as some additional symmetries which appear.

3.1 Contour integral formulas for Z_P

We begin by stating a contour integral formula for Z_P.

Theorem 3.1. *Let $x = (x_1, \dots, x_M)$ and $y = (y_1, \dots, y_N)$, and let C be a positively oriented circle centered at 0. Then*

$$Z_P(n;x,y;u,q) = c_n \oint_C \frac{dz_1}{2\pi i z_1}\cdots\oint_C \frac{dz_n}{2\pi i z_n}\prod_{i,j}(1+z_i^{-1}x_j)\prod_{i,j}(1+z_i y_j)\widetilde{\Delta}(z;q,u), \tag{3.1}$$

where

$$c_n = \frac{(1-uq)^n}{n!(1-q)^n(1-u)^n}, \qquad \widetilde{\Delta}(z;q,u) = \prod_{i\neq j}\frac{(1-quz_iz_j^{-1})(1-z_iz_j^{-1})}{(1-qz_iz_j^{-1})(1-uz_iz_j^{-1})}. \tag{3.2}$$

Theorem 1.1 follows immediately as a consequence of Theorem 3.1, since the integrand and prefactor are both manifestly symmetric in u and q. Another easy consequence is that $\left(\prod_j y_j^{-n}\right) Z_P(n;x,y;u,q)$ is symmetric in the combined alphabet (x, y^{-1}), as after factoring out $\prod_i z_i^N$ from the second product in the integrand, it clearly exhibits the same symmetry.

The key tool in proving Theorem 3.1 is the following complementation formula for q-Whittaker functions, which appears to be new.

Proposition 3.2. *Let $N \geq 1$, $x = (x_1, \ldots, x_N)$, and let $\mu \subseteq \lambda$ such that $\lambda_1 \leq n$. Then*

$$\frac{(q;q)_{n-\mu_1}}{(q;q)_{n-\lambda_1}} Q_{\lambda/\mu}(x;q,0) = P_{(n^N,\mu)/\lambda}(x^{-1};q,0) \prod_{i=1}^{N} x_i^n. \tag{3.3}$$

Proof sketch of Theorem 3.1. The case $N = 0$ can be handled separately, so we assume $N \geq 1$. After applying Proposition 3.2, one can then apply the branching rule to the sum to obtain an expression with a single q-Whittaker function:

$$\sum_{\mu:\mu_1 \leq n} \frac{1}{(q;q)_{n-\mu_1}} u^{|\mu|} P_{(n^N,\mu)/\mu}(x;q,0). \tag{3.4}$$

After applying the Macdonald involution to turn the q-Whittaker functions into Hall–Littlewood functions, we can use Lemma 2.1 to express the sum as a contour integral. The Macdonald involution can then be undone, giving the desired expression. □

3.2 Contour integral formulas for Z_{FB}

We begin by noting the following freedom to take $a = b = c = d = 0$:

Lemma 3.3. *Let x be an alphabet, and let (x, a, b, c, d) denote the combined alphabet consisting of x plus the four variables (a, b, c, d). Then*

$$Z_{FB}(n; x; u, q, a, b, c, d) = Z_{FB}(n; (x, a, b, c, d); u, q, 0, 0, 0, 0). \tag{3.5}$$

Sketch of proof. We prove the lemma separately for the variables (a, b) and (c, d) via the branching rule. For (a, b), this essentially follows from Theorem 4.1 of [22]. For (c, d), this is a direct computation. □

We now state a contour integral formula for Z_{FB} in the case $a = b = c = d = 0$, from which we can recover the general formula. The proof is somewhat similar to that of Theorem 3.1, except that we can skip the first step. In particular, it follows from the contour integral formula given by Lemma 2.1 (after applying the Macdonald involution to both sides) and a Littlewood identity for q-Whittaker polynomials.

Theorem 3.4. *Let $x = (x_1, \ldots, x_N)$, and let C be a positively oriented circle centered at 0 of radius r, with $1 < r < u^{-1/2}$. Then $Z_{FB}(n; x; u, q, 0, 0, 0, 0)$ equals*

$$\begin{aligned} \frac{1}{n!(1-q)^n} \oint_C \frac{dz_1}{2\pi i z_1} \cdots \oint_C \frac{dz_n}{2\pi i z_n} \prod_{i,j} (1 + z_i x_j) \prod_{i \neq j} \frac{z_i - z_j}{z_i - q z_j} \\ \times \prod_i \frac{z_i^2}{(z_i^2 - 1)(1 - u z_i^2)} \prod_{i<j} \frac{(z_i z_j - q)}{(z_i z_j - 1)} \frac{(1 - uq z_i z_j)}{(1 - u z_i z_j)}. \end{aligned} \tag{3.6}$$

Notice that the integrand is no longer symmetric in u and q, but nevertheless Theorem 1.2 can be derived from Theorem 3.4 by studying its residues inductively. We now sketch the proof of Theorem 1.2 given Theorem 3.4.

