Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #137, 12 pp.

Proceedings of the 37^{th} Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

Crystal skeletons and their axioms

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Abstract. Crystal skeletons were introduced by Maas-Gariépy in 2023 by contracting quasi-crystal components in a crystal graph. On the representation theoretic level, crystal skeletons model the expansion of Schur functions into Gessel's quasisymmetric functions. Motivated by questions of Schur positivity, we give a new axiomatic approach to crystal skeletons in analogy to the local Stembridge axioms for crystals. In addition, we provide a combinatorial description of crystal skeletons, and prove many new properties, including a conjecture by Maas-Gariépy that crystal skeletons generalize dual equivalence graphs.

Keywords: crystal graphs, Lusztig involution, dual equivalence graphs

1 Introduction

Crystal graphs provide combinatorial tools to study the representation theory of Lie algebras. For instance, crystals are well-behaved with respect to taking tensor products and hence can be used to give combinatorial interpretations for Littlewood–Richardson coefficients. In type A, the character of an irreducible crystal $B(\lambda)$ of highest weight λ is the Schur function s_{λ}. See [4] for a detailed introduction.

It is an important problem in representation theory and algebraic combinatorics to deduce the Schur function expansion of a symmetric function whose expansion in terms of Gessel's fundamental quasisymmetric function [6] F_{α} is known. For example, combinatorial expressions for the quasisymmetric expansion of LLT polynomials, modified Macdonald polynomials [8], characters of higher Lie modules (or Thrall's problem) [7]

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or the plethysm of two Schur functions [10] exist, yet their Schur expansions are in general still elusive. It is thus desirable to develop methods to deduce the Schur expansions from these quasisymmetric expansions. Whereas Schur functions are characters of irreducible crystals in type A, Gessel's fundamental quasisymmetric functions are characters of *quasi-crystals* [11, 5], which are certain subcomponents of a crystal (see §2).

In [11], Maas-Gariépy introduced the *crystal skeleton* $\mathsf{CS}(\lambda)$ by contracting each quasi-crystal in $B(\lambda)$ to a vertex. Since there is a unique standard tableau $T \in \mathsf{SYT}(\lambda)$ in each quasi-crystal, it is natural to label the vertices of the crystal skeleton by standard tableaux. The crystal skeleton construction is the crystal analogue of Gessel's formula [6]

$$s_\lambda = \sum_{T \in \mathsf{SYT}(\lambda)} F_{\mathsf{Des}(T)},$$

where $\mathsf{Des}(T)$ is the descent composition of the standard tableau T (see §3.1). As such, crystal skeletons have the potential to serve as a powerful tool in deriving Schur expansions from quasisymmetric expansions.

Our goal is to initiate this project by characterizing the crystal skeleton both combinatorially (see Section 3) and axiomatically (see Section 4), in analogy to the local Stembridge axioms for crystals [15]. Stembridge axioms have played a crucial role in crystal theory and have facilitated proofs of Schur positivity using crystals. For example, in [12] the Schur expansion of Stanley symmetric functions was analyzed by defining a crystal structure on the combinatorial objects underlying Stanley symmetric functions (decreasing factorizations of a permutation); there, the crystal structure was proved using Stembridge's axioms. We anticipate that our axioms will have similar applications for Schur positivity in cases where the quasisymmetric expansion is known.

Beyond applications to positivity questions, our analysis shows that the combinatorics of the crystal skeleton is interesting in its own right. We prove self-similarity properties, symmetries, and branching rules. We show that the edges of the crystal skeleton are labeled by certain intervals, and that including only minimal length intervals recovers the *dual equivalence graph* developed by Assaf [2, 1] and Roberts [13, 14]. This implies that crystal skeletons generalize dual equivalence graphs, verifying a conjecture of Maas-Gariépy. Our crystal skeleton axioms should give new axioms for the dual equivalence graphs. In addition, even though crystal skeletons are obtained by contracting quasicrystals, we prove that they contain crystal graphs as subgraphs. Hence we expect our new axioms to give a new perspective on crystal axioms as well. Proofs are available in [3].

2 Crystals and quasi-crystals

Let $\mathsf{SSYT}(\lambda)_n$ be the set of semistandard Young tableaux of shape λ over the alphabet $[n] := \{1, 2, \ldots, n\}$ and $\mathsf{SYT}(\lambda)$ be the set of standard Young tableaux of shape λ.

