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A Geometric Interpretation and Skewing Formula for the Delta Theorem

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Abstract. We show that the symmetric function $\Delta'_{e_{k-1}} e_n$ appearing in the Delta Theorem can be obtained from the symmetric function in the integer-slope Rectangular Shuffle Theorem by applying a Schur skewing operator. This generalizes a formula by the first and third authors for the Delta Theorem at $t = 0$, and follows from work of Blasiak, Haiman, Morse, Pun, and Seelinger. We also provide a combinatorial proof of this identity, giving a new proof of the Rise Delta Theorem from the Rectangular Shuffle Theorem.

We then introduce a variety $Y_{n,k}$, which we call the *affine* Δ*-Springer fiber*, generalizing the affine Springer fiber studied by Hikita, whose Borel–Moore homology has an S_n action and a bigrading that corresponds to the Delta Theorem symmetric function $\mathrm{rev}_q\, \omega \Delta'_{e_{k-1}} e_n$ under the Frobenius character map. To prove this, we first similarly provide a geometric interpretation for the Rectangular Shuffle Theorem, and then use a geometric skewing identity along with the skewing formula above to obtain our results on $Y_{n,k}$.

Keywords: Schur functions, Delta Conjecture, parking functions, affine Springer fibers

1 Introduction and background

The Shuffle Theorem, which was conjectured in [11] and proven in [5], gives a beautiful combinatorial formula for ∇e_n, the evaluation of the Macdonald eigenoperator ∇ on the elementary symmetric function e_n, in terms of labeled Dyck paths. Two prominent generalizations of the Shuffle Theorem are the Rectangular Shuffle Theorem [1] (which also generalizes the 'Rational Shuffle Theorem'), which gives a combinatorial formula for $E_{ka,kb} \cdot 1$ where $E_{ka,kb}$ is a certain operator from the elliptic Hall Algebra, and the Delta Theorem [12], which gives a combinatorial formula for $\Delta'_{e_{k-1}} e_n$ where Δ'_f is a class

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of Macdonald eigenoperators generalizing ∇. In this abstract, we establish a formula connecting these two generalizations, as well as a new geometric interpretation for each.

Given $k \le n$, a **stack** S of boxes in an $n \times k$ grid is a subset of the grid boxes such that there is one element of S in each row, at least one in each column, and each box in S is weakly to the right of the one below it. A (word) **stacked parking function** with respect to S is a labeled up-right path D such that each box of S lies below D, and the labeling is strictly increasing up each column. See the image on the right side of Figure 1 for an example with $k = 3$ and $n = 5$. Let $\mathrm{WLD}^{\mathrm{stack}}_{n,k}$ be the set of word stacked parking functions ranging over all stacks S in the $n \times k$ grid.

Theorem 1.1 (Rise Delta Theorem [2, 6]). *We have*

$$\Delta'_{e_{k-1}} e_n = \sum_{P \in \mathrm{WLD}^{\mathrm{stack}}_{n,k}} q^{\mathrm{area}(P)} t^{\mathrm{hdinv}(P)} x^P. \tag{1.1}$$

Here, the **area** of an element of $\mathrm{WLD}^{\mathrm{stack}}_{n,k}$, written $\mathrm{area}(P)$, is the number of boxes between the path and the stack S. The statistic $\mathrm{hdinv}(P)$ counts inversions among certain pairs of labels in P, see [12].

On the other hand, the Rectangular Shuffle Theorem concerns Dyck paths in a rectangle. Let $K = k(n - k + 1)$, and consider rational Dyck paths of height K and width k, that stay weakly above the northeast diagonal in the grid. A **word parking function** P is a labeling of the vertical runs of the Dyck path by positive integers such that the labeling strictly increases up each vertical run (but letters may repeat between columns). See the image on the left side of Figure 1 for an example with $k = 3$ and $n = 5$. We let $\mathrm{WPF}_{K,k}$ be the set of word parking functions whose path is a rational Dyck path in the $K \times k$ grid.

