

Monotonicity for generalized binomial coefficients and Jack positivity

Hong Chen ^{*1} and Siddhartha Sahi ^{†1}

¹*Department of Mathematics, Rutgers University, NJ, USA*

Abstract. Binomial formulas for Schur polynomials and Jack polynomials were studied by Lascoux in 1978, and Kaneko, Okounkov–Olshanski and Lassalle in the 1990s. We prove that the associated binomial coefficients are monotone and derive some symmetric function inequalities, in particular, a Schur positivity and Jack positivity result. These inequalities are similar to those studied by Newton, Muirhead, Gantmacher, Cuttler–Greene–Skandera, Sra and Khare–Tao.

Keywords: Symmetric functions, Schur polynomials, Jack polynomials, binomial coefficients, Schur positivity

1 Introduction

Schur polynomials are ubiquitous in math, in particular, in algebra, representation theory and combinatorics. They form an orthonormal basis of the ring of symmetric polynomials, correspond to the irreducible (polynomial) representations of the symmetric group and the general linear group, and are the generating functions of semi-standard Young tableaux [24, 30, 17].

A symmetric polynomial is called *Schur positive*, if its expansion coefficients in terms of Schur polynomials are all non-negative. Schur positivity has long been a curious and difficult question. The first Schur positivity result is that products of Schur polynomials are Schur positive via the celebrated *Littlewood–Richardson rule*:

$$s_\mu s_\nu = \sum_{\lambda} c_{\mu\nu}^{\lambda} s_{\lambda}, \quad c_{\mu\nu}^{\lambda} \geq 0.$$

There are many proofs and combinatorial interpretations of this rule, e.g., [2, 12, 30, 17].

Jack polynomials are a one-parameter deformation of Schur polynomials, defined over the base field $\mathbb{F} = \mathbb{Q}(\tau)$. The Jack polynomial $P_{\lambda}(x; \tau)$ reduces to the Schur polynomial $s_{\lambda}(x)$ when $\tau = 1$ (and to many other bases for other τ). Jack polynomials originated from multivariate statistics [6, 19], but later turned out to have rich combinatorial structures [29, 11, 17, 1] and play an important role in the study of Selberg integrals [8, 7], the Calogero–Sutherland model [14] and random partitions [22].

^{*}hc813@math.rutgers.edu

[†]sahi@math.rutgers.edu

1.1 Symmetric function inequalities

In this extended abstract, we prove the following Schur positivity and Jack positivity result. For the full details and proofs, see [4].

Theorem 1.1. *Let λ and μ be partitions of length at most n , $x = (x_1, \dots, x_n)$ and $\mathbf{1} = (1^n)$. Then the following are equivalent:*

- (1) $\lambda \supseteq \mu$, i.e., $\lambda_i \geq \mu_i$, $1 \leq i \leq n$.
- (2) $\frac{s_\lambda(x + \mathbf{1})}{s_\lambda(\mathbf{1})} - \frac{s_\mu(x + \mathbf{1})}{s_\mu(\mathbf{1})}$ is **Schur positive**.
- (3) $\frac{P_\lambda(x + \mathbf{1}; \tau)}{P_\lambda(\mathbf{1}; \tau)} - \frac{P_\mu(x + \mathbf{1}; \tau)}{P_\mu(\mathbf{1}; \tau)}$ is **Jack positive**.

Here, the Jack positivity is over the cone $\mathbb{F}_{\geq 0} = \{f/g \in \mathbb{Q}[\tau] \mid f, g \in \mathbb{Z}_{\geq 0}[\tau], g \neq 0\} \subset \mathbb{F}$.

Example 1.2. Write $S(\lambda)(x) = s_\lambda(x)/s_\lambda(\mathbf{1})$ and $\tilde{S}(\lambda)(x) = S(\lambda)(x + \mathbf{1})$ for Schur polynomials and similarly $P^*(\lambda)$ and $\tilde{P}^*(\lambda)$ for Jack polynomials, then we have

