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Counting homomorphisms in antiferromagnetic graphs via Lorentzian polynomials

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Abstract. An edge-weighted graph G, possibly with loops, is said to be *antiferromagnetic* if it has nonnegative weights and at most one positive eigenvalue, counting multiplicities. The number of graph homomorphisms from a graph H to an antiferromagnetic graph G generalises various important parameters in graph theory, including the number of independent sets and proper vertex colourings.

We obtain a number of new homomorphism inequalities for antiferromagnetic target graphs G. In particular, we prove that various graphs H satisfy the inequality

$$|\operatorname{Hom}(H, G)|^2 \leq |\operatorname{Hom}(H \times K_2, G)|$$

for any antiferromagnetic G, where $H \times K_2$ denotes the tensor product of H and K_2. As a corollary, this confirms conjectures of Zhao and of Sah, Sawhney, Stoner and Zhao for complete graphs K_d and adds many more instances. Our method uses the emerging theory of Lorentzian polynomials due to Brändén and Huh, which may be of independent interest.

Keywords: graph homomorphism inequalities, Lorentzian polynomials, antiferromagnetic graphs

1 Introduction

For graphs G and H, a *homomorphism* from H to G is a vertex map that preserves adjacency. There are numerous concepts in graph theory that can be rephrased in terms of homomorphisms, which include two fundamental examples: independent sets and proper vertex colourings with q colours, or simply q-colourings. Indeed, there is a natural bijection between independent sets of a graph H and homomorphisms from H to

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$G =$. Similarly, each q-coloring of H corresponds to a homomorphism from H to $G = K_q$, the complete graph on q vertices.

This correspondence translates extremal problems on the number of independent sets or q-colourings to homomorphism inequalities. For instance, the Kahn–Zhao theorem [9, 14] states that the complete bipartite graph $K_{d,d}$ has the maximum number of independent sets amongst d-regular graphs. More precisely, if H is d-regular, then

$$\hom(H, \;)^{1/v(H)} \leq \hom(K_{d,d}, \;)^{1/(2d)},$$

where $\hom(H, G) = |\mathrm{Hom}(H, G)|$ denotes the number of homomorphisms from H to G and $v(H)$ is the number of vertices in H. There have been a lot of exciting developments along these lines of research for the last decade or two, which touches upon information theory [7] and statistical physics [5]. For a survey on the topic, see [16]. Amongst many results, one of the strongest homomorphism inequalities is the 'reverse Sidorenko' inequality of Sah, Sawhney, Stoner, and Zhao [11].

Theorem 1.1 (Theorem 1.9 in [11]). *Let H be a d-regular triangle-free graph and let G be a graph possibly with loops. Then*

$$\hom(H, G)^{1/v(H)} \leq \hom(K_{d,d}, G)^{1/(2d)}. \tag{1.1}$$

We remark that the original statement of [11, Theorem 1.9] allows distinct degrees in H, which generalises Theorem 1.1 to arbitrary triangle-free graphs. On the other hand, the triangle-freeness condition is essential in the sense that, for every H that contains a triangle, there exists G that breaks the inequality (1.1). In contrast, the Kahn–Zhao theorem requires no condition on H while an extremely specific target graph $G =$ is chosen. For another example, when $G = K_q$, [11, Theorem 1.7] generalises a series of previous results [6, 15] to give that, for any d-regular graph H,

$$\hom(H, K_q)^{1/v(H)} \leq \hom(K_{d,d}, K_q)^{1/(2d)}.$$

Thus, it is natural to wonder whether an extra condition on G that generalises both $G =$ and $G = K_q$ allows us to obtain an inequality of the form (1.1) for arbitrary d-regular graphs. Sah, Sawhney, Stoner, and Zhao conjectured that G, as a symmetric matrix, having at most one positive eigenvalue may be the correct condition to add on.

Conjecture 1.2 (Conjecture 1.16 in [11]). Let H be a d-regular graph and let G be a graph possibly with loops that has at most one positive eigenvalue. Then

$$\hom(H, G)^{1/v(H)} \leq \hom(K_{d,d}, G)^{1/(2d)}.$$

In [11], the conjecture is confirmed for the particular case of *semiproper colourings*, i.e., complete graphs G possibly with loops, which can be seen as a common generalisation of and K_q, albeit weaker than the conjecture itself.

Identifying a graph G by its adjacency matrix naturally generalises to an arbitrary symmetric matrix with nonnegative entries, or equivalently a *weighted graph*, whose edges are weighted by the corresponding entry in the symmetric matrix. Then the homomorphism count $\hom(H, G)$ is also weighted in the sense that

$$\hom(H, G) = \sum_{\phi \in \mathrm{Hom}(H,G)} \prod_{uv \in E(H)} G(\phi(u), \phi(v)),$$

where $G(x, y)$ denotes the corresponding entry of the edge $xy \in E(G)$. In fact, Conjecture 1.2 was already stated in terms of weighted graphs G in [11].