We use French notation for partitions and tableaux, where the sizes of the rows weakly decrease from bottom to top. Work of Robinson–Schensted–Knuth defines a bijection RSK between words in the alphabet $[n]$ and pairs (P, Q) of a semistandard tableau P over $[n]$ and a standard tableau Q of the same shape. If $\mathsf{RSK}(w) = (P, Q)$, then P is known as the *insertion tableau* and Q as the *recording tableau*.

A semistandard tableau $b \in \mathsf{SSYT}(\lambda)_n$ gives rise to several combinatorial objects:

- The (row) *reading word* $\mathsf{row}(b)$ is the word obtained from b by reading rows left to right, top to bottom.
- The *weight* $\mathsf{wt}(b)$ is the tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ with α_j the number of letters j in b.
- The *standardization* $\mathsf{std}(b)$ is obtained from b and $\mathsf{wt}(b)$ by replacing the letters i in b from left to right by
$$\alpha_1 + \alpha_2 + \cdots + \alpha_{i-1} + 1, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_i \text{ for all } 1 \leqslant i \leqslant \ell.$$
The standardization of a word w can be defined similarly. By construction, $\mathsf{std}(b) \in \mathsf{SYT}(\lambda)$ and $\mathsf{std}(w)$ is a permutation.

Example 2.1.

$$\text{Suppose} \quad b = \begin{array}{|c|cc} \hline 4 & & \\ \hline 2 & \multicolumn{1}{c|}{4} & \\ \hline 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \end{array}\,. \qquad \text{Then} \qquad \mathsf{std}(b) = \begin{array}{|c|cc} \hline 5 & & \\ \hline 2 & \multicolumn{1}{c|}{6} & \\ \hline 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array}\,,$$

$\mathsf{row}(b) = 424133$, and $\mathsf{wt}(b) = (1, 1, 2, 2)$. Note that $\mathsf{row}(\mathsf{std}(b)) = 526134 = \mathsf{std}(\mathsf{row}(b))$.

2.1 Crystals and quasi-crystals on tableaux

We briefly review the crystal of type A_{n-1} on tableaux. More details can be found in [4, Chapters 3, 8]. The *crystal* $B(\lambda)_n$ is the set $\mathsf{SSYT}(\lambda)_n$ together with the maps

$$\begin{aligned} \mathsf{wt}\colon B(\lambda)_n &\to \mathbb{Z}^n_{\geqslant 0}, \\ e_i, f_i\colon B(\lambda)_n &\to B(\lambda)_n \cup \{\varnothing\} \qquad \text{for } i \in I = \{1, 2, \ldots, n-1\}. \end{aligned} \tag{2.1}$$

The *crystal raising* and *crystal lowering* operators e_i and f_i are defined as follows. The operators e_i and f_i act on the subword of $w = \mathsf{row}(b)$ containing only the letters i and $i + 1$, denoted by $w^{(i)}$. Successively bracket (i.e. group) letters $i + 1$ to the left of i. The subword of unbracketed letters is of the form $i^r (i + 1)^s$. On this subword

$$e_i(i^r (i+1)^s) = \begin{cases} i^{r+1}(i+1)^{s-1} & \text{if } s > 0, \\ \varnothing & \text{else,} \end{cases} \qquad f_i(i^r (i+1)^s) = \begin{cases} i^{r-1}(i+1)^{s+1} & \text{if } r > 0, \\ \varnothing & \text{else.} \end{cases}$$

These crystal operators on words preserve the recording tableau under RSK insertion. Hence they are well-defined on tableaux as well.