$$\begin{aligned}
 \tilde{S}(\begin{smallmatrix} \square & \square & \square \\ \square & & \end{smallmatrix}) - \tilde{S}(\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}) &= S(\begin{smallmatrix} \square & \square & \square \\ \square & & \end{smallmatrix}) + \frac{4}{3}S(\begin{smallmatrix} \square & \square & \square \\ \square & & \end{smallmatrix}) + \frac{8}{3}S(\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}) \\
 &+ 3S(\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}) + 2S(\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}) + 2S(\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}) ; \\
 \tilde{P}^*(\begin{smallmatrix} \square & \square & \square \\ \square & & \end{smallmatrix}) - \tilde{P}^*(\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}) &= P^*(\begin{smallmatrix} \square & \square & \square \\ \square & & \end{smallmatrix}) + \frac{2\tau+2}{\tau+2}P^*(\begin{smallmatrix} \square & \square & \square \\ \square & & \end{smallmatrix}) + \frac{2\tau+6}{\tau+2}P^*(\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}) \\
 &+ \frac{4\tau+2}{\tau+1}P^*(\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}) + \frac{\tau+3}{\tau+1}P^*(\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}) + 2P^*(\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}) .
 \end{aligned}$$

All coefficients are in $\mathbb{Q}_{\geq 0}$ or $\mathbb{F}_{\geq 0}$.

Note that when $\tau = 1$, the coefficients for Jack polynomials specialize to those for Schur polynomials. By specializing τ to other values, one can get monomial positivity, elementary positivity and Zonal positivity results, see [Theorem 4.1](#).

Our Schur positivity (and monomial positivity and elementary positivity) result is closely related to the inequalities studied by Cuttler–Greene–Skandera, Sra and Khare–Tao in [5, 28, 9], which we now recall.

Proposition 1.3. *Let λ , μ , x , and $\mathbf{1}$ be as in [Theorem 1.1](#).*

- (1) (CGS–Sra) Suppose $|\lambda| = |\mu|$. Then λ dominates μ if and only if

$$\frac{s_\lambda(x)}{s_\lambda(\mathbf{1})} - \frac{s_\mu(x)}{s_\mu(\mathbf{1})} \geq 0, \quad x \in [0, \infty)^n.$$

- (2) (KT) λ weakly dominates μ if and only if

$$\frac{s_\lambda(x + \mathbf{1})}{s_\lambda(\mathbf{1})} - \frac{s_\mu(x + \mathbf{1})}{s_\mu(\mathbf{1})} \geq 0, \quad x \in [0, \infty)^n.$$

Here λ *weakly dominates* μ means that $\lambda_1 + \cdots + \lambda_i \geq \mu_1 + \cdots + \mu_i$ for $1 \leq i \leq n$; and λ *dominates* μ means, in addition, that $|\lambda| = |\mu|$.

In [5], the authors studied inequalities of means for various symmetric polynomials. The first inequality above, in the case of the monomial symmetric function, is the classical Muirhead's inequality; in the case of elementary symmetric function and power-sum, generalizes some inequalities due to Newton and Gantmacher, respectively. They made a conjecture for Schur polynomials, which was later proved by Sra in [28] using the HCIZ integral and the AM–GM inequality.

In [9], the authors studied entrywise-functions that preserve the positive semidefiniteness of matrices and provided a complete characterization of the sign patterns of the higher-order Maclaurin coefficients of such functions. Using a Schur positivity result of Lam–Postnikov–Pylyavskyy, they turned from a qualitative existence result to quantitative bounds on the coefficients. As an application, they gave a new characterization of weak majorization in terms of Schur polynomials (the second part of Proposition 1.3).

Each of statements in Theorem 1.1 and Proposition 1.3 is a characterization of certain partial order on partitions. The two inequalities in Proposition 1.3 are *evaluation positivity*, while in Theorem 1.1, we have *expansion positivity*. Since Schur polynomials are evaluation positive, it follows that the difference in Theorem 1.1 is also positive when evaluating at the positive orthant $[0, \infty)^n$. In fact, Theorem 1.1, together with the first inequality in Proposition 1.3, implies the second one, see Section 4.

1.2 Binomial formulas

The numerator $s_\lambda(x + \mathbf{1})$ in Theorem 1.1 was first considered by Lascoux in [15] in order to compute the Chern classes of the exterior and symmetric squares of a vector bundle, see also [17, page 47 Example 10]. This is a natural generalization of the well-known Newton's binomial formula to symmetric polynomials.

Later, in the 1990s, binomial formulas for Schur polynomials, Jack polynomials, Macdonald polynomials (and their non-symmetric counterparts) and Koornwinder polynomials were studied by Lassalle, Kaneko, Sahi, Okounkov–Olshanski and others, in [16, 8, 23, 20, 21, 26]. See also [13] for a good survey.