The **area** of P is the number of whole boxes lying between the path and the diagonal such that the diagonal does not pass through the interior of the box. A **diagonal** of the $K \times k$ rectangle is a set of boxes whose centers lie on the same line with slope K/k. Given two boxes a, b above the main diagonal, we say (a, b) are an **attacking pair** if either a and b are on the same diagonal and a is to the left, or a and b are on adjacent diagonals and b is to the left and on the higher diagonal. The **tdinv** (for "temporary diagonal inversions") of P is the number of attacking pairs (a, b) such that a and b are labeled and a has a strictly smaller label than b. Given a Dyck path D, the **maxtdinv** of D is the maximum tdinv over all word parking functions on D. The **dinv** statistic (for "diagonal inversions") on $\mathrm{WPF}_{K,k}$ is defined as

$$\mathrm{dinv}(P) = \mathrm{pathdinv}(D) + \mathrm{tdinv}(P) - \mathrm{maxtdinv}(D)$$

where D is the Dyck path of P. See [1] for the definition of $\mathrm{pathdinv}(D)$ which counts a certain set of boxes above D according to their arm and leg.

Theorem 1.2 (Rectangular Shuffle Theorem [15]).

$$E_{K,k} \cdot 1 = \sum_{P \in \mathrm{WPF}_{K,k}} q^{\mathrm{area}(P)} t^{\mathrm{dinv}(P)} x^P. \tag{1.2}$$

In Section 2, we sketch our two proofs of the fact that the two formulas are directly related by a Schur skewing operator.

Theorem 1.3 ([7]). *Letting $K = k(n-k+1)$ and $\lambda = (k-1)^{n-k}$, we have*

$$\Delta'_{e_{k-1}} e_n = s_\lambda^\perp (E_{K,k} \cdot 1), \tag{1.3}$$

where $s_\lambda^\perp$ is the adjoint to multiplication by the Schur function s_λ.

Furthermore, we show that this skewing formula can be used to give geometric meaning to the Delta Theorem. In Section 3, we give geometric realizations $X_{n,k}$ and $Y_{n,k}$ of both the Rectangular Shuffle Theorem in the (K,k) case and for the Delta Conjecture, respectively, in terms of affine Springer fibers.

Theorem 1.4 ([8]). *There are varieties $X_{n,k}$ and $Y_{n,k}$ whose Borel–Moore homology have S_K and S_n actions, respectively, such that*

$$\mathrm{grFrob}(H_*^{\mathrm{BM}}(X_{n,k}); q, t) = \mathrm{rev}_q \omega(E_{K,k} \cdot 1),$$
$$\mathrm{grFrob}(H_*^{\mathrm{BM}}(Y_{n,k}); q, t) = \mathrm{rev}_q \omega(\Delta'_{e_{k-1}} e_n),$$

where $K = k(n-k+1)$.

The connection between the two constructions is a geometric version of the skewing formula which we derive from work of Borho and MacPherson on partial resolutions of nilpotent varieties [4] after checking rational smoothness.

Theorem 1.4 generalizes work of Hikita [14] in the case of ∇e_n. Pawlowski–Rhoades [17] and Griffin–Levinson–Woo [10] have given two different geometric models for the symmetric function $\Delta'_{e_{k-1}} e_n$ at $t = 0$, and our construction directly generalizes the latter. We also note that Haiman has given a different geometric interpretation of $\Delta'_{e_{k-1}} e_n$ in terms of derived global sections of a particular vector bundle on the punctual Hilbert scheme of points in the plane [13].

We summarize the results of the paper in the following table:

Degree	Algebra	Combinatorics	Geometry	Module
K	$E_{K,k} \cdot 1$	$\mathrm{PF}_{K,k}$	$X_{n,k}$	$H_*^{BM}(X_{n,k}) \circlearrowleft S_K$
n	$\Delta'_{e_{k-1}} e_n$	$\mathrm{WLD}_{n,k}^{\mathrm{stack}}$	$Y_{n,k}$	$H_*^{BM}(Y_{n,k}) \circlearrowleft S_n$

2 Skewing formula and combinatorial proof

We outline two proofs of Theorem 1.3, following the work in [7].

2.1 Algebraic proof of the skewing formula

Our first proof of the skewing formula, Theorem 1.3, relies on results of Blasiak, Haiman, Morse, Pun, and Seelinger [2] and Negut [16] relating the Delta Conjecture and Rectangular Shuffle Theorems to raising operators. On the one hand, Blasiak et al. prove that

$$\omega\Delta'_{e_{k-1}}e_n(x_1,\ldots,x_k) = H^k_{q,t}\left(\frac{x_1\cdots x_k h_{n-k}(x_1,\ldots,x_k)}{\prod(1-qtx_i/x_{i+1})}\right)_{\text{pol}},$$

where $H^k_{q,t}$ is a certain raising operator on Laurent polynomials, and the subscript pol means that one must expand first in rational characters of GL_k (Schur polynomials) and then truncate the series to the terms corresponding to polynomial characters.