Proof sketch of Theorem 1.2. With Lemma 3.3, the symmetry of (a, b, c, d) is clear, and for the (u, q) symmetry, we may take $a = b = c = d = 0$ without loss of generality, which we assume for the rest of the proof. We may assume that the contours C are circles with radius $r < q^{-1/2}$.

Let $I(n; z)$ denote the integrand in (3.6) (including the prefactor). We will show the symmetry of u and q by inductively evaluating the residues of $I(n; z)$. In fact, for technical reasons we induct on the statement that the integral of $F(z)I(n; z)$ is symmetric in (u, q) for any meromorphic function $F(z)$ such that: 1) $F(z)$ is symmetric in the z_i, 2) $F(z)$ is symmetric in (u, q), 3) the poles of $F(z)$ in the variable z_i are independent of z_j for all j, and lie outside the contour C.

The poles in the variable z_n of $F(z)I(n; z)$ lying within C fall into one of three categories: 1) $z_n = qz_j$, 2) $z_n = \pm 1$, 3) $z_n = z_j^{-1}$. We show that the residues of z_n at these poles are symmetric in (u, q) in each case.

In case 1), it is not hard to see that the residue actually contains no poles in z_j within the contour C. Thus, the residue is 0 and may be ignored. In case 2), we can check that $\frac{\mathrm{res}_{z_n=\pm 1} F(z)I(n;z)}{I(n-1;z_1,\ldots,z_{n-1})}$ satisfies conditions 1), 2), and 3), and so by induction is symmetric in (u, q). Finally, in case 3), we can use symmetry to assume $j = n - 1$, and check that $\frac{\mathrm{res}_{z_n=z_{n-1}^{-1}} F(z)I(n;z)}{I(n-2;z_1,\ldots,z_{n-2})}$ satisfies conditions 1), 2), and 3), and so by induction the integral over $z_1, \ldots, z_{n-2}$ is (u, q) symmetric. But then the final integral over z_{n-1} must also have this property. □

Remark 3.5. In fact, a more careful analysis of the proof reveals that there are contour integral formulas for Z_{FB} that exhibit all the expected symmetries. Due to lack of space, we have not included them here, but they will appear in [13].

It is no longer obvious from the formula that Z_{FB} exhibits additional symmetries in the variables x, so we prove this separately.

Proposition 3.6. *We have that*

$$\left(\prod_i x_i^{-n/2}\right) Z_{FB}(n; x; u, q, a, b, c, d) \tag{3.7}$$

is invariant under the simultaneous inversion of both x_i and a.

Remark 3.7. Note that Proposition 3.6 implies that $\left(\prod_i x_i^{-n/2}\right) Z_{FB}(n; x; u, q, a, b, c, d)$ exhibits type D symmetry in the variables x_i, rather than the type BC symmetry that appears in Theorem 1.7. We do not have a good a priori explanation for this.

4 Vertex model interpretations

In this section, we give interpretations for Z_P and Z_{FB} in terms of vertex models, after applying a Macdonald involution to replace q-Whittaker with Hall–Littlewood functions.

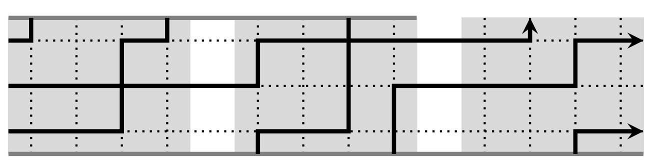
4.1 Six vertex model

Let u, t be formal variables, and let $x = (x_1, \ldots, x_M)$ and $y = (y_1, \ldots, y_N)$ be finite alphabets. Let $p(z) = \frac{1-z}{1-tz}$. The *six vertex model* is a measure on configurations of arrows along the edges of a lattice. The allowed configurations at a vertex, along with their weights, are given below

$p(z)$ $\quad 1-p(z)$ $\quad tp(z)$ $\quad 1-tp(z)$ $\quad 1$ $\quad 1$

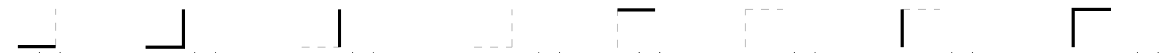
Here, the parameter z is allowed to change from vertex to vertex. We will now define two specific latticestwo specific lattice with some unusual geometries that our models will live on. with some unusual geometries that our models will live on.

Consider a periodic lattice formed by taking $M \times N$ lattices (where all vertices have degree 4) and joining the left and bottom of one to the right and top of the next. A *configuration* of the *quasi-periodic $M \times N$ six vertex model* is an assignment of arrows traveling through the edges of this lattice, with the requirement that arrows enter from infinity on the left, i.e. the only local configuration of the six below allowed to occur infinitely often is the first. We write $S_P(M, N)$ for the set of all configurations. See below for an example when $M = 4$ and $N = 3$, where the dark gray line indicates the identification of the top and bottom edges and the light gray indicates the different 4×3 lattices:

$\cdots$  $W(\sigma) = 4$

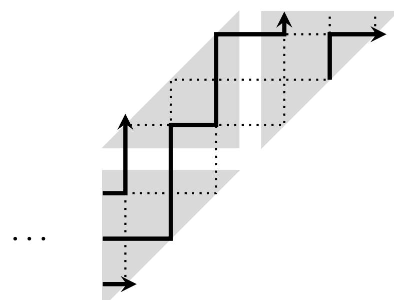
Given $\sigma \in S_P(M, N)$, we let $W(\sigma)$ denote the number of times arrows exit the top of any rectangle including the first one. In the above example, $W(\sigma) = 4$ (assuming arrows remain horizontal from infinity). We define the *weight* $\mathrm{wt}(\sigma)$ of a configuration to be the infinite product over all vertices of local weights given above, where in the kth rectangle at the (i, j)th vertex, we set $z = u^{k-1} x_i y_j$ (we number rectangles from right to left, and the graph is infinite only to the left).

Let u, t, a, b, c, d be formal variables, and let $x = (x_1, \ldots, x_N)$ be a finite alphabet. Let $r_1(z) = \frac{1-z^2}{(1+az)(1+bz)}$ and $r_2(z) = \frac{1-z^2}{(1+cz/\sqrt{u})(1+dz/\sqrt{u})}$. For our next lattice, we need to introduce boundary vertices where arrows may enter or exit the system. They have the following local weights:



$r_1(z)$ $\quad 1-r_1(z)$ $\quad -abr_1(z)$ $\quad 1+abr_1(z)$ $\quad r_2(z)$ $\quad 1-r_2(z)$ $\quad -cdr_2(z)$ $\quad 1+cdr_2(z)$

A *triangular lattice* of size N is obtained by taking the subset of an $N \times N$ lattice on or below the diagonal (so the vertices on the diagonal have degree 2). Now, join triangular lattices of size N along with their reflections across the diagonal in an alternating fashion, with the left joined to the right or the bottom joined to the top. A *configuration* of the *quasi-open six vertex model* of size N is an assignment of arrows traveling through the edges of the lattice like before, except that they may now enter and exit at the vertices of degree 2. We again require they enter from infinity on the left, which means that eventually, the arrows will enter from the left diagonal of one triangle and exit the right of the previous triangle. We write $S_{FB}(N)$ for the set of all configurations. See below for an example, where the light gray triangles indicate the different triangular lattices:



$W(\sigma) = 5$

Given $\sigma \in S_{FB}(N)$, we let $W(\sigma)$ denote the number of times arrows exit the top of any triangle plus the number of times arrows do not exit the right of any triangle (this does not include any diagonal boundary). In the above example, $W(\sigma) = 5$ (assuming arrows remain horizontal from infinity). We define the *weight* $\mathrm{wt}(\sigma)$ to be the infinite product over all vertices of the local weights given above, where in the kth triangle at the (i,j)th vertex, we set $z = u^{(k-1)/2} x_i x_j$ (again numbering right to left).

4.2 A vertex model interpretation

We now express Z_P and Z_{FB} as sums over six vertex model configurations.

Theorem 4.1. *Let $x = (x_1, \ldots, x_M)$ and $y = (y_1, \ldots, y_N)$. We have*

$$(u;u)_\infty (t;t)_\infty \sum_{\substack{\sigma \in S_P(M,N), k,l \in \mathbb{N} \\ W(\sigma)+k+l \leq n}} \frac{u^k}{(u;u)_k} \frac{t^l}{(t;t)_l} \mathrm{wt}(\sigma; x, y; t) = \omega_{t,0} \left(\frac{Z_P(n; x, y; u, t)}{Z_P(\infty; x, y; u, t)} \right). \quad (4.1)$$

Theorem 4.2. *Let $x = (x_1, \ldots, x_M)$. We have*

$$\sum_{\substack{\sigma \in S_{FB}(N), k \in \mathbb{N} \\ W(\sigma)+k \leq n}} \gamma_k \, \mathrm{wt}(\sigma) = \omega_{t,0} \left(\frac{Z_{FB}(n; x; u, t, a, b, c, d)}{Z_{FB}(\infty; x; u, t, a, b, c, d)} \right), \quad (4.2)$$

where

$$\gamma_k = [z^k]\frac{F(z;u,t,a,b,c,d)}{F(1;u,t,a,b,c,d)} \tag{4.3}$$

with $F(z;u,t,a,b,c,d)$ given by

$$\frac{(abcdz^2;u,t)_\infty}{(uz;u)_\infty(qz;q)_\infty(uqz;u,q)_\infty(abcdz;u,q)_\infty}\prod_{s\in\left\{\substack{ab,ac,ad,\\ bc,bd,cd}\right\}}\frac{1}{(sz;u,q)_\infty}. \tag{4.4}$$

The proofs of these theorems involve vertex model arguments using the Yang–Baxter and reflection equations.

Remark 4.3. Setting $n=0$ and applying the Macdonald involution, these identities can be used to compute the partition functions $Z_P(\infty;x,y;u,q)$ and $Z_{FB}(\infty;x;u,q,a,b,c,d)$. In particular, the proofs only require that the right hand sides are normalized so that their $n\to\infty$ limit is 1.

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