It was shown in [20] for Jack polynomials that

$$\frac{P_\lambda(x + \mathbf{1}; \tau)}{P_\lambda(\mathbf{1}; \tau)} = \sum_\nu \binom{\lambda}{\nu}_\tau \frac{P_\nu(x; \tau)}{P_\nu(\mathbf{1}; \tau)},$$

where the *generalized binomial coefficients* $\binom{\lambda}{\nu}_\tau$ are given by evaluations of the *interpolation Jack polynomials*, first studied in [25, 10]. Later in [27], it was shown that $\binom{\lambda}{\nu}_\tau$ is always positive (that is, in $\mathbb{F}_{\geq 0}$). In this paper, we prove that $\binom{\lambda}{\nu}_\tau$ is increasing in λ , which implies Theorem 1.1. See Section 3 for a sketch of the proof.

Theorem 1.4. *If $\lambda \supseteq \mu$, then $\binom{\lambda}{\nu}_\tau - \binom{\mu}{\nu}_\tau \in \mathbb{F}_{\geq 0}$ for all ν .*

2 Background

Throughout the paper, let $n \geq 1$ be the number of variables and $x = (x_1, \dots, x_n)$. Recall the following notions in [24, 30, 17].

2.1 Partitions

A *partition* of length at most n is a tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. The *size* of λ is $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_n$, and the *length* $\ell(\lambda)$ is the number of nonzero components. Let $m_i(\lambda)$ be the number of components equal to i in λ . Let $\mathcal{P}_n$ be the set of all partitions of length at most n .

For two partitions λ and μ , consider the following partial orders:

We say λ *contains* μ and write $\lambda \supseteq \mu$ if $\lambda_i \geq \mu_i$ for $1 \leq i \leq n$.

We say λ *weakly dominates* or *weakly majorizes* μ and write $\lambda \succ_w \mu$ if $\sum_{i=1}^k \lambda_i \geq \sum_{i=1}^k \mu_i$ for $1 \leq k \leq n$.

We say λ *dominates* or *majorizes* μ and write $\lambda \geq \mu$ if $|\lambda| = |\mu|$ and $\lambda \succ_w \mu$.

2.2 Diagrams and tableaux

For a partition λ , we usually identify it with its *Young diagram* (or Ferrers diagram), which is a set of lattice points (usually drawn as boxes), $\lambda = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq j \leq \lambda_i, 1 \leq i \leq \ell(\lambda)\}$. A *tableau* T of shape λ and rank N is a function $T : \lambda \rightarrow \{1, 2, \dots, N\}$. A *semi-standard reverse tableau* (RT, for short) is one such that $T(i, j) > T(i+1, j)$ and $T(i, j) \geq T(i, j+1)$ for $(i, j), (i+1, j), (i, j+1) \in \lambda$.

For example, for the partition $(3, 2)$, its diagram and all RTs of rank 2 are as follows:

$$\begin{array}{|c|c|c|} \hline & & \\ \hline & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 2 & 2 & 1 \\ \hline 1 & 1 & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|} \hline 2 & 2 & 2 \\ \hline 1 & 1 & \\ \hline \end{array}.$$

The *conjugate* of a partition λ is the partition λ' associated to the transpose of the diagram of λ . In the example above, $(3, 2)' = (2, 2, 1)$.

2.3 Symmetric polynomials

Denote by $\Lambda = \Lambda_n := \mathbb{Z}[x_1, \dots, x_n]^{S_n}$ the ring of symmetric polynomials over $\mathbb{Z}$ in n variables and $\Lambda_{\mathbb{F}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{F}$.

The *monomial symmetric polynomial* m_λ is defined as $m_\lambda = \sum_{\alpha \sim \lambda} x^\alpha$, where the sum runs over permutations of λ and $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$.

For each $r \geq 1$, the *elementary symmetric polynomial* e_r and the *power-sum* p_r are

$$e_r = m_{(r)} = \sum_{1 \leq i_1 < \dots < i_r \leq n} x_{i_1} \cdots x_{i_r}, \quad p_r = m_{(1^r)} = \sum_{1 \leq i \leq n} x_i^r. \quad (2.1)$$

Define multiplicatively, $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_l}$ and $p_\lambda = p_{\lambda_1} \cdots p_{\lambda_l}$, where $l = \ell(\lambda)$.

