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Centralizers in the plactic monoid

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Abstract. Let u be a word over the positive integers. Motivated in part by a question from representation theory, we study the centralizer set of u which is

$$C(u) = \{w \mid uw \text{ is Knuth-equivalent to } wu\}.$$

In particular, we give various necessary conditions for w to be in $C(u)$. We also characterize $C(u)$ when u has few letters, when it has a single repeated entry, or when it is a certain type of decreasing sequence. We consider $c_{n,m}(u)$, the number of $w \in C(u)$ of length n with $\max w \le m$. We prove that for $|u| = 1$ the value of this function depends only on the relative sizes of u and m and not on their actual values. And for various u we use Stanley's theory of poset partitions to show that, for fixed n, $c_{n,m}(u)$ is a polynomial in m with certain degree and leading coefficient. We end with various conjectures and directions for further research.

Keywords: centralizer, jeu-de-taquin, Knuth equivalence, partition, plactic monoid, Robinson–Schensted–Knuth correspondence

1 Introduction

Let $\mathbb{P} = \{1, 2, 3, \ldots\}$ and $\mathbb{N} = \mathbb{P} \uplus \{0\}$ denote the positive and nonnegative integers, respectively. For $n \in \mathbb{N}$ we let

$$[n] = \{1, 2, \ldots, n\}.$$

In addition, for any set S we will use either $\#S$ or $|S|$ to denote the cardinality of S. We apply the same notation to words w over S and call $|w|$ the *length* of w. Finally, we let S^* be the *Kleene closure* of S, that is, all words with elements from S.

We will assume the reader is familiar with the Robinson–Schensted–Knuth (RSK) correspondence as well as Schützenberger's jeu-de-taquin (jdt). Background on these operations can be found in the texts of Sagan [5, 6] or Stanley [11]. In particular, if $w \in \mathbb{P}^*$ then we will let $P(w)$ denote the insertion tableau of w under RSK. Recall that $v, w \in \mathbb{P}^*$ differ by a *Knuth transposition* if there are words x, y and elements a, b, c such that either

$$v = xacby \text{ and } w = xcaby \text{ with } a \le b < c,$$

or

$$v = xbacy \text{ and } w = xbcay \text{ with } a < b \le c.$$

Furthermore, we say that v, w are *Knuth-equivalent*, written $v \equiv w$, if one can obtain w from v by applying a sequence of Knuth transpositions. When Knuth introduced this equivalence relation [3], he proved that

$$v \equiv w \text{ if and only if } P(v) = P(w).$$

The *plactic monoid* is $\mathbb{P}^*$ modulo Knuth equivalence. It was first considered from this perspective by Lascoux and Schützenberger [4].

Given a word $u \in \mathbb{P}^*$, our primary object of study will be the *centralizer of* u in the plactic monoid which is

$$C(u) = \{w \mid uw \equiv wu\},$$

or equivalently

$$C(u) = \{w \mid P(uw) = P(wu)\}.$$

In particular, we wish to characterize $C(u)$ for certain u and also consider the enumerative properties of the integers

$$c_{n,m}(u) = \#\{w \in C(u) \mid \#w = n \text{ and } \max w \leq m\}. \tag{1.1}$$

Beside the fact that $C(u)$ is a natural set to study, our research is motivated by work in preparation by the second author and Nate Harman concerning commuting crystal structures on "lexicographic bitableaux," semistandard tableaux filled with entries in $[m] \times [n]$ ordered lexicographically. In this setting, it is natural for one crystal operation to transform a reading word of the tableau by cutting out a subword and pasting it in a different location. In order for this crystal operation to commute with a classical crystal operation, the transformed word must be Knuth-equivalent to the original reading word.

The rest of this paper is organized as follows. In the next section we will collect some necessary conditions for w to be in $C(u)$ which will prove useful in the sequel. In Section 3 we will characterize the $w \in C(u)$ for certain u with $\#u \leq 3$. In particular, we will describe $C(u)$ for any u of length 1. Next, in Section 4 we will describe $C(u)$ for certain special u of arbitrary length such as those which consist of a single repeated integer or are of the form $m(m-1)\ldots 1$ for some $m \in \mathbb{P}$. Section 5 is devoted to the study of the $c_{n,m}(u)$ as defined in (1.1). In particular, if $|u| = 1$ we show that their values depend only on the relative sizes of m and u. Furthermore, we use Stanley's theory of poset partitions to prove that for certain u and fixed n, they are polynomials in m. We end with a section containing open problems and conjectures.