There is a good reason to consider weighted graphs with at most one positive eigenvalue as a common generalisation of and K_q. In statistical physics, such a weighted graph is called *antiferromagnetic*, a term that originates from the fundamental Ising and Potts models. In what follows, an antiferromagnetic graph always means a weighted one.

Our main result is a homomorphism inequality that compares the number of complete graphs K_d and its 'bipartisation' $K_{d,d} \setminus M$, the complete bipartite graph $K_{d,d}$ minus a perfect matching M, in an arbitrary antiferromagnetic graph.

Theorem 1.3. *Let G be an antiferromagnetic graph. Then*

$$\hom(K_d, G)^{1/d} \leq \hom(K_{d,d} \setminus M, G)^{1/(2d)}. \tag{1.2}$$

As $K_{d,d} \setminus M$ is bipartite, we have $\hom(K_{d,d} \setminus M, G)^{1/(2d)} \leq \hom(K_{d-1,d-1}, G)^{1/(2d-2)}$ by Theorem 1.1. Combining this with (1.2) then confirms Conjecture 1.2 for complete graphs. Recall that, due to Theorem 1.1, it is enough to prove Conjecture 1.2 for H that contains a triangle. Theorem 1.3 is the first result that confirms Conjecture 1.2 for graphs H that contain a triangle and arbitrary antiferromagnetic graphs G.

The graph $K_{d,d} \setminus M$ can also be seen as a 'double cover' $K_d \times K_2$ of K_d, where $H \times G$ denotes the tensor product of H and G. In [15, Conjecture 2.7], Zhao conjectured that for every d-regular graph H and every $q \geq 2$, the inequality

$$\hom(H, K_q)^2 \leq \hom(H \times K_2, K_q) \tag{1.3}$$

holds. In fact, Theorem 1.3 is a consequence of a stronger correlation inequality, which generalises (1.3) for 'clique-blow-ups' of bipartite graphs. We postpone the precise definition of the graph class until Section 4.

Theorem 1.4. *Let H be a clique-blow-up of a bipartite graph and let G be an antiferromagnetic graph. Then*

$$\hom(H, G)^2 \leq \hom(H \times K_2, G).$$

In particular, one can take $H = K_d$ or $H = K_{r+s} \setminus K_r$.

Antiferromagnetism also appears in mathematical contexts with different names and their generalisations. For example, antiferromagnetism for an n-vertex graph G is equivalent to 'hyperbolicity' of its $n \times n$ adjacency matrix $A = A_G$, i.e., $(\mathbf{x}^T A\mathbf{x})(\mathbf{y}^T A\mathbf{y}) \leq (\mathbf{x}^T A\mathbf{y})^2$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. As a higher-order generalisation of this, Brändén and Huh [3] (and independently, Anari et al. [1]) developed the theory of *Lorentzian polynomials*, which applies to settle several conjectures that relate to log-concavity of combinatorial or geometric sequences. In particular, the theory explains why log-concavity appears in the mixed volumes and mixed discriminants in Euclidean spaces and, as a consequence, recovers the classical Alexandrov–Fenchel inequality [12, Section 7.3].

Our main idea in proving Theorems 1.3 and 1.4 is to apply this emerging theory of Lorentzian polynomials. First, we find a new class of Lorentzian polynomials given by homomorphism counts of K_d. We then obtain a 'discrete' analogue of the Alexandrov–Fenchel inequality as a corollary, which is our key tool. To the best of our knowledge, this is the first application of the theory of Lorentzian polynomials in extremal combinatorics, although a recent result of Matherne, Morales, and Selover [10] shows that some chromatic symmetric functions are Lorentzian.

This proceeding is organised as follows. In Section 2, some standard results on graph homomorphisms and the theory of Lorentzian polynomials are introduced. In Section 3, we obtain a new family of Lorentzian polynomials that come from homomorphism counts of K_t. As a consequence, our main results, Theorems 1.3 and 1.4, follow in the subsequent section. For the expository purpose, some proofs are written in a simplified form that requires extra assumptions.

2 Preliminaries

In what follows, n, q, and t are positive integers. Write $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ for the set of nonnegative integers and let $[n] := \{1, 2, \dots, n\}$. For a finite set I, let $\binom{I}{r}$ be the collection of its subsets of size r.