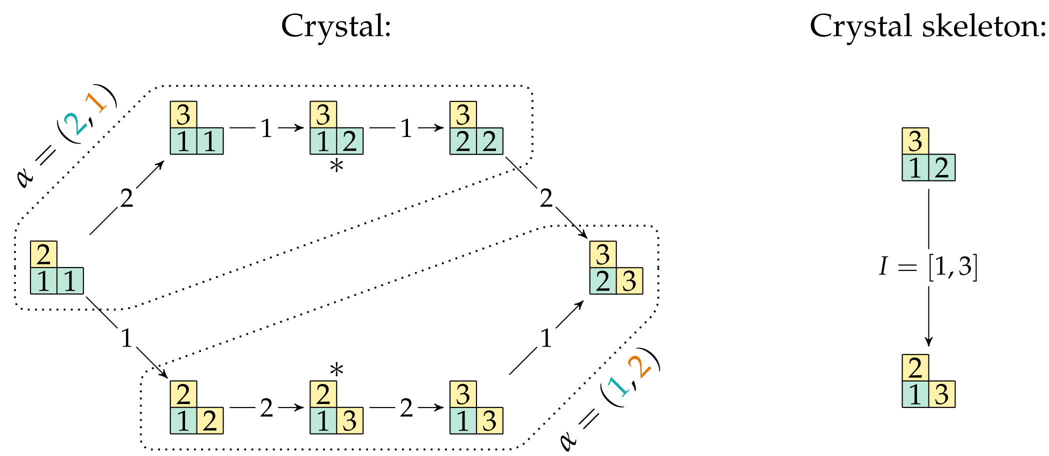


Figure 1: Left: Crystal $B(2,1)_3$ with two quasi-crystal components indicated with dotted lines and standard tableaux indicated by $*$. Right: Corresponding crystal skeleton.

Example 2.2. In $\mathsf{SSYT}(10,9,3,1)_4$, if

$$b = \begin{array}{llllllllll} 4 \\ 3 & 3 & 4 \\ 2 & 2 & 2 & 3 & 3 & 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 4 \end{array} \qquad \text{then} \qquad f_3 b = \begin{array}{llllllllll} 4 \\ 3 & 3 & 4 \\ 2 & 2 & 2 & 3 & 4 & 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 4 \end{array}.$$

One can define a *quasi-crystal* inside of the crystal $B(\lambda)_n$ as follows.

Definition 2.3. For a given crystal $B(\lambda)_n$, we define the *quasi-crystal* Q_T associated to a standard tableau $T \in \mathsf{SYT}(\lambda)$ by all elements $b \in B(\lambda)_n$ such that $\mathsf{std}(b) = T$.

Example 2.4. Suppose $T \in \mathsf{SYT}(3,2,1)$ has $\mathsf{row}(T) = 645123$. Then $w = 322111$ is in the same quasi-component as T in $B(3,2,1)_6$, since $\mathsf{std}(w) = \mathsf{row}(T)$.

It was shown in [11, Theorem 1] that the components Q_T are connected in $B(\lambda)_n$[1]. Each quasi-crystal Q_T contains exactly one standard tableau, namely T. It follows from [5, Section 2.5.2] that for $b \in Q_T \subseteq B(\lambda)_n$ with $f_i(b) \in B(\lambda)_n$, we have $f_i(b) \in Q_T$ if and only if no $i+1$ and i are bracketed in $\mathsf{row}(b)$.

Example 2.5. Figure 1 shows the crystal $B(2,1)_3$ and its two quasi-crystal components.

The crystal $B(\lambda)_n$ of type A_{n-1} enjoys a symmetry under the Schützenberger or Lusztig involution (see for example [4, p. 79]). Let $B(\lambda)_n$ be the crystal of type A_{n-1} with highest weight element u_λ of highest weight λ and lowest weight element v_λ; that is, the unique elements u_λ and v_λ such that $e_i u_\lambda = \emptyset$ and $f_i v_\lambda = \emptyset$ for all $1 \leqslant i < n$.

[1]Note that in [11] the quasi-crystal components are defined in a slightly different way by fixing the descent composition. This is not quite accurate since the descent composition does not uniquely specify a quasi-crystal component.

Definition 2.6. The *Lusztig involution* $\eta \colon B(\lambda)_n \to B(\lambda)_n$ is defined as follows. Any $b \in B(\lambda)_n$ can be obtained from u_λ by applying a sequence of lowering operators, that is, $b = f_{i_1} \cdots f_{i_k} u_\lambda$ for some sequence $1 \leqslant i_j < n$. We define

$$\eta(b) := e_{n-i_1} \cdots e_{n-i_k} v_\lambda.$$

In particular, $\eta(u_\lambda) = v_\lambda$ and $\eta(v_\lambda) = u_\lambda$.

Remark 2.7. For the crystal of tableaux in type A, the Lusztig involution is given by *evacuation*. For $T \in \mathsf{SSYT}(\lambda)$ and $w = w_1 \dots w_\ell = \mathsf{row}(T)$, $\mathsf{evac}(T)$ is the RSK insertion tableau of the word $w^\# = (n+1-w_\ell)\dots(n+1-w_1)$.

We will see later that crystal skeletons are also invariant under the Lusztig involution, and in fact this invariance is one of the axioms which characterize them (see Axiom **A4**).

3 Crystal skeletons

In [11], Maas-Gariépy introduced the *crystal skeleton* by contracting the quasi-crystals in $B(\lambda)_n$ to a vertex, assuming that n is sufficiently large. Since there is a unique standard tableau $T \in \mathsf{SYT}(\lambda)$ in each quasi-crystal in $B(\lambda)_n$, it is natural to label the vertices of the crystal skeleton by standard tableaux.