A similar formula holds for the Rectangular Shuffle Theorem. It follows from [16] that

$$\omega E_{K,k}(1)(x_1,\ldots,x_k) = H^k_{q,t}\left(\frac{x_1^{n-k+1}\cdots x_k^{n-k+1}}{\prod(1-qtx_i/x_{i+1})}\right)_{\text{pol}}.$$

We then prove that the identity (1.3) holds when restricted to k variables by relating the two raising operator formulas using the Schur skewing operator $s_\lambda^{\perp,k}$ (where the superscript k indicates skewing of symmetric polynomials in k variables). We then show that the statement for k variables implies the identity of symmetric functions.

2.2 Combinatorial proof of the skewing formula

Our second proof of Theorem 1.3 involves a direct comparison of the combinatorial sides of (1.3). That is, we show that $s_\lambda^\perp$ applied to the right-hand side of (1.2) yields the right-hand side of (1.1). We first express the right hand side of (1.2) in terms of polynomials f_D defined as follows. Fix a $K\times k$ Dyck path D and denote $f_D = \sum_{\pi\in\mathrm{WPF}_{K,k}(D)} q^{\mathrm{tdinv}(\pi)}x^\pi$ where the sum is over column-strict word parking functions on D. We then wish to compute $s^\perp_{(k-1)^{n-k}}f_D$. By general theory of plethystic substitution and the Hall inner product,

$$s^\perp_{(k-1)^{n-k}}f_D = \langle s_{(k-1)^{n-k}}[Y], f_D[X+Y;q]\rangle,$$

and we wish to compute the latter. To do so, let $1,\ldots,n$ be **small labels** and $n+1,\ldots,K$ be **big labels**, where again $K = k(n-k+1)$. Given a (K,k)-parking function π with numbers $1,2,\ldots,K$ used exactly once, we write:

- $\mathrm{tdinv}_{\mathrm{small}}(\pi)$ is the number of temporary diagonal inversions between small labels.
- $d(\underline{s},\underline{b})$ is the number of pairs (c,c') such that c has a small label, c' has a big label, and (c,c') are attacking (in that order).
- $\mathrm{tdinv}_{\mathrm{big}}(\pi)$ is the number of temporary diagonal inversions between big labels.

Note that $d(\underline{s}, \underline{b})$ depends only on the positions of big and small labels, but not on a specific parking function π. Also,

$$\text{tdinv}(\pi) = \text{tdinv}_{\text{small}}(\pi) + d(\underline{s}, \underline{b}) + \text{tdinv}_{\text{big}}(\pi). \tag{2.1}$$

We generalize these notions to word parking functions. Given a $K \times k$ Dyck path D and any weak composition $\underline{b} = (b_1, \ldots, b_k)$ of $(n-k)(k-1)$ such that b_i is at most the number of vertical steps in column i, we will say the **big boxes** of D are the top b_i boxes in column i that are immediately to the right of a vertical step of D. Similarly, the *small boxes* of D are the remaining boxes to the right of vertical steps. We define

$$f_{D,\underline{b}} = \sum_{\pi_{\text{big}} \in \text{WPF}(D,b)} q^{\text{tdinv}_{\text{big}}(\pi_{\text{big}})} x^{\pi_{\text{big}}}, \quad f_{D,\underline{s}} = \sum_{\pi_{\text{small}} \in \text{WPF}(D,s)} q^{\text{tdinv}_{\text{small}}(\pi_{\text{small}})} x^{\pi_{\text{small}}}$$

where $\text{tdinv}_{\text{big}}$ measures the number of diagonal inversions of big boxes of π_{big}, which is a partial word parking function that only labels big boxes. We similarly define $\text{tdinv}_{\text{small}}$ and π_{small}. We also still write $d(\underline{s}, \underline{b})$ for the number of attacking pairs between big and small boxes such if the big box is labeled with a larger number than the small box, it would form an inversion.