Graphs. A *graph* has no parallel edges but may have loops with nonempty edge set. We use $v(G)$ and $e(G)$ for the number of vertices and edges of a graph G, respectively. A *weighted graph* G is a graph with positive edge weights on edges and zero weights on non-edges. We write $G(u, v)$ for the corresponding weight on uv by identifying G as its adjacency matrix with corresponding weights. A graph without specified edge weights corresponds to a canonical $\{0, 1\}$-weight. The *support* of a weighted graph G, denoted by $\mathrm{supp}(G)$, is the graph G' where $V(G') := V(G)$ and $uv \in E(G')$ if and only if $G(u, v) > 0$. Let $N_G(v)$ be the set of neighbours of the vertex v in G.

To recall, a weighted graph is *antiferromagnetic* if it has at most one positive eigenvalue, counting multiplicities. Recall also that an antiferromagnetic graph always refers to a weighted graph. We say that a symmetric matrix with nonnegative entries is antiferromagnetic if the corresponding weighted graph is antiferromagnetic. By the Cauchy interlacing theorem, vertex deletions, i.e., principal minors, preserve antiferromagnetism.

Proposition 2.1. *An induced subgraph of an antiferromagnetic graph is also antiferromagnetic.*

Discrete mixed volume. If a graph H is q-colourable, then the *q-chromatic symmetric polynomial*, due to Stanley [13], is defined by

$$X_H(\mathbf{x}) := \sum_{\phi \in \mathrm{Hom}(H,K_q)} \prod_{v \in V(H)} x_{\phi(v)}.$$

We generalise this definition by replacing K_q by an arbitrary weighted graph. Let G be an n-vertex weighted graph on the vertex set $[n]$ and let $\mathbf{x} = (x_1, \ldots, x_n)$ be an n-tuple of variables. A *G-chromatic function of H* is an n-variable homogeneous polynomial

$$h_H(\mathbf{x}; G) = h_H(x_1, \ldots, x_n; G) := \sum_{\phi\colon V(H) \to V(G)} \prod_{uv \in E(H)} G(\phi(u), \phi(v)) \prod_{v \in V(H)} x_{\phi(v)},$$

which has degree $v(H)$. Note that the coefficient $\prod_{uv \in E(H)} G(\phi(u), \phi(v))$ is nonzero if and only if ϕ is a homomorphism from H to the support of G. In particular, the K_q-chromatic function of H is the q-chromatic symmetric polynomial of H. We however remark that $h_H(\mathbf{x}; G)$ is *not* symmetric in general. Indeed, $h_H(x_{\sigma(1)}, \ldots, x_{\sigma(n)}; G) = h_H(x_1, \ldots, x_n; G)$ holds for an automorphism σ of $\mathrm{supp}(G)$ that preserves edge weights, but this may *not* be true for an arbitrary permutation σ on $[n]$.

The *G-volume* of H is a multilinear generalisation of the G-chromatic function. Let $V(H) = [t]$ and let $\mathbf{x}_i = (x_{i,1}, \ldots, x_{i,n})$, $i = 1, 2, \ldots, t$ be n-tuples of variables. The G-volume of H is a t-variable real function on $(\mathbb{R}^n)^t$ defined by

$$V_H(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_t; G) := \sum_{\phi\colon V(H) \to V(G)} \prod_{uv \in E(H)} G(\phi(u), \phi(v)) \prod_{u \in V(H)} x_{u,\phi(u)}.$$

In particular, $V_H(\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x}; G) = h_H(\mathbf{x}; G)$. If each $\mathbf{x}_i$ is the indicator vector for a vertex subset $U_i \subseteq V(G)$, then $V_H(\mathbf{x}_1, \ldots, \mathbf{x}_t; G)$ counts all the homomorphisms ϕ that embed the vertex $i \in V(H)$ into U_i. Although the G-volume is not symmetric in general, we are mostly interested in properties of the G-volume of K_t, which is a symmetric function.

Resembling the definition of mixed volume in Euclidean geometry, the identity

$$h_H(\lambda_1 \mathbf{x}_1 + \cdots + \lambda_r \mathbf{x}_r; G) = \sum_{(i_1, \ldots, i_t) \in [r]^t} V_H(\mathbf{x}_{i_1}, \ldots, \mathbf{x}_{i_t}) \lambda_{i_1} \cdots \lambda_{i_t}$$

holds for $\mathbf{x}_1, \ldots, \mathbf{x}_r \in \mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$. By regarding $\lambda_1, \ldots, \lambda_r$ as variables, we see

$$\sum_{\sigma \in S_t} V_H(\mathbf{x}_{\sigma(1)}, \ldots, \mathbf{x}_{\sigma(t)}; G) = \partial_{\lambda_1} \cdots \partial_{\lambda_t} h_H(\lambda_1 \mathbf{x}_1 + \cdots + \lambda_t \mathbf{x}_t; G),$$

where S_t is the symmetric group on $[t]$. In particular, as $V_{K_t}(\mathbf{x}_1, \ldots, \mathbf{x}_t; G)$ is symmetric,

$$V_{K_t}(\mathbf{x}_1, \ldots, \mathbf{x}_t; G) = \frac{1}{t!}\partial_{\lambda_1} \cdots \partial_{\lambda_t} h_{K_t}(\lambda_1 \mathbf{x}_1 + \cdots + \lambda_t \mathbf{x}_t; G). \tag{2.1}$$