Definition 3.1. Let λ be a partition and consider the ambient crystal $B(\lambda)_n$ for $n \geqslant |\lambda|$. The *crystal skeleton* $\mathsf{CS}(\lambda)$ is an edge-labeled, directed graph whose vertices are in $\mathsf{SYT}(\lambda)$. For distinct $T, T' \in \mathsf{SYT}(\lambda)$, there is an edge from T to T' in $\mathsf{CS}(\lambda)$ if there exist $b \in Q_T$ and $b' \in Q_{T'}$ such that $f_i(b) = b'$ for some $1 \leqslant i < n$.

This definition is rather abstract. One goal of our work is to give a concrete, combinatorial description of $\mathsf{CS}(\lambda)$ as a graph. In addition, we will give a natural way to label the edges with intervals different from the edge-labels in [11]. See Figure 1 for an example.

3.1 Vertices of the crystal skeleton

As discussed in Definition 3.1, the vertices of $\mathsf{CS}(\lambda)$ are naturally labeled by the set $\mathsf{SYT}(\lambda)$. We will also label the vertices of $\mathsf{CS}(\lambda)$ by certain compositions of n as follows.

For $T \in \mathsf{SYT}(\lambda)$, the letter i is a *descent* if the letter $i+1$ is in a higher row of T. Denote the descents of T by $d_1 < d_2 < \cdots < d_k$. The *descent composition* is defined as

$$\mathsf{Des}(T) = (d_1, d_2 - d_1, \dots, d_k - d_{k-1}, n - d_k), \quad \text{where } n = |\lambda|.$$

Example 3.2. We color $T \in \mathsf{SYT}(3,2,1)$ by its descent composition below:

$$T = \begin{array}{|c|c|c|} \cline{1-1} 5 \\ \cline{1-2} 2 & 4 \\ \hline 1 & 3 & 6 \\ \hline \end{array}\,.$$

In particular, $\mathsf{Des}(T) = (1,2,1,2)$.

In what follows, it will be useful to label the vertices of $\mathsf{CS}(\lambda)$ by both $T \in \mathsf{SYT}(\lambda)$ and its corresponding descent composition $\mathsf{Des}(T)$. In particular, our axiomatic description of crystal skeletons in Section 4 labels vertices by the compositions $\mathsf{Des}(T)$.

3.2 Edges of the crystal skeleton

The edges of the crystal skeleton are more subtle. In [11], the edges of $\mathsf{CS}(\lambda)$ are indexed by the minimal index j such that $f_j(b) = b'$ for b and b' in the respective quasicrystal components. We give two alternative characterizations of the edges in $\mathsf{CS}(\lambda)$.

First, let $\pi = \mathsf{row}(T)$ for $T \in \mathsf{SYT}(\lambda)$ with $n = |\lambda|$. Let $I = [i, i+2m] \subseteq [n]$ be an interval of length $2m+1 \geqslant 3$ and $\pi|_I$ be the subword of π restricted to the letters in I.

Definition 3.3. The letters I in π form a *Dyck pattern interval* if the RSK insertion tableau of $\pi|_I$ has shape $(m+1, m)$ and the RSK insertion tableau of $\pi|_{[i,i+m]}$ has shape $(m+1)$.

We show that the edges of $\mathsf{CS}(\lambda)$ are labeled by Dyck pattern intervals.

Theorem 3.4 ([3, Theorem 3.9])**.** *There is a bijection between the edges in $\mathsf{CS}(\lambda)$ and Dyck pattern intervals that occur in $\mathsf{row}(T)$ for $T \in \mathsf{SYT}(\lambda)$.*

Example 3.5. We describe in detail the edge below found in $\mathsf{CS}(3,2,1)$ in Figure 2:

$$T = \begin{array}{|c|c|c|} \hline 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 4 & 5 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 3 \\ \hline \end{array} \qquad \xrightarrow{I=[1,5]} \qquad T' = \begin{array}{|c|c|c|} \hline 6 & \multicolumn{2}{c}{} \\ \cline{1-2} 3 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 5 \\ \hline \end{array}.$$

In this case $\pi = \mathsf{row}(T) = 645123$, $\pi|_{[1,5]} = 45123$, which has RSK insertion tableau of shape $(3,2)$, and $\pi|_{[1,3]} = 123$ is increasing. Thus I is a Dyck pattern interval on π.