It is easy to see from (2.1) (compare [3, Equation (11)]) that

$$f_D[X+Y;q] = \sum_{\underline{b},\underline{s}} q^{d(\underline{s},\underline{b})} f_{D,\underline{s}}[X;q] f_{D,\underline{b}}[Y;q]$$

where the sum is over all possible decompositions $\underline{b}, \underline{s}$. Now

$$s^{\perp}_{(k-1)^{n-k}} f_D = \langle s_{(k-1)^{n-k}}(Y), f_D[X+Y;q] \rangle = \sum_{\underline{b},\underline{s}} q^{d(\underline{s},\underline{b})} f_{D,\underline{s}}[X;q] \langle s_{(k-1)^{n-k}}(Y), f_{D,\underline{b}}[Y;q] \rangle.$$

A key step now is to compute $\langle s_{(k-1)^{n-k}}(Y), f_{D,\underline{b}}[Y;q] \rangle$, which we do below in Theorem 2.1. We define $\underline{b}$ to be **admissible** if all $b_i \leq n-k$, and define the statistic

$$c_{D,\underline{b}} = \text{maxtdinv}(D) - \text{pathdinv}(D) - d(\underline{s}, \underline{b}). \tag{2.2}$$

Theorem 2.1 ([7]). *Let D be a Dyck path and $\underline{b}$ be an admissible sequence for D. Then*

$$\langle f_{D,\underline{b}}[X;q], s_{(k-1)^{n-k}} \rangle = \langle \omega f_{D,\underline{b}}[X;q], s_{(n-k)^{k-1}} \rangle = q^{c_{D,\underline{b}}}.$$

If $\underline{b}$ is not admissible, then $\langle f_{D,\underline{b}}[X;q], s_{(k-1)^{n-k}} \rangle = 0$.

Proof. The proof goes in several steps.

Step 1: Given a composition $\widetilde{\alpha} = (\widetilde{\alpha}_1, \ldots, \widetilde{\alpha}_\ell)$, the pairing $\langle h_{\widetilde{\alpha}}, f_{D,\underline{b}}[X;q] \rangle$ equals the coefficient of the monomial symmetric function $m_{\widetilde{\alpha}}$ in $f_{D,\underline{b}}[X;q]$, which counts the column-strict fillings of $(D, \underline{b})$ with content $\widetilde{\alpha}$. We denote this set of fillings by $P_{D,\underline{b},\widetilde{\alpha}}$ and write

$$\langle h_{\widetilde{\alpha}}, f_{D,\underline{b}}[X;q] \rangle = \sum_{P \in P_{D,\underline{b},\widetilde{\alpha}}} q^{\text{tdinv}_{\text{big}}(P)} \tag{2.3}$$

Note that $(D, \underline{b})$ has at most k vertical runs, so in a column-strict filling of $(D, \underline{b})$ any label is repeated at most k times. Therefore, for $\widetilde{\alpha}_i \geq k+1$ there are no such fillings.

Step 2: We expand the Schur function using the Jacobi–Trudi formula, where we replace any h_j with 0 if $j \geq k+1$:

$$s_{(k-1)^{n-k}} = \det(h_{k-1-i+j})_{i,j} \mod (h_j, j \geq k+1). \tag{2.4}$$

For a composition $\widetilde{\alpha}$ such that $h_{\widetilde{\alpha}} = h_{\widetilde{\alpha}_1} \cdot \cdots \cdot h_{\widetilde{\alpha}_{n-k}}$ appearing in the expansion of (2.4) modulo h_j for $j \geq k+1$, we write α to be the complementary composition where $\alpha_i = k - \widetilde{\alpha}_i$ for all i. We call all resulting compositions α **allowable contents**, and rewrite (2.4) as

$$s_{(k-1)^{n-k}} = \sum_{\alpha \text{ allowable}} (-1)^{\mathrm{sgn}(\widetilde{\alpha})} h_{\widetilde{\alpha}} \mod (h_j, j \geq k+1).$$

The quantity $\mathrm{sgn}(\widetilde{\alpha})$ is the sign of the permutation used to obtain $h_{\widetilde{\alpha}}$ in (2.4). By combining this with (2.3), we get

$$\langle s_{(k-1)^{n-k}}, f_{D,\underline{b}}[X;q] \rangle = \sum_{\alpha \text{ allowable}} \sum_{P \in P(D,\underline{b},\widetilde{\alpha})} (-1)^{\mathrm{sgn}(\widetilde{\alpha})} q^{\mathrm{tdinv}_{\mathrm{big}}(P)}. \tag{2.5}$$

Also, since $\widetilde{\alpha}$ has $(n-k)$ parts, the set $P(D, \underline{b}, \widetilde{\alpha})$ is empty for all $\widetilde{\alpha}$ if $b_i > n-k$ for some i (that is, $\underline{b}$ is not admissible). This implies that $\langle f_{D,\underline{b}}[X;q], s_{(k-1)^{n-k}} \rangle = 0$ whenever $\underline{b}$ is not admissible. From now on we assume $b_i \leq n-k$.