Lorentzian polynomials. Let f be a polynomial on n variables $x_1, \ldots, x_n$. The Hessian $\mathcal{H}f$ of f is the $n \times n$ matrix where $(\mathcal{H}f)_{i,j} = \partial_i \partial_j f$ for $1 \leq i, j \leq n$. The *support* $\operatorname{supp} f$ of f is the set of tuples $(a_1, \ldots, a_n) \in \mathbb{N}_0^n$ such that $x_1^{a_1} \cdots x_n^{a_n}$ has a nonzero coefficient in f.

Following Brändén and Huh [3], Lorentzian polynomials f are defined recursively by using partial derivatives $\partial_i f$ and a combinatorial property of the support of f, the so-called M-*convexity*. A set of vectors $S \subseteq \mathbb{N}_0^n$ is M-*convex* if the following *exchange property* holds: for any vectors $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$ in S, whenever $a_i > b_i$ for some $i \in [n]$, there is $j \in [n]$ such that $a_j < b_j$ and $\mathbf{a} - \mathbf{e}_i + \mathbf{e}_j \in S$. Here $\mathbf{e}_i$ and $\mathbf{e}_j$ are the i-th and j-th standard unit vectors in $\mathbb{N}_0^n$, respectively. Roughly speaking, M-convexity of S defines a base of a discrete polymatroid, a multiset analogue of a matroid. We refer the reader to [8] for further discussions about polymatroids.

Although there is no harm in saying that all homogeneous linear polynomials with nonnegative coefficients are Lorentzian, it only causes extra technicalities to carry on. Thus, we restrict ourselves onto those homogeneous polynomials of degree at least two.

A homogeneous polynomial f with nonnegative coefficients of degree $d \geq 2$ is said to be *Lorentzian* if:

(1) for $d = 2$, the Hessian $\mathcal{H}f$ is antiferromagnetic;

(2) for $d > 2$, each partial derivative $\partial_i f$ is Lorentzian and $\operatorname{supp} f$ is M-convex.

We remark that, while not stated explicitly above, a quadratic Lorentzian polynomial f also has M-convex support. In other words, $\operatorname{supp}(f)$ is M-convex if $\mathcal{H}f$ is antiferromagnetic. We refer to [4, Theorem 5.3] and [2, Theorem 3.2] for the proof.

An n-vertex weighted graph or its adjacency matrix, denoted by G, naturally corresponds to the quadratic n-variable polynomial $Q_G(\mathbf{x}) := \frac{1}{2}\mathbf{x}^T G \mathbf{x}$, whose Hessian is exactly G. Then G is antiferromagnetic if and only if Q_G is Lorentzian. Let $G' = \operatorname{supp}(G)$ for an antiferromagnetic graph G. Then $\operatorname{supp}(Q_{G'}) = \operatorname{supp}(Q_G)$ is M-convex, so $Q_{G'}$ is Lorentzian by [3, Lemma 3.11], whence G' is also antiferromagnetic.

Let $V(G) = \{v_1, \ldots, v_n\}$ and integers $r_1, \ldots, r_n \geq 1$ be given. A *blow-up* of G is a weighted graph G' on the vertex set $\{v_i^{(j)} : i \in [n], j \in [r_i]\}$ with the edge weights $G'(v_i^{(j)}, v_k^{(\ell)}) := G(v_i, v_k)$ for $i, k \in [n]$, $j \in [r_i]$, and $\ell \in [r_k]$. Here, the vertices $v_i^{(1)}, \ldots, v_i^{(r_i)}$ can be seen as 'clones' of v_i. If v_i is a looped vertex, its clones $v_i^{(1)}, \ldots, v_i^{(r_i)}$ form a complete graph K_{r_i} plus r_i loops. In contrast, clones of a non-looped vertex form an independent set. The following lemma, which generalises Proposition 2.1, states that a nonnegative linear change of variables preserves the Lorentzian property. It is a direct consequence of [3, Theorem 2.10] and its proof therein, so we omit the proof.

Lemma 2.2. *Let G' be a blow-up or an induced subgraph of G. Then the following holds:*

(1) if $h_H(-;G)$ has M-*convex support, then so does $h_H(-;G')$;*

(2) if $h_H(-;G)$ is Lorentzian, then so is $h_H(-;G')$.