The *Schur polynomial* s_λ can be defined by the bialternant formula

$$s_\lambda = \frac{\det(x_i^{\lambda_j + n - j})_{1 \leq i, j \leq n}}{\det(x_i^{n - j})_{1 \leq i, j \leq n}}. \quad (2.2)$$

The *Hall inner product* on $\Lambda_{\mathbb{Q}}$ is defined by

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} z_\lambda, \quad (2.3)$$

where $z_\lambda = \prod_{i \geq 1} i^{m_i(\lambda)} m_i(\lambda)!$ and $\delta_{\lambda\mu}$ is the Kronecker delta function.

It is a well-known fact that Schur polynomials form an *orthonormal* basis of Λ_n that is *unitriangular* under the monomial basis with respect to the dominance order:

$$\langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}, \quad s_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda\mu} m_\mu. \quad (2.4)$$

Jack polynomials are deformations of Schur polynomials. The base field is $\mathbb{F} = \mathbb{Q}(\tau)$, the field of rational polynomials in the parameter τ . (Note: our τ corresponds to $1/\alpha$ for the parameter α in [17, Section VI.10].) Define an inner product on $\Lambda_{\mathbb{F}}$ by

$$\langle p_\lambda, p_\mu \rangle_\tau = \delta_{\lambda\mu} z_\lambda \tau^{-l(\lambda)}. \quad (2.5)$$

Then, similar to Schur polynomials, Jack polynomials are uniquely determined by the following *orthogonality* and *unitriangularity* conditions:

$$\langle P_\lambda(x; \tau), P_\mu(x; \tau) \rangle_\tau = 0, \quad \lambda \neq \mu; \quad P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu} m_\mu. \quad (2.6)$$

The normalization here is sometimes called *monic*.

Jack polynomials specialize to many families. For example, when $\tau = 0$, $P_\lambda = m_\lambda$; when $\tau = 1$, $P_\lambda = s_\lambda$; when $\tau = \infty$, $P_\lambda = e_\lambda$. Also, when $\tau = 1/2$ or 2 , P_λ specializes to the Zonal polynomial Z_λ , see [17, Chapter VII].

2.4 Binomial formulas and interpolation polynomials

The classical Newton's binomial formula states that in the *univariate* case

$$(x + 1)^k = \sum_{m \geq 0} \binom{k}{m} x^m, \quad (2.7)$$

where the binomial coefficient $\binom{k}{m} = \frac{k(k-1)\cdots(k-m+1)}{m!}$ is a polynomial in k .

In [20], the following binomial formula for Jack polynomials was proved:

$$\frac{P_\lambda(x + \mathbf{1}; \tau)}{P_\lambda(\mathbf{1}; \tau)} = \sum_{\nu} \binom{\lambda}{\nu}_{\tau} \frac{P_{\nu}(x; \tau)}{P_{\nu}(\mathbf{1}; \tau)}, \quad (2.8)$$

where $\mathbf{1} = (1^n) = (1, 1, \dots, 1)$. The coefficients $\binom{\lambda}{\nu}_{\tau}$ are called the *generalized binomial coefficients*. They, too, are given by evaluations of certain polynomials, called *interpolation Jack polynomials* (or shifted Jack polynomials), first studied in [25, 10].

To define the interpolation Jack polynomials, we need the following result about interpolation in [10]. Let $\bar{\lambda}_i = \lambda_i + (n - i)\tau$.

Proposition 2.1. Fix $d \geq 0$ and any function $\bar{f} : \{\lambda \in \mathcal{P}_n \mid |\lambda| \leq d\} \rightarrow \mathbb{F}$. Then there is a unique symmetric polynomial $f \in \Lambda_{\mathbb{F}}$, of degree at most d , such that $f(\bar{\lambda}) = \bar{f}(\lambda)$ for $|\lambda| \leq d$.