Step 3: In [7] we define a sign-reversing involution φ on the set of column-strict fillings with allowable contents, which has the following properties:

- φ is determined by its action on the (complemented) reading word of the filling,
- φ has a unique fixed point, which we denote by $P^0_{D,\underline{b}}$, that has positive sign,
- φ preserves $\mathrm{tdinv}_{\mathrm{big}}$ and reverses the sign $(-1)^{\mathrm{sgn}(\widetilde{\alpha})}$ for every element except $P^0_{D,\underline{b}}$.

As an example of the involution on (complemented) reading words, on words of length 3 with allowable contents α, the involution pairs

$$\begin{array}{ll} 132 \leftrightarrow 122 & 213 \leftrightarrow 113 \\ 231 \leftrightarrow 131 & 312 \leftrightarrow 212 \\ 321 \leftrightarrow 221 & 311 \leftrightarrow 111 \end{array}$$

and leaves 123 as a fixed point.

Step 4: By the previous step, the terms in (2.5) cancel in pairs according to the involution φ, and we are left with a single term corresponding to the fixed point:

$$\langle s_{(k-1)^{n-k}}, f_{D,\underline{b}}[X;q] \rangle = q^{\mathrm{tdinv}_{\mathrm{big}}(P^0_{D,\underline{b}})}.$$

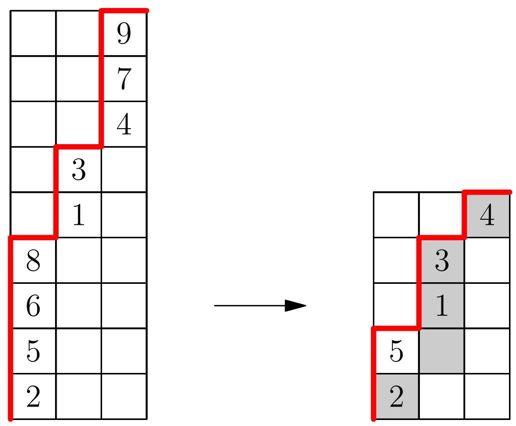

Figure 1: Mapping a standard (K,k)-parking function to a stacked parking function by removing the big labels, that is, those that are greater than n. Here $k=3$ and $n=5$, so $K=9$. The tdinv of the parking function at left is 7, because there are 7 attacking pairs contributing to tdinv: $(1,7)$, $(3,9)$, $(5,7)$, $(6,9)$, $(4,5)$, $(1,6)$, $(3,8)$

To complete the proof, we prove that $\mathrm{tdinv}_{\mathrm{big}}(P^0_{D,\underline{b}}) = c_{D,\underline{b}}$. By (2.2), this translates to showing the combinatorial fact that

$$\mathrm{tdinv}_{\mathrm{big}}(P^0_{D,\underline{b}}) = \mathrm{maxtdinv}(D) - \mathrm{pathdinv}(D) - d(\underline{s},\underline{b}) \tag{2.6}$$

This is highly nontrivial and involves multiple induction arguments [7]. □

Assuming all of the above steps, the main combinatorial result can now be proven.

Theorem 2.2 ([7]). *We have*

$$s^{\perp}_{(k-1)^{n-k}} \sum_{\pi \in \mathrm{WPF}_{K,k}} t^{\mathrm{area}(\pi)} q^{\mathrm{dinv}(\pi)} x^{\pi} = \sum_{P \in \mathrm{WLD}^{\mathrm{stack}}_{n,k}} t^{\mathrm{area}(P)} q^{\mathrm{hdinv}(P)} x^{P}.$$

where the sums are over column-strict parking functions that may have repeats between columns. In other words, Theorem 1.3 holds combinatorially.