3 Lorentzian G-chromatic functions

Our first step in proving Theorem 1.3 is to show that $h_{K_t}(\mathbf{x};G)$ is Lorentzian whenever G is antiferromagnetic.

Theorem 3.1. *Let G be an antiferromagnetic graph and let $t \geq 2$. Then $h_{K_t}(\mathbf{x};G)$ is Lorentzian.*

By the recursive definition of Lorentzian polynomials, there are two facts to check: first, $\partial_i h_{K_t}(\mathbf{x};G)$ is Lorentzian for every $i = 1,\dots,n$, where $n = v(G)$; second, the support of $h_{K_t}(\mathbf{x};G)$ is M-convex. The first is rather straightforward to verify. For the expository purpose, we prove Theorem 3.1 for the particular case when G is unweighted and loopless.

We use induction on t, where the base case follows from $h_{K_2}(\mathbf{x};G) = Q_G(\mathbf{x})$. Let $V(K_t) = \{w_1,\dots,w_t\}$ and fix $v \in V(G) = [n]$. Let g_v be the polynomial obtained by summing all the monomials of $h_{K_t}(\mathbf{x};G)$ that contain x_v. Since $v \in V(G)$ is not looped, the pre-image $\phi^{-1}(v)$ of v for a homomorphism $\phi \in \mathrm{Hom}(K_t,G)$ has size at most one. Thus, $g_v(\mathbf{x}) = x_v \cdot \partial_v h_{K_t}(\mathbf{x};G)$. On the other hand,

$$g_v(\mathbf{x}) = \sum_{i=1}^{t} \sum_{\substack{\phi\in\mathrm{Hom}(K_t,G)\\ \phi(w_i)=v}} \prod_{j=1}^{t} x_{\phi(w_j)} = x_v \sum_{i=1}^{t} \sum_{\substack{\phi\in\mathrm{Hom}(K_t,G)\\ \phi(w_i)=v}} \prod_{j\in[t]\setminus\{i\}} x_{\phi(w_j)}.$$

Once we embed the vertex w_i into v, the other vertices in $V(K_t)\setminus\{w_i\}$ must map to vertices in $N_G(v)$ so that the $t-1$ vertices induce a copy of K_{t-1} in $G[N_G(v)]$. That is, with the tuple of variables $\mathbf{y}^{(v)} := (x_i : i \in N_G(v))$,

$$\sum_{\substack{\phi\in\mathrm{Hom}(K_t,G)\\ \phi(w_i)=v}} \prod_{j\in[t]\setminus\{i\}} x_{\phi(w_j)} = h_{K_{t-1}}(\mathbf{y}^{(v)}; G[N_G(v)]).$$

Here $G[N_G(v)]$ is antiferromagnetic too by Proposition 2.1 and thus, by the induction hypothesis, $\partial_v h_{K_t}(\mathbf{x};G) = t \cdot h_{K_{t-1}}(\mathbf{y}^{(v)}; G[N_G(v)])$ is Lorentzian.

To prove that $\mathrm{supp}(h_{K_t}(\mathbf{x};G))$ is M-convex, we provide a structural characterisation of antiferromagnetic graphs G. Let K_q° denote the complete graph on q vertices with exactly one vertex looped; recall that this corresponds to a semiproper colouring.

Theorem 3.2. *Let G be a (unweighted) graph without isolated vertices. Then the following are equivalent:*

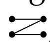
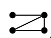
(1) G is antiferromagnetic;

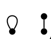
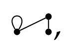
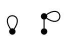
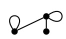
(2) Q_G is Lorentzian;

(3) $\mathrm{supp}(Q_G)$ is M*-convex;*

(4) there exist disjoint vertex sets V_1 and V_2 such that $V_1 \cup V_2 = V(G)$, V_1 induces a complete multipartite graph, and V_2 consists of looped vertices that connect to all the vertices in G.

That is, G is obtained by blowing up K_q° and taking its connected induced subgraph.

Proof. It was already mentioned in Section 2 that (1), (2), and (3) are equivalent by [3, Lemma 3.11].
(1) $\implies$ (4). If G is antiferromagnetic, then so are all its induced subgraphs by Proposition 2.1. Thus, the graphs , , , and  are forbidden as an induced subgraph. It follows that non-looped vertices form a complete multipartite graph and looped vertices are pairwise adjacent.