To see why there is an edge from T to T' in $\mathsf{CS}(3,2,1)$, note that the word $w = 622111$ is in the same quasi-component of $B(3,2,1)_6$ as π, since $\mathsf{std}(w) = \pi$. Similarly, $w' = 622112$ is in the same quasi-component as $\pi' := \mathsf{row}(T') = 634125$ in $B(3,2,1)_6$, since $\mathsf{std}(w') = \pi'$. Next, observe that in $B(3,2,1)_6$, we have

$$f_1(w) = f_1(622111) = 622112 = w'.$$

The idea behind Theorem 3.4 is that the Dyck pattern interval detects this edge in $B(\lambda)_n$. Note that $\mathsf{Des}(T) = (3,2,1)$ and $\mathsf{Des}(T') = (2,3,1)$. The way descent compositions change between edges in $\mathsf{CS}(\lambda)$ is described more generally in Axiom **A2** in Section 4.

Second, note that in Example 3.5, one can obtain T' from T by the left action of the 3-cycle (543) on T. We show in [3] that this is true in general: given an edge between T and T' in $\mathsf{CS}(\lambda)$ labeled by the interval $I = [i, i+2m]$, one has

$$(j+m, j+m-1, \ldots, j+1, j) \cdot T = T',$$

where $j \in [i, i+m]$. Refer to [3, Section 3.2.2] for details on how to specify j.

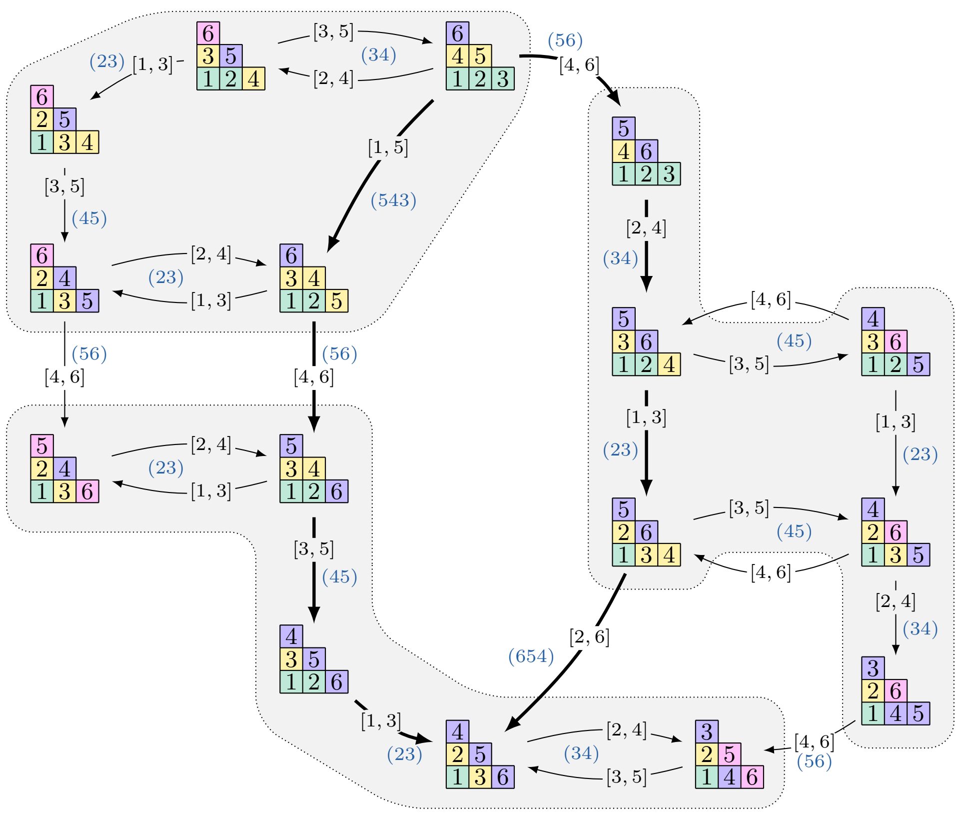


Figure 2: The crystal skeleton $\mathsf{CS}(3,2,1)$ with edges labeled by the intervals and cycles. Thick arrows highlight the top subcrystal as in Axiom **A5**, and the outlined components are the connected components in $G_{[1,5]}$ as in (4.1).