Definition 2.2. The interpolation Jack polynomial $h_{\mu}(x; \tau)$ is the unique symmetric polynomial of degree $|\mu|$ that interpolates the Kronecker delta function (for $|\lambda| \leq |\mu|$), i.e.,

$$h_{\mu}(\bar{\lambda}; \tau) = \delta_{\lambda\mu}, \quad \forall \lambda \in \mathcal{P}_n, |\lambda| \leq |\mu|, \quad (2.9)$$

$$\deg h_{\mu} = |\mu|. \quad (2.10)$$

The generalized binomial coefficients above are given by evaluations:

$$\binom{\lambda}{\mu}_{\tau} := h_{\mu}(\bar{\lambda}; \tau). \quad (2.11)$$

The normalization here is called *unital* in the sense that $\binom{\mu}{\mu} = 1$. The *monic* normalization, denoted by h_{μ}^{monic} , is the one such that the coefficient of m_{μ} (or P_{μ}) is 1.

It is well-known that Schur polynomials and Jack polynomials can be given by tableau-sum formulas. Okounkov found a similar formula for monic interpolation Jack polynomials, which we now recall. (We are following the notations in [13, 4].)

Proposition 2.3. Schur polynomials, Jack polynomials and interpolation Jack polynomials can be given by the following tableau-sum formulas:

$$S : s_{\lambda}(x) = \sum_T \prod_{s \in \lambda} x_{T(s)}, \quad (2.12)$$

$$J : P_{\lambda}(x; \tau) = \sum_T \psi_T(\tau) \prod_{s \in \lambda} x_{T(s)}, \quad (2.13)$$

$$AJ : h_{\lambda}^{\text{monic}}(x; \tau) = \sum_T \psi_T(\tau) \prod_{s \in \lambda} \left(x_{T(s)} - (a'_{\lambda}(s) + (n - T(s) - l'_{\lambda}(s))\tau) \right), \quad (2.14)$$

where S and J stand for Schur and Jack polynomials and AJ stands for (type A) interpolation Jack polynomials. Each sum runs over RTs of shape λ and rank n . The weight ψ_T , the statistics $a'_{\lambda}(s)$ and $l'_{\lambda}(s)$ can be found in [4, Section 2].

Example 2.4. For example, for $\mu = (3, 2)$ and $n = 2$, we have

$$s_\mu = P_\mu = m_\mu = x_1^3 x_2^2 + x_1^2 x_2^3,$$

and

$$\begin{aligned} h_\mu^{\text{monic}} &= x_2 x_1 (x_2 - 1)(x_1 - 1)(x_1 - 2 - \tau) + x_2 x_1 (x_2 - 1)(x_1 - 1)(x_2 - 2) \\ &= x_1 x_2 (x_1 - 1)(x_2 - 1)(x_1 + x_2 - \tau - 4). \end{aligned}$$

The definition involves the following 12 partitions

$$(0, 0), (1, 0), (1, 1), (2, 0), (2, 1), (3, 0), (2, 2), (3, 1), (4, 0), (3, 2), (4, 1), (5, 0).$$

One can easily verify that h_μ^{monic} indeed vanishes at $\bar{\lambda} = (\lambda_1 + \tau, \lambda_2)$ for all but $\overline{(3, 2)}$. One also sees that h_μ^{monic} vanishes at $\overline{(m, 0)}$ and $\overline{(m-1, 1)}$ for $m \geq 6$, more than required in the definition.

This phenomenon is true in general, called the *extra vanishing property*, proved in [10].

Proposition 2.5. We have $\binom{\lambda}{\mu}_\tau = 0$ unless $\lambda \supseteq \mu$.

2.5 Positivity

In this subsection, we recall the positivity cone $\mathbb{F}_{\geq 0}$ for Jack polynomials [27, 4]. The base field $\mathbb{F}$ is $\mathbb{Q}(\tau)$. Let

$$\mathbb{F}_{\geq 0} := \left\{ \frac{f}{g} \mid f, g \in \mathbb{Z}_{\geq 0}[\tau], g \neq 0 \right\}, \quad \mathbb{F}_{> 0} := \mathbb{F}_{\geq 0} \setminus \{0\}. \quad (2.15)$$

Then $\mathbb{F}_{\geq 0}$ is a convex multiplicative cone, i.e., it is closed under addition, multiplication, and scalar multiplication by $\mathbb{Q}_{\geq 0}$. When we view τ as a real number instead of an indeterminate, we have $f(\tau) \geq 0$ if $\tau \geq 0$ for $f \in \mathbb{F}_{\geq 0}$. For example, by [17, (VI.10.20)], we have (our τ is equal to $1/\alpha$)

$$P_\lambda(\mathbf{1}; \tau) = \prod_{(i,j) \in \lambda} \frac{n + \alpha(j-1) - (i-1)}{\alpha(\lambda_i - j) + \lambda'_j - i + 1} = \prod_{(i,j) \in \lambda} \frac{(n-i+1)\tau + j-1}{(\lambda'_j - i + 1)\tau + (\lambda_i - j)} \in \mathbb{F}_{> 0}. \quad (2.16)$$

As mentioned in Section 1, it is proved in [27] that the binomial coefficients $\binom{\lambda}{\mu}_\tau$ is positive. More precisely, we have the following.