Suppose G has  as an induced subgraph. As G has no isolated vertex, G must have , , , or  as an induced subgraph. However, none of these are antiferromagnetic, which contradicts antiferromagnetism of G. Thus, each looped vertex connects to all the non-looped vertices. Together with the above paragraph, this characterises the structure of G as given in (4).
(4) $\implies$ (1). By direct calculation, K_q° is antiferromagnetic. By Lemma 2.2, so is G. □

Theorem 3.2 can be seen as a generalisation of [4, Corollary 5.4], which states that a loopless graph without isolated vertices is antiferromagnetic if and only if it is a complete multipartite graph. The only difference is that 'apex' loops, which connect to all the vertices, may appear.

Let us briefly show M-convexity of $\mathrm{supp}(h_{K_t}(\mathbf{x}; G))$ for $G = K_q$. In this case, the support of $h_{K_t}(\mathbf{x}; G)$ consists of $\{0,1\}$-vectors, each of which is the indicator of a set in $\binom{[q]}{t}$. Thus, its M-convexity directly follows from the base exchange property of uniform matroids. For general G, some case analysis proves M-convexity of $\mathrm{supp}(h_{K_t}(\mathbf{x}; G))$. In fact, even for any H other than K_t, it is enough to analyse $\mathrm{supp}(h_H(\mathbf{x}; K_q))$ only. We omit the proof.

Proposition 3.3. *Let H be a connected graph on $t \geq 2$ vertices. If $h_H(-; K_q)$ has* M*-convex support for some $q \geq 3t$, then so does $h_H(-; G)$ for all antiferromagnetic graphs G.*

This may be potentially useful in proving M-convexity of G-chromatic functions for arbitrary antiferromagnetic graphs G.

4 Alexandrov–Fenchel-type inequalities for G-volumes

[3, Proposition 4.5] states that one can obtain an Alexandrov–Fenchel-type inequality from any Lorentzian polynomial.

Proposition 4.1 (Proposition 4.5 in [3]). *Let f be a homogeneous polynomial of degree d in n variables. Let $F_f\colon (\mathbb{R}^n)^d \to \mathbb{R}$ be defined by*

$$F_f(v_1, \dots, v_d) := \frac{1}{d!}\partial_{x_1} \cdots \partial_{x_d} f(x_1 v_1 + \cdots + x_d v_d).$$

If f is Lorentzian, then for any $v_1 \in \mathbb{R}^n$ and $v_2, \dots, v_d \in (\mathbb{R}_{\geq 0})^n$,

$$F_f(v_1, v_2, v_3, \dots, v_d)^2 \geq F_f(v_1, v_1, v_3, \dots, v_d) \cdot F_f(v_2, v_2, v_3, \dots, v_d).$$

For example, f can be the volume of the Minkowski sum $x_1 K_1 + \cdots + x_d K_d$ of convex bodies $K_1, \dots, K_d$, a Lorentzian polynomial given in [3]. Then Proposition 4.1 recovers the classical Alexandrov–Fenchel inequality for mixed volumes.

In our setting, choosing $f = h_{K_t}(\mathbf{x}; G)$ gives $F_f = V_G(\mathbf{a}_1, \dots, \mathbf{a}_t; G)$, the G-volume of K_t due to (2.1). As $f = h_{K_t}(\mathbf{x}; G)$ is Lorentzian by Theorem 3.1, Proposition 4.1 yields an Alexandrov–Fenchel-type inequality for the G-volume of a complete graph. That is, for $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_t \in (\mathbb{R}_{\geq 0})^n$,

$$V_{K_t}(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \dots, \mathbf{a}_t; G)^2 \geq V_{K_t}(\mathbf{a}_1, \mathbf{a}_1, \mathbf{a}_3, \dots, \mathbf{a}_t; G) \cdot V_{K_t}(\mathbf{a}_2, \mathbf{a}_2, \mathbf{a}_3, \dots, \mathbf{a}_t; G). \tag{4.1}$$

Applying (4.1) iteratively gives the following inequality.

Corollary 4.2. *Let G be an n-vertex antiferromagnetic graph and let $\mathbf{a}, \mathbf{b} \in (\mathbb{R}_{\geq 0})^n$. Then*

$$V_{K_t}(\mathbf{b}, \mathbf{a}, \dots, \mathbf{a}; G) \cdot V_{K_t}(\mathbf{a}, \mathbf{b}, \dots, \mathbf{b}; G) \geq V_{K_t}(\mathbf{a}, \mathbf{a}, \dots, \mathbf{a}; G) \cdot V_{K_t}(\mathbf{b}, \mathbf{b}, \dots, \mathbf{b}; G).$$

We are now ready to prove our main result of this section, which implies Theorem 1.3 by setting $\mathbf{a} = \mathbf{b} = (1, 1, \dots, 1)$.