To finish the proof of the combinatorial skewing formula, we use a bijection F between small parking functions π in the union of $\mathrm{WPF}(D,\underline{s})$ running over all D and admissible $\underline{s}$ sequence, and stacked parking functions. (See Figure 1.) It has the property that $\mathrm{tdinv}_{\mathrm{small}}(\pi) = \mathrm{hdinv}(F(\pi))$ and $\mathrm{area}(D) = \mathrm{area}(F(\pi))$, so

$$\begin{aligned}\sum_{D \in \mathrm{Dyck}(K,k)} \sum_{\underline{b},\underline{s}} t^{\mathrm{area}(D)} f_{D,\underline{s}} &= \sum_{D \in \mathrm{Dyck}(K,k)} \sum_{\underline{b},\underline{s}} \sum_{\pi \in \mathrm{WPF}(D,\underline{s})} t^{\mathrm{area}(\pi)} q^{\mathrm{tdinv}_{\mathrm{small}}(\pi)} x^{\pi} \\ &= \sum_{F(\pi) \in \mathrm{WLD}^{\mathrm{stack}}_{n,k}} t^{\mathrm{area}(F(\pi))} q^{\mathrm{hdinv}(F(\pi))} x^{F(\pi)}.\end{aligned}$$

This connects the skewing operation, using Theorem 2.1, to the Delta formula (1.1).

The combinatorial proof described in this section can alternatively be interpreted in the following way.

Corollary 2.3. *We have a new combinatorial proof of the Rise Delta Theorem, assuming the Rectangular Shuffle Theorem and the skewing formula (Theorem 1.3, where we start with the simpler algebraic proof).*

3 Geometric skewing

In this section, we give geometric realizations of both the integer-slope Rectangular Shuffle Theorem and the Delta Theorem. Precisely, we construct varieties $X_{n,k}$ and $Y_{n,k}$ such that the Borel–Moore homology of $X_{n,k}$ has an action of S_K so that its Frobenius character coincides with $E_{K,k} \cdot 1$ (up to a minor twist). Similarly, $Y_{n,k}$ will be a variety whose Borel–Moore homology has an action of S_n. We then use a geometric version of our skewing formula 1.3 to show so that its Frobenius character coincides with $\Delta'_{e_{k-1}} e_n$ (up to a minor twist).

Given $\eta = (\eta_1, \ldots, \eta_m)$ a composition of K, let $\widetilde{\mathrm{Fl}}_\eta$ be the **partial affine flag variety** of partial flags of lattices in $\mathbb{C}^K((\epsilon))$,

$$\Lambda_\bullet = (\Lambda_0 \supset \Lambda_{\eta_1} \supset \Lambda_{\eta_1+\eta_2} \supset \cdots \supset \Lambda_{\eta_1+\cdots+\eta_m} = \Lambda_K)$$

with the periodicity property $\Lambda_K = \epsilon\Lambda_0$ and $\dim_{\mathbb{C}}(\Lambda_i/\Lambda_j) = j - i$. We write $\widetilde{\mathrm{Fl}} = \widetilde{\mathrm{Fl}}_{(1^K)}$ for the space of complete flags of lattices. Let $\mathrm{pr}_\eta : \widetilde{\mathrm{Fl}} \to \widetilde{\mathrm{Fl}}_\eta$ be the projection map which forgets parts of the flag.

Definition 3.1. An (extended) **affine permutation** in $\widetilde{S_K}$ is a bijection $\omega : \mathbb{Z} \to \mathbb{Z}$ such that, for all $i \in \mathbb{Z}$,

$$\omega(i + K) = \omega(i) + K.$$

We will often write the affine permutations in **window notation**

$$\omega = [\omega(1), \ldots, \omega(K)] .$$

Each affine permutation $\omega \in \widetilde{S_K}$ corresponds to a **Schubert cell**, $C_\omega = I^-\omega I^-/I^- \subseteq \widetilde{\mathrm{Fl}}$, and the Schubert cells partition the affine flag variety, $\widetilde{\mathrm{Fl}} = \bigsqcup_{\omega \in \widetilde{S_K}} C_\omega$.

The **degree** of an affine permutation ω is

$$\deg \omega = \frac{1}{K} \sum_{i=1}^{K} (\omega(i) - i) = \frac{1}{K} \left(\sum_{i=1}^{K} \omega(i) - \frac{K(K+1)}{2} \right). \tag{3.1}$$

One can check that $\deg \omega$ is always an integer, and it is well known that the connected components of $\widetilde{\mathrm{Fl}}$ are indexed by $\mathbb{Z}$, given by the union of the Schubert cells for ω with a fixed degree.