Example 3.6. Figure 2 shows the crystal skeleton $\mathsf{CS}(3,2,1)$. Each edge is labeled by the Dyck pattern interval in black, and corresponding cycle in blue. The vertices of $\mathsf{CS}(3,2,1)$ are indexed by the set $\mathsf{SYT}(3,2,1)$; each $T \in \mathsf{SYT}(3,2,1)$ is colored by its corresponding descent composition as in Example 3.2. The gray components correspond to the pieces G_{λ^-} obtained from the branching rules discussed in (4.1) in Section 4.

Remark 3.7. Using Theorem 3.4, we prove that $\mathsf{CS}(\lambda)$ exhibits *self-similarity* in the following sense. For $T \in \mathsf{SYT}(\lambda)$ and an interval $[a,b]$, we define $T_{[a,b]}$ to be the skew tableau T restricted to the interval $[a,b]$. For a skew tableau T, let $\mathsf{jdt}(T)$ be the jeu de taquin straightening of T, and write $\mu = \mathsf{shape}(\mathsf{jdt}(T_{[a,b]}))$. We show that $\mathsf{CS}(\mu)$ is a subgraph of $\mathsf{CS}(\lambda)$ with the following edge relabeling: if I is an edge label in $\mathsf{CS}(\lambda)$, then the corresponding edge label in $\mathsf{CS}(\mu)$ is $\{i - a + 1 \mid i \in I\}$. See [3, Section 4.3].

3.3 Generalizing dual equivalence graphs

Dual equivalence graphs were first introduced by Haiman [9]. The vertices of the *dual equivalence graph* $\mathsf{DE}(\lambda)$ are also indexed by $\mathsf{SYT}(\lambda)$. The edges in $\mathsf{DE}(\lambda)$ are undirected and given by the elementary dual equivalence relations D_i $(1 < i < |\lambda|)$ defined on permutations as follows:

$$\begin{aligned} \dots i \dots i+1 \dots i-1 \dots &\overset{i}{\longleftrightarrow} \dots i-1 \dots i+1 \dots i \dots \\ \dots i \dots i-1 \dots i+1 \dots &\overset{i}{\longleftrightarrow} \dots i+1 \dots i-1 \dots i \dots \end{aligned} \tag{3.1}$$

The operator D_i is not defined for other configurations of the letters $i-1, i, i+1$ in the permutation. Note that descents in the permutation do not change under D_i. Hence D_i is defined on a standard Young tableau T as well using the reading word $\mathsf{row}(T)$.

We prove that crystal skeletons generalize dual equivalence graphs.

Theorem 3.8 ([11, Conjecture 5.3], [3, Theorem 4.1])**.** *The dual equivalence graph* $\mathsf{DE}(\lambda) = (V, E)$ *is a subgraph of the crystal skeleton* $\mathsf{CS}(\lambda)$ *(disregarding edge labels and directions), where*

$$V = \{T \in \mathsf{SYT}(\lambda)\}, \qquad E = \left\{ \textit{Dyck pattern intervals } I \textit{ in } \mathsf{CS}(\lambda) \;\middle|\; |I| = 3 \right\}.$$

In Figure 2, $\mathsf{DE}(3,2,1)$ is obtained by including the edges labeled by intervals of the form $I = [i, i+2]$, or equivalently, the edges where the corresponding cycle is a transposition.

4 Axiomatic characterization of the crystal skeleton

We now give an axiomatic characterization of the crystal skeleton. We state the axioms in Section 4.1. In Section 4.2, we show that these axioms characterize crystal skeletons.

4.1 The axioms

Fix $n \in \mathbb{Z}_{\geqslant 1}$. Let G be a finite, directed, vertex- and edge-labeled graph, with underlying vertex set V and edge set E, satisfying the following:

- The vertices are labeled by compositions $\alpha = (\alpha_1, \dots, \alpha_\ell)$ of n, so that the labeled vertex set is $V_L = \{(v, \alpha) \mid v \in V\}$ with $\alpha \models n$. Sometimes we identify α with the partition $(\alpha^{(1)}, \dots, \alpha^{(\ell)})$ of $[n]$, where $\alpha^{(i)} = \{\alpha_1 + \cdots + \alpha_{i-1} + 1, \dots, \alpha_1 + \cdots + \alpha_i\}$.
- The edges are labeled by (odd-length) intervals $I \subseteq [n]$, so that the labeled edge set is $E_L = \{(vw, I) \mid vw \in E\}$.

With G as above, we define the Lusztig involution on G as follows.