Theorem 4.3. *Let G be an n-vertex antiferromagnetic graph and let $t \geq 2$. Write the bipartition of $K_t \times K_2$ as $V_1 \sqcup V_2$. For $\mathbf{a}, \mathbf{b} \in (\mathbb{R}_{\geq 0})^n$,*

$$V_{K_t}(\underbrace{\mathbf{a}, \dots, \mathbf{a}}_{t}; G) \cdot V_{K_t}(\underbrace{\mathbf{b}, \dots, \mathbf{b}}_{t}; G) \leq V_{K_t \times K_2}(\underbrace{\mathbf{a}, \dots, \mathbf{a}}_{t}, \underbrace{\mathbf{b}, \dots, \mathbf{b}}_{t}; G),$$

where in the right-hand side, $\mathbf{a}$ and $\mathbf{b}$ correspond to the vertices in V_1 and V_2, respectively.

Proof. We provide a simplified proof for the case when $\mathbf{a}$, $\mathbf{b}$ are $\{0, 1\}$-valued so that they represent vertex subsets and G is an unweighted loopless graph. Let $V(G) = [n]$. For

$A \subseteq [n]$, denote by $\mathbf{1}_A \in \{0,1\}^n$ the indicator vector of A. For $A, B \subseteq [n]$ and $1 \leq \ell \leq t$, write

$$V_{K_t}(A;t;G) := V_{K_t}(\underbrace{\mathbf{1}_A, \ldots, \mathbf{1}_A}_{t}; G), \quad V_{K_t}(A,B;\ell;G) := V_{K_t}(\underbrace{\mathbf{1}_A, \ldots, \mathbf{1}_A}_{\ell}, \underbrace{\mathbf{1}_B, \ldots, \mathbf{1}_B}_{t-\ell}; G),$$

$$\text{and} \quad V_{K_t \times K_2}(A;B;G) := V_{K_t \times K_2}(\underbrace{\mathbf{1}_A, \ldots, \mathbf{1}_A}_{t}, \underbrace{\mathbf{1}_B, \ldots, \mathbf{1}_B}_{t}; G).$$

Thus, our goal is to show that for $A, B \subseteq [n]$,

$$V_{K_t}(A;t;G) \cdot V_{K_t}(B;t;G) \leq V_{K_t \times K_2}(A,B;t;G).$$

We use induction on t. As $K_2 \times K_2 = K_2 \sqcup K_2$, the base case $t = 2$ reduces to

$$V_{K_2}(A,A;G) \cdot V_{K_2}(B,B;G) \leq V_{K_2}(A,B;G)^2 = V_{K_2 \times K_2}(A,A,B,B;G),$$

which follows from Corollary 4.2. Suppose $t \geq 3$. We may assume that $K_t \times K_2$ is the bipartite graph with the bipartition $V_1 \sqcup V_2 = ([t] \times \{1\}) \sqcup ([t] \times \{2\})$ and the edge set $\{(i,1)(j,2) : i,j \in [t],\ i \neq j\}$.

We compute the G-volume $V_{K_t \times K_2}(A,B;t;G)$ by counting homomorphisms ϕ from $K_t \times K_2$ to G recursively as follows: first, choose $\phi(t,1) = r \in A$ and $\phi(t,2) = s \in B$; next, find a homomorphic copy of $K_{t-1} \times K_2$ such that the vertices in A and B are adjacent to s and r, respectively. These two-step embeddings give us the identity

$$V_{K_t \times K_2}(A;B;G) = \sum_{r \in A,\, s \in B} V_{K_{t-1} \times K_2}(A \cap N_G(s); B \cap N_G(r); G)$$

By the induction hypothesis, it follows that

$$\begin{aligned} V_{K_t \times K_2}(A;B;G) &\geq \sum_{r \in A,\, s \in B} V_{K_{t-1}}(A \cap N_G(s); t-1; G) \cdot V_{K_{t-1}}(B \cap N_G(r); t-1; G) \\ &= \sum_{s \in B} V_{K_{t-1}}(A \cap N_G(s); t-1; G) \cdot \sum_{r \in A} V_{K_{t-1}}(B \cap N_G(r); t-1; G). \end{aligned}$$

The term $V_{K_{t-1}}(A \cap N_G(s); t-1; G)$ counts the number of labelled copies of K_{t-1} in $N_G(s)$ whose vertices are also in A. By summing this over $s \in B$, we get the number of labelled copies of K_t whose first vertex is in B and the others are in A. In other words,

$$\sum_{s \in B} V_{K_{t-1}}(A \cap N_G(s); t-1; G) = V_{K_t}(A,B;t-1;G),$$

and, symmetrically,

$$\sum_{r \in A} V_{K_{t-1}}(B \cap N_G(r); t-1; G) = V_{K_t}(A,B;1;G).$$

Therefore, together with Corollary 4.2, we conclude that

$$V_{K_t \times K_2}(A;B;G) \geq V_{K_t}(A,B;t-1;G) \cdot V_{K_t}(A,B;1;G) \geq V_{K_t}(A;t;G) V_{K_t}(B;t;G). \qquad \square$$