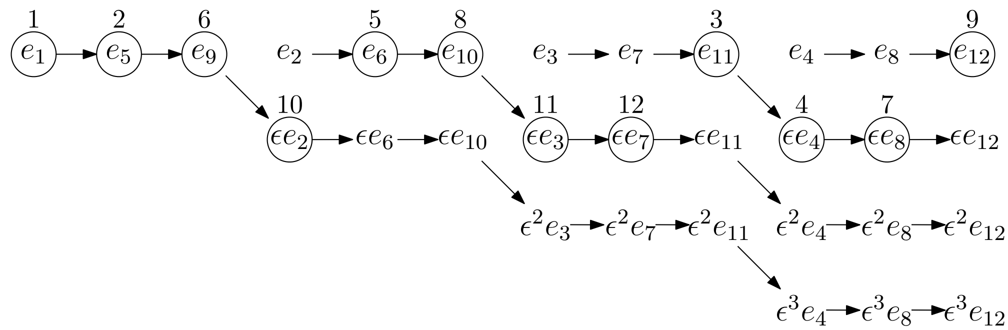

Figure 2: The staircase diagram of the fixed point ωI^- in $X_{n,k,N} = \mathrm{Sp}_\gamma \cap C$ in the case when $n = 6$, $k = 4$, $K = 12$, N is arbitrary and where $\omega = [1, 5, 11, 16, 6, 9, 20, 10, 12, 14, 15, 19]$.

Definition 3.2. We say that an affine permutation $\omega = [\omega(1), \dots, \omega(K)]$ is **positive** if $\omega(i) > 0$ for $i = 1, \dots, K$. We denote the set of positive affine permutations by $\widetilde{S}_K^+$.

Definition 3.3. We say that an affine permutation ω is **normalized**, if $1 \le \omega^{-1}(1) \le K$, that is, the values $\omega(1), \dots, \omega(K)$ in the window contain 1. We denote by $\widetilde{S}_K^0$ the set of normalized affine permutations, and by $\widetilde{S}_K^{+,0}$ the set of positive and normalized affine permutations.

Define the following unions of Schubert cells,

$$C = \bigcup_{\omega \in \widetilde{S}_K^{+,0}} C_w \subset \widetilde{\mathrm{Fl}}, \quad C' = \mathrm{pr}_{(K-n,1^n)}(C) \subset \widetilde{\mathrm{Fl}}_{(K-n,1^n)}. \tag{3.2}$$

Our varieties depend on the following operator γ on $\mathbb{C}^K[[\epsilon]]$. Its definition depends on a parameter $N \ge k$, but the varieties themselves turn out to be independent of $N >> 0$.

Definition 3.4. Let $\gamma = \gamma_{n,k,N}$ be the $\mathbb{C}[[\epsilon]]$-linear operator on $\mathbb{C}^K[[\epsilon]]$ defined by

$$\gamma e_i = e_{\gamma(i)} = \begin{cases} e_{i+k} & \text{if } 1 \le i \le (n-k)k \\ e_{i+k+1} & \text{if } (n-k)k < i < K \\ \epsilon^{N+1} e_1 & \text{if } i = K. \end{cases}$$

Definition 3.5. We define the following two varieties,

$$X_{n,k,N} := C \cap \{\Lambda_\bullet \in \widetilde{\mathrm{Fl}}_{(1^K)} \mid \gamma\Lambda_i \subseteq \Lambda_i \, \forall i\}, \tag{3.3}$$

$$Y_{n,k,N} := C' \cap \{\Lambda_\bullet \in \widetilde{\mathrm{Fl}}_{(K-n,1^n)} \mid \gamma\Lambda_i \subseteq \Lambda_i \, \forall i, \, \mathrm{JT}(\gamma|_{\Lambda_0/\Lambda_{K-n}}) \le (n-k)^{k-1}\}. \tag{3.4}$$

The variety $X_{n,k}$ gives a geometric model of the (K, k) Rectangular Shuffle Theorem, see Theorem 3.11 below. We call $Y_{n,k}$ the **affine Δ-Springer fiber**, since it gives a geometric model of the Delta conjecture, see Theorem 3.12 below.