Proposition 2.6 (Positivity). The binomial coefficient $\binom{\lambda}{\mu}_\tau \in \mathbb{F}_{> 0}$ if and only if $\lambda \supseteq \mu$.

As a corollary of the binomial formula Equation (2.8) and the positivity, we have:

Corollary 2.7. $P_\lambda(x + \mathbf{1}; \tau)$ is Jack positive over $\mathbb{F}_{\geq 0}$. In particular, $s_\lambda(x + \mathbf{1})$ is Schur positive, $m_\lambda(x + \mathbf{1})$ is monomial positive, and $e_{\lambda'}(x + \mathbf{1})$ is elementary positive.

3 Proof of Theorem 1.4

After the positivity of the binomial coefficients, one asks for the monotonicity, i.e., [Theorem 1.4](#). In this section, we give a sketch of its proof. See [4] for full details.

We may assume that λ is obtained from μ by adding one box in row i_0 . The normalizing factor $H_\nu(\tau) := h_\nu^{\text{monic}}(\bar{\nu}; \tau)$ and the weight $\psi_T(\tau)$ can be easily seen to be in $\mathbb{F}_{>0}$. Now apply the tableau-sum formula ([Proposition 2.3](#)) to write $h_\nu^{\text{monic}}(\bar{\lambda}; \tau) - h_\nu^{\text{monic}}(\bar{\mu}; \tau)$ as a sum over RTs. For each T , the factors for λ and μ are almost identical, apart from the ones with $T(s) = i_0$. For such $s \in \nu$, one can show that the difference is positive.

4 Symmetric function inequalities

Using the binomial formula [Equation \(2.8\)](#) and the monotonicity ([Theorem 1.4](#)), we can find many interesting inequalities for symmetric functions.

Theorem 4.1. Fix $\tau_0 \in [0, \infty]$. The following statements are equivalent:

- (1) $\lambda \supseteq \mu$.
- (2) $\frac{P_\lambda(x + \mathbf{1}; \tau)}{P_\lambda(\mathbf{1}; \tau)} - \frac{P_\mu(x + \mathbf{1}; \tau)}{P_\mu(\mathbf{1}; \tau)}$ is **Jack positive** over $\mathbb{F}_{\geq 0}$.
- (3) $\frac{P_\lambda(x + \mathbf{1}; \tau_0)}{P_\lambda(\mathbf{1}; \tau_0)} - \frac{P_\mu(x + \mathbf{1}; \tau_0)}{P_\mu(\mathbf{1}; \tau_0)}$, is τ_0 -**Jack positive** over $\mathbb{R}_{\geq 0}$ (or over $\mathbb{Q}_{\geq 0}$ if $\tau_0 \in \mathbb{Q}$).

As special cases, $\frac{s_\lambda(x + \mathbf{1})}{s_\lambda(\mathbf{1})} - \frac{s_\mu(x + \mathbf{1})}{s_\mu(\mathbf{1})}$ is **Schur positive**, $\frac{m_\lambda(x + \mathbf{1})}{m_\lambda(\mathbf{1})} - \frac{m_\mu(x + \mathbf{1})}{m_\mu(\mathbf{1})}$ is **monomial positive**, $\frac{Z_\lambda(x + \mathbf{1})}{Z_\lambda(\mathbf{1})} - \frac{Z_\mu(x + \mathbf{1})}{Z_\mu(\mathbf{1})}$ is **Zonal positive**, and $\frac{e_{\lambda'}(x + \mathbf{1})}{e_{\lambda'}(\mathbf{1})} - \frac{e_{\mu'}(x + \mathbf{1})}{e_{\mu'}(\mathbf{1})}$ is **elementary positive** (when expressed in $\{e_{\nu'} \mid \nu \in \mathcal{P}_n\}$).