Let $\mathcal{H}$ be a class of graphs that includes K_1. An *$\mathcal{H}$-blow-up* H of a k-vertex graph F is obtained by replacing all k vertices $v_1, \ldots, v_k$ of F by $H_1, \ldots, H_k$ in $\mathcal{H}$, respectively, and the edges of the form $v_i v_j \in E(F)$ by a complete bipartite graph $K_{s,t}$, where $s = v(H_i)$ and $t = v(H_j)$. Note that a vertex blown up by K_1 can be considered unchanged. A *clique-blow-up* of a graph F is an $\mathcal{H}$-blow-up with the set $\mathcal{H}$ of all complete graphs. For instance, $K_{r+s} \setminus K_s$ is a clique-blow-up of $K_{1,s}$. Theorem 1.4 then follows from Theorem 4.3.

Proof of Theorem 1.4. We give a brief sketch, again assuming G is unweighted. Let H be a clique-blow-up of a k-vertex bipartite graph F, where each vertex v_i is replaced by a copy of K_{r_i}. Let H_1 and H_2 be vertex-disjoint copies of H. In H_j, $j = 1, 2$, $V_i^{(j)}$, $i = 1, 2, \ldots, k$, denotes the vertex set of the copy of K_{r_i} that replace $v_i \in V(F)$.

Let $L_0 = H_1 \sqcup H_2$ and let L_ℓ, $\ell = 1, 2, \ldots, k$ be the graph obtained by replacing the copy of $K_{r_\ell} \sqcup K_{r_\ell}$ induced on $V_\ell^{(1)} \cup V_\ell^{(2)}$ in $L_{\ell-1}$ by $K_{r_1} \times K_2$, whose bipartition is again $V_\ell^{(1)} \cup V_\ell^{(2)}$. It is easy to check that L_k is isomorphic to $H \times K_2$ by using bipartiteness of F. Repeated applications of Theorem 4.3 then give

$$\hom(H, G)^2 = \hom(L_0, G) \le \hom(L_1, G) \le \cdots \le \hom(L_k, G) = \hom(H \times K_2, G).$$

Indeed, to compare L_ℓ and $L_{\ell-1}$, fix all homomorphic images of $V_i^{(1)} \cup V_i^{(2)}$, $i \neq \ell$, where L_ℓ and $L_{\ell-1}$ induce isomorphic subgraphs, and denote by ϕ this partial embedding. Then there exist $A_\phi, B_\phi \subseteq V(G)$ such that finding a homomorphic copy of K_{r_ℓ} in A_ϕ and in B_ϕ, respectively, extends ϕ to a homomorphic copy of $L_{\ell-1}$. In contrast, finding a homomorphic copy of $K_{r_\ell} \times K_2$ across $A_\phi \cup B_\phi$ extends ϕ to a homomorphic copy of L_ℓ. Thus, $\hom(L_{\ell-1}, G) \le \hom(L_\ell, G)$ reduces to summing the inequality

$$V_{K_{r_\ell}}(A_\phi; r_\ell; G) V_{K_{r_\ell}}(B_\phi; r_\ell; G) \le V_{K_{r_\ell} \times K_2}(A_\phi, B_\phi; r_\ell; G)^2$$

for all partial embeddings ϕ, which follows from Theorem 4.3. $\square$

The essence of the proof of Theorem 1.4 leverages on a weaker statement than the Alexandrov–Fenchel-type inequality. Namely, it only requires, with $k = v(K)$,

$$V_K(A; k; G) V_K(B; k; G) \le V_{K \times K_2}(A, B; k; G)^2 \tag{4.2}$$

holds for any $A, B \subseteq V(G)$ to blow-up a vertex of F by using a copy of K.

For $G = K_q$, we show that complete multipartite graphs or paths K also satisfy (4.2), so more graphs can be added to $\mathcal{H}$ to blow up vertices of a bipartite graph F. We omit the proof of this result, which relies on multiple reductions and certain log-concavity between complete multipartite graph counts in K_q. As a consequence, we have slightly more options to blow up a bipartite graph F than Theorem 1.4 offers if $G = K_q$.

Theorem 4.4. *Let $\mathcal{H}$ be the class of graphs that consist of all complete multipartite graphs and paths. For any $\mathcal{H}$-blow-up H of a bipartite graph F and any $q \geq 2$,*

$$\hom(H, K_q)^2 \le \hom(H \times K_2, K_q).$$

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