Definition 4.1 (Lusztig involution)**.** The *Lusztig involution* $\mathcal{L}_n$ on G is defined by

- relabeling the vertices by replacing (v, α) by $(v, \mathsf{rev}(\alpha))$, and
- reversing all edge directions and changing the edge label

$$I = [a, b] \qquad \text{to} \qquad I^{\mathcal{L}} := [n+1-b, n+1-a].$$

Example 4.2. Suppose $n = 6$. Then $\mathcal{L}_6$ acts by

$$\mathcal{L}_6 : \Big((v, (3,2,1)) \xrightarrow{[4,6]} (w, (3,1,2)) \Big) \longmapsto \Big((w, (2,1,3)) \xrightarrow{[1,3]} (v, (1,2,3)) \Big).$$

We will also restrict our graphs via *branched graphs* as follows.

Definition 4.3 (Branched graph)**.** Define $G_{[1,n-1]} = (V'_L, E'_L)$, where

$$V'_L = \{(v, \alpha \setminus \{n\}) \mid (v, \alpha) \in V_L\} \quad \text{and} \quad E'_L = \{(vw, I) \mid (vw, I) \in E_L,\ I \subseteq [n-1]\}.$$

Example 4.4. In Figure 2, the graph $G_{[1,5]}$ is isomorphic to the portion of $\mathsf{CS}(3,2,1)$ shaded in gray. To obtain $G_{[1,5]}$ from the gray subgraph, keep the edge labels the same and replace the vertex $\mathsf{Des}(T)$ for $T \in \mathsf{SYT}(3,2,1)$ with $\mathsf{Des}(T|_{[1,5]})$ as in Remark 3.7.

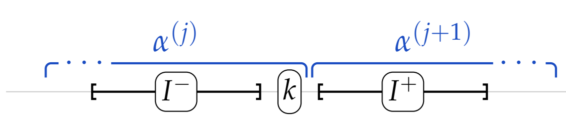
We are now ready to state the axioms for crystal skeletons. Graph isomorphisms are considered to preserve vertex and edge labels.

Axiom 4.5 (Axioms for crystal skeletons)**.** Let n be a positive integer and G be a finite, connected, vertex- and edge-labeled graph with labeled vertex set V_L and labeled edge set E_L as above. We call G a CS-*graph* if the following axioms hold:

A0. (*Intervals*) Suppose $(v, \alpha) \xrightarrow{I} (w, \beta)$ is an edge in G. Then the interval $I \subseteq [n]$ satisfies

(a) $I = I^- \cup \{k\} \cup I^+$ where $|I^-| = |I^+| > 0$; and

(b) $I^- \cup \{k\} \subseteq \alpha^{(j)}$ and $I^+ \subseteq \alpha^{(j+1)}$ for some $1 \leqslant j < \ell$, where ℓ is the length of α.



A1. (*Outgoing edges*) For each $(v, \alpha) \in V_L$ and each interval I satisfying **A0**, exactly one of the following holds: Either

(i) there is exactly one outgoing edge $(v, \alpha) \xrightarrow{I} \boxed{}$ labeled by I; or

(ii) there is an incoming edge $(u, \gamma) \xrightarrow{J} (v, \alpha)$ with $J \subseteq I$ and γ dominating α.

A2. (*Labels*) Let $(v,\alpha) \xrightarrow{I} (w,\beta)$ be an edge as in **A0**. Given α and I, then β must be of the form

$$\beta = (\alpha^{(1)}, \dots, \alpha^{(j-2)}, \underbrace{\qquad ⊛ \qquad}, \alpha^{(j+2)}, \dots, \alpha^{(\ell)}),$$

where ⊛ is one of

- **I.** (*length-preserving*) $⊛ = (\alpha^{(j-1)},\ \alpha^{(j)} \setminus \{k\},\ \alpha^{(j+1)} \cup \{k\})$;
- **II.** (*length-increasing*) $⊛ = (\alpha^{(j-1)},\ \alpha^{(j)} \setminus \{k\},\ I^+ \cup \{k\},\ \alpha^{(j+1)} \setminus I^+)$; or
- **III.** (*length-decreasing*) $⊛ = (\alpha^{(j-1)} \cup I^-,\ \alpha^{(j+1)} \cup \{k\})$, given $\alpha^{(j)} = I^- \cup \{k\}$.