- (4) $\frac{p_\lambda(x + \mathbf{1})}{p_\lambda(\mathbf{1})} - \frac{p_\mu(x + \mathbf{1})}{p_\mu(\mathbf{1})}$ is **power-sum positive** (when expressed in $\{p_{\nu'} \mid \nu \in \mathcal{P}_n\}$).

Proof. Note that $P_\lambda(\mathbf{1}; \tau) \in \mathbb{F}_{>0}$ and $P_\lambda(\mathbf{1}; \tau_0) > 0$.

We first show that (1) $\implies$ (2). If $\lambda \supseteq \mu$, then by the binomial formula [Equation \(2.8\)](#),

$$\frac{P_\lambda(x + \mathbf{1}; \tau)}{P_\lambda(\mathbf{1}; \tau)} - \frac{P_\mu(x + \mathbf{1}; \tau)}{P_\mu(\mathbf{1}; \tau)} = \sum_{\nu} \left(\binom{\lambda}{\nu}_\tau - \binom{\mu}{\nu}_\tau \right) \frac{P_\nu(x; \tau)}{P_\nu(\mathbf{1}; \tau)}.$$

The coefficient $\binom{\lambda}{\nu}_\tau - \binom{\mu}{\nu}_\tau$ is in $\mathbb{F}_{\geq 0}$ by [Theorem 1.4](#).

(2) $\implies$ (3) is clear since functions in $\mathbb{F}_{\geq 0}$ have non-negative evaluation at $\tau_0 \in [0, \infty]$.

(3) $\implies$ (1): Assume that $\lambda \not\geq \mu$, then the difference $\frac{P_\lambda(x+1;\tau)}{P_\lambda(1;\tau)} - \frac{P_\mu(x+1;\tau)}{P_\mu(1;\tau)}$ would contain the term $-\frac{P_\mu(x;\tau_0)}{P_\mu(1;\tau_0)}$, contradicting (3).

(1) $\iff$ (4) is some easy computation, see [4, Theorem 6.4]. $\square$

These inequalities are related to the ones studied by Muirhead, Cuttler–Greene–Skandera, Sra and Khare–Tao [18, 5, 28, 9], which we now recall.

Proposition 4.2. *Suppose $|\lambda| = |\mu|$. The following are equivalent:*

- | | |
|--|--|
| (1) λ dominates μ . | (4) $\frac{p_\lambda(x)}{p_\lambda(1)} - \frac{p_\mu(x)}{p_\mu(1)} \geq 0$ for $x \in [0, \infty)^n$. |
| (2) $\frac{m_\lambda(x)}{m_\lambda(1)} - \frac{m_\mu(x)}{m_\mu(1)} \geq 0$ for $x \in [0, \infty)^n$. | (5) $\frac{s_\lambda(x)}{s_\lambda(1)} - \frac{s_\mu(x)}{s_\mu(1)} \geq 0$ for $x \in [0, \infty)^n$. |
| (3) $\frac{e_{\lambda'}(x)}{e_{\lambda'}(1)} - \frac{e_{\mu'}(x)}{e_{\mu'}(1)} \geq 0$ for $x \in [0, \infty)^n$. | |

Proposition 4.3. *The following are equivalent:*

- | | |
|--|--|
| (1) λ weakly dominates μ . | (2) $\frac{s_\lambda(x)}{s_\lambda(1)} - \frac{s_\mu(x)}{s_\mu(1)} \geq 0$ for $x \in [1, \infty)^n$. |
|--|--|

Other such inequalities in Proposition 4.3 follow from the previous two types Theorem 4.1 and Proposition 4.2.

Corollary 4.4. *The following are equivalent:*

- | | |
|--|--|
| (1) λ weakly dominates μ . | (4) $\frac{p_\lambda(x)}{p_\lambda(1)} - \frac{p_\mu(x)}{p_\mu(1)} \geq 0$ for $x \in [1, \infty)^n$. |
| (2) $\frac{m_\lambda(x)}{m_\lambda(1)} - \frac{m_\mu(x)}{m_\mu(1)} \geq 0$ for $x \in [1, \infty)^n$. | (5) $\frac{s_\lambda(x)}{s_\lambda(1)} - \frac{s_\mu(x)}{s_\mu(1)} \geq 0$ for $x \in [1, \infty)^n$. |
| (3) $\frac{e_{\lambda'}(x)}{e_{\lambda'}(1)} - \frac{e_{\mu'}(x)}{e_{\mu'}(1)} \geq 0$ for $x \in [1, \infty)^n$. | |