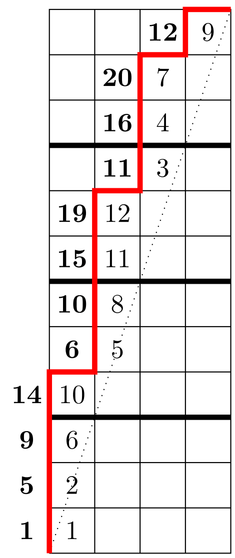

Figure 3: A labeled rational Dyck path π. The ranks are in bold to the left of the Dyck path, and the parking function is to the right of the path.

One can visualize a permutation lattice ωI^- (torus fixed point) in $X_{n,k,N}$ as follows: List the vectors $\epsilon^j e_i$ for $0 \leq j \leq (i-1) \bmod k$ with a directed arrow from $\epsilon^j e_i$ to $\gamma(\epsilon^j e_i)$. Given $\omega I^- = \epsilon^\lambda \mathbf{w} I^- \in X_{n,k}$, with $\mathbf{w} \in S_K$, circle the vector $\epsilon^{\lambda_i} e_i$ and label it by $\mathbf{w}^{-1}(i)$. See Figure 2 for an example of a staircase diagram. Alternatively, we have the following characterization of the permutation lattices ωI^- in $X_{n,k,N}$.

Definition 3.6. We say that an affine permutation ω is γ**-restricted** if:

- ω is positive and normalized, and
- $\omega^{-1}(x) < \omega^{-1}(\gamma(x))$ for all $x \in \mathbb{Z}$.

It follows from the definitions that $X_{n,k,N} \cap C_\omega \neq \emptyset$ if and only if ω is γ-restricted.

Definition 3.7. To a parking function π we associate an affine permutation ω_π such that $\omega_\pi(i)$ is the rank of the box in the same row as, and just to the left of, the parking function label i.

Example 3.8. *The parking function π in Figure 3 corresponds to the affine permutation*

$$\omega = \omega_\pi = [1, 5, 11, 16, 6, 9, 20, 10, 12, 14, 15, 19].$$

which is γ-restricted as shown above.

Lemma 3.9. *The map $\pi \mapsto \omega_\pi$ is a bijection between the set of (K, k) parking functions and the set of γ-restricted affine permutations, for any fixed $N \geq 0$.*

Theorem 3.10 ([8]). *For $N \geq k$ the space $X_{n,k} = X_{n,k,N}$ does not depend on N and admits an affine paving with cells $X_{n,k} \cap C_\omega$ for ω which are γ-restricted. Moreover, under the bijection in Lemma 3.9, the dimension of a cell can be computed in terms of inversions of π:*

$$X_{n,k} \cap C_{\omega_\pi} \cong \mathbb{C}^{(k-1)K/2 - \operatorname{dinv}(\pi)}.$$

Theorem 3.11 ([8]). *For all N, the Borel–Moore homology of $X_{n,k,N}$ has an action of S_K. For $N \geq k$ the Frobenius character of this action equals*

$$\operatorname{Frob}(H_*^{BM}(X_{n,k}); q, t) = \operatorname{rev}_q \omega(E_{K,k} \cdot 1).$$

where the q parameter keeps track of homological degree and the t grading keeps track of the connected component of $\widetilde{\mathrm{Fl}}$.

Theorem 3.12 ([8]).

(a) For all $N \geq k$, there is an action of S_n in the Borel–Moore homology of $Y_{n,k} = Y_{n,k,N}$, and we have the identity of Frobenius characters

$$q^{\binom{k-1}{2}(n-k)} \operatorname{Frob}(H_*^{BM}(Y_{n,k}); q, t) = s_{\lambda'}^{\perp} \operatorname{Frob}(H_*^{BM}(X_{n,k}); q, t)$$

where $\lambda' = (n-k)^{k-1}$.

(b) We have

$$\operatorname{Frob}\left(H_*^{BM}(Y_{n,k}); q, t\right) = \operatorname{rev}_q \omega(\Delta'_{e_{k-1}} e_n),$$

where the q parameter keeps track of homological degree and the t grading keeps track of the connected component of $\widetilde{\mathrm{Fl}}_{(K-n,1^n)}$.

The proof of part (a) relies on the results in [9], which relates Δ-Springer varieties (in the non-affine setting) to work of Borho and MacPherson [4] on partial resolutions of nilpotent varieties. The proof of part (b) then follows by combining part (a) with Theorem 1.3.

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