A3. (*Fans*) Suppose $(v,\alpha) \xrightarrow{I} (w,\beta)$ is an edge in G with β satisfying axiom **A2 II**. Then this edge is part of a *fan*

$$\begin{tikzcd}
(w_1,\beta_1) \arrow[r, "J_1"] & (w_2,\beta_2) \arrow[r] & \cdots \arrow[r] & (w_{m-1},\beta_{m-1}) \arrow[r, "J_{m-1}"] & (w_m,\beta_m) \arrow[dl, bend left, "J_m"] \\
& & \cdots & (v,\alpha) \arrow[ulll, "I_1"] \arrow[ull, "I_2"] \arrow[u, "I_{m-1}"] \arrow[ur, "I_m"] &
\end{tikzcd}$$

where (w,β) is one of the (w_j,β_j) and

$$I_1 = [i, i+2m], \quad I_2 = [i+1, i+2m-1], \dots, I_m = [i+m-1, i+m+1]$$
$$J_1 = [i+2m-1, i+2m+1], \quad J_2 = [i+2m-2, i+2m], \dots, J_m = [i+m, i+m+2]$$

for some i and m. The edges labeled $I_1, \dots, I_m$ satisfy axiom **A2 II**, the edges labeled $J_1, \dots, J_{m-1}$ satisfy axiom **A2 I**, and the edge labeled J_m satisfies axiom **A2 III**.

A4. (*Lusztig involution*) G and $G_{[1,n-1]}$ are invariant under Lusztig involution:

(a) $\mathcal{L}_n(G) \cong G$; and

(b) $\mathcal{L}_{n-1}(G_{[1,n-1]}) \cong G_{[1,n-1]}$.

A5. (*Top subcrystal*) Let $s = \min\{\mathsf{len}(\alpha) \mid (v,\alpha) \in V_L\}$ be the minimal length of all vertex labels. Let G^s be the induced subgraph of G with vertex set

$$\{(v,\alpha) \in V_L \mid \mathsf{len}(\alpha) = s\}.$$

Then G^s is isomorphic to the crystal graph $B(\lambda)_s$ for some partition λ with $\ell(\lambda) = s$.

Remark 4.6. Note that the crystal $B(\lambda)_s$ appearing in axiom **A5** has a unique vertex (u,λ) with label λ. Hence G itself has a unique such vertex and so we will write G_λ for the CS-graph G containing $B(\lambda)_s$.

Remark 4.7. We refer to Axioms 4.5 as GL_n-axioms. There are several alternative but equivalent axioms defined in [3]. First, one can replace Axioms **A4** and **A5** by assuming a branching condition holds (see (4.1)), as well as a certain connectivity condition. We refer to these as S_n-axioms. Second, we give local axioms most closely related to Stembridge axioms for crystals [15] using commutation relations. See [3, Section 5].

4.2 The characterization

Our main results state that the axioms in Section 4.1 are necessary and sufficient conditions for a graph to be a crystal skeleton. We first prove that the crystal skeleton from Definition 3.1 is indeed a CS-graph.

Theorem 4.8 ([3, Theorem 5.7])**.** *For any partition λ, the crystal skeleton* $\mathsf{CS}(\lambda)$ *is a* CS*-graph with vertex labeling given by descent compositions and edge labeling given by Dyck pattern intervals.*

The proof of Theorem 4.8 uses the combinatorial properties of $\mathsf{CS}(\lambda)$ to show that it satisfies each of the conditions in Axiom 4.5.

Example 4.9. In Figure 2, the edges drawn in bold show a graph isomorphic to the crystal $B(3,2,1)_3$. More generally, the graph $\mathsf{CS}(\lambda)$ will contain the crystal $B(\lambda)_{\ell(\lambda)}$.

Next, we show that *any* CS-graph is a crystal skeleton $\mathsf{CS}(\lambda)$ for some λ.

Theorem 4.10 ([3, Section 5.5])**.** *Axioms 4.5 uniquely characterize G_λ. In particular, $G_\lambda \cong$* $\mathsf{CS}(\lambda)$.

CS-graphs exhibit branching rules mirroring the symmetric group:

$$G_{[1,n-1]} \cong \bigcup_{\lambda^- \subseteq \lambda, |\lambda/\lambda^-|=1} G_{\lambda^-}, \tag{4.1}$$

where each of the G_{λ^-} are CS-graphs. An example of this decomposition is shown in Figure 2; each connected component G_{λ^-} is shaded in gray.

Proof idea of Theorem 4.10. Our proof is by induction, using (4.1). In particular, we show that given the collection of $\{G_{\lambda^-}\}$ from (4.1), the graph G_λ can be uniquely recovered. This argument is quite subtle, and heavily utilizes Axioms **A3**, **A4** and **A5**. □

Acknowledgements

This work began at the conference "Community in Algebraic and Enumerative Combinatorics" in January 2024. We thank the organizers of this workshop and the Banff International Research Station for hosting it.

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