Proof. For brevity, We only show (1) $\implies$ (5). There exists ν such that $\lambda \supseteq \nu$ and $\nu \succ \mu$.

$$\frac{s_\lambda(x+1)}{s_\lambda(1)} - \frac{s_\mu(x+1)}{s_\mu(1)} = \left(\frac{s_\lambda(x+1)}{s_\lambda(1)} - \frac{s_\nu(x+1)}{s_\nu(1)} \right) + \left(\frac{s_\nu(x+1)}{s_\nu(1)} - \frac{s_\mu(x+1)}{s_\mu(1)} \right).$$

Both differences are positive for $x \in [0, \infty)^n$ by Theorem 4.1 and Proposition 4.3. Each of the implications (1) $\implies$ (2)–(4) is similar. Each of the reverse implications (2)–(5) $\implies$ (1) is by some degree consideration. $\square$

Moreover, our inequalities work for Jack polynomials as well. It is natural to extend the inequalities of the second and third types to Jack polynomials:

Conjecture 4.5. Fix $\tau_0 \in [0, \infty]$, and suppose $|\lambda| = |\mu|$. Then the following statements are equivalent:

- (1) λ dominates μ .
- (2) $\frac{P_\lambda(x; \tau)}{P_\lambda(\mathbf{1}; \tau)} - \frac{P_\mu(x; \tau)}{P_\mu(\mathbf{1}; \tau)} \in \mathbb{F}_{\geq 0}^{\mathbb{R}} = \{f/g \mid f(\tau) \in \mathbb{R}_{\geq 0}[\tau], g \in \mathbb{Z}_{\geq 0}[\tau] \setminus 0\}, \text{ for } x \in [0, \infty)^n.$
- (3) $\frac{P_\lambda(x; \tau_0)}{P_\lambda(\mathbf{1}; \tau_0)} - \frac{P_\mu(x; \tau_0)}{P_\mu(\mathbf{1}; \tau_0)} \geq 0, \text{ for } x \in [0, \infty)^n.$

Conjecture 4.6. Fix $\tau_0 \in [0, \infty]$. The following statements are equivalent:

- (1) λ weakly dominates μ .
- (2) $\frac{P_\lambda(x; \tau)}{P_\lambda(\mathbf{1}; \tau)} - \frac{P_\mu(x; \tau)}{P_\mu(\mathbf{1}; \tau)} \in \mathbb{F}_{\geq 0}^{\mathbb{R}}, \text{ for } x \in [1, \infty)^n.$
- (3) $\frac{P_\lambda(x; \tau_0)}{P_\lambda(\mathbf{1}; \tau_0)} - \frac{P_\mu(x; \tau_0)}{P_\mu(\mathbf{1}; \tau_0)} \geq 0, \text{ for } x \in [1, \infty)^n.$

Note that (2) $\implies$ (3) $\implies$ (1) in both conjectures are clear; and the implication (1) $\implies$ (2) of [Conjecture 4.5](#) implies that of [Conjecture 4.6](#) via the argument of [Corollary 4.4](#).

Example 4.7. To illustrate Sra's inequality and the first conjecture, let $n = 2$, $\lambda = (4, 0)$ and $\mu = (3, 1)$, then we have

$$\begin{aligned} \frac{s_\lambda(x)}{s_\lambda(\mathbf{1})} - \frac{s_\mu(x)}{s_\mu(\mathbf{1})} &= \frac{1}{15}(x_1 - x_2)^2(3x_1^2 + 4x_1x_2 + 3x_2^2), \\ \frac{P_\lambda(x; \tau)}{P_\lambda(\mathbf{1}; \tau)} - \frac{P_\mu(x; \tau)}{P_\mu(\mathbf{1}; \tau)} &= \frac{(\tau + 3)(x_1 - x_2)^2}{4(2\tau + 1)(2\tau + 3)} \left(\tau(x_1 + x_2)^2 + 2(x_1^2 + x_1x_2 + x_2^2) \right). \end{aligned}$$

Clearly both are positive when evaluating at $[0, \infty)^n$.

The authors, together with Apoorva Khare, are currently working on these conjectures and some further generalizations in [3].

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