2 Necessary Conditions

In this section, we collect results giving general constraints on the tableaux $P = P(w)$ for $w \in C(u)$. In particular, we will give a criterion which will permit us to bound the size

$$P = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 1 & 1 & 3 & 4 & 4 \\ \hline 2 & 3 & 3 & 4 & 5 \\ \cline{1-5} 3 & 5 & 5 \\ \cline{1-3} 4 \\ \cline{1-1}\end{array}$$

Figure 1: A semistandard Young tableau (SSYT), P

of the elements in the first few rows of P by the maximum value in u. Our principal tool here and going forward will be to compare the computation of $P(wu)$ using RSK with the computation of $P(uw)$ using jdt. In the former, the elements of u are inserted into $P(w)$ using the usual RSK bumping procedure. In the latter, a skew tableau is formed with $P(u)$ in the southwest and $P(w)$ in the northeast. The tableau is then brought to left-justified shape using jdt slides.

Given any sequence R and any element a we let

$$m_a(R) = \text{ the multiplicity of } a \text{ in } R.$$

Also, for a semistandard Young tableau (SSYT) P with rows $R_1, R_2, \ldots$, we consider the weak composition

$$\alpha_a(P) = (m_a(R_1),\ m_a(R_2),\ \ldots).$$

For example, if P is the tableau in Figure 1, then

$$\alpha_4(P) = (2, 1, 0, 1, 0, 0, \ldots).$$

We will compare weak compositions $\alpha = (\alpha_1, \alpha_2, \ldots)$ and $\beta = (\beta_1, \beta_2, \ldots)$ using *dominance order* where $\alpha \preceq \beta$ if for all $i \geq 1$,

$$\alpha_1 + \alpha_2 + \cdots + \alpha_i \leq \beta_1 + \beta_2 + \cdots + \beta_i.$$

The fact that entries of a tableau P are bumped to lower rows under RSK and slid up to higher rows under jdt slides leads to the following lemma.

Lemma 2.1. *Let $a \neq b$ be distinct positive integers and let $w \in \mathbb{P}^*$. Then*

$$\alpha_b(P(wa)) \preceq \alpha_b(P(w)) \preceq \alpha_b(P(aw)).$$

From this we obtain our first necessary condition for when $w \in C(u)$.

Corollary 2.2. *If $w \in C(u)$ and $b \notin u$ then*

$$\alpha_b(P(wu)) = \alpha_b(P(w)) = \alpha_b(P(uw)).$$

In particular, no $b \notin u$ can be bumped by the insertion of u into $P(w)$ to form $P(wu)$. And such an element b cannot slide between two rows in the computation of $P(uw)$ by jdt.

We can now bound the size of certain elements in $P(w)$ for $w \in C(u)$ in terms of the maximum value in u.

Lemma 2.3. *Given u and $w \in C(u)$ we let $P = P(w)$ have rows R_i for $i \geq 1$. Also let $m = \max u$. If u contains a subsequence $m, m-1, \ldots, m-k+1$, then for $1 \leq i \leq k$,*

$$\max R_i \leq m.$$

Proof sketch. First we show that if a semistandard tableau T contains an a in a higher row than an $a+1$, then this will continue to be the case after any insertion into T. Using the assumed subsequence of u we then show that in forming $P(wu)$ from P we must have the elements $m, m-1, \ldots, m-k+1$ from u in separate rows with m in the lowest row. This means m must have traveled through at least the first k rows to its present position. But if one of these rows contains an element from P larger then m, then m would bump the smallest such element to the next row. This contradicts Corollary 2.2 which completes the demonstration. □

3 Commuting with short u

Given a row R of a tableau and a condition I on integers we let

$$R(I) = \text{ multiset of elements of } R \text{ satisfying } I.$$

For example, if u is an integer then $R(\leq u)$ would be all elements of R which are at most u. More specifically, if R is the second row of the the SSYT, P, in Figure 1 then

$$R(\leq 3) = \{\{2, 3, 3\}\}.$$

We say that cell (i, j) in row i and column j is *adjacent* to the cells $(i, j+1)$ and $(i+1, j)$ (that is, those which could be next in a jeu-de-taquin path) and similarly with the elements of a tableau in those cells.

Theorem 3.1. *Suppose u consists of a single integer which we also denote by u. Also, use $R_1, R_2, \ldots, R_l$ to denote the rows of $P = P(w)$. Then the set $C(u)$ is all w such that $P = P(w)$ satisfies*

(a) $\max R_1 \leq u$, *and*

(b) for $i \geq 1$ *we have* $\#R_i(< u) = \#R_{i+1}(\leq u)$.

Proof sketch. We first prove that if P satisfies the given restrictions then $w \in C(u)$. That is, we need to prove $P(wu) = P(uw)$. But by (a), $P(wu)$ is obtained from P by appending u to the first row and so has rows $R_1 u, R_2, \ldots, R_l$.

To compute $P(uw)$, we perform jeu-de-taquin on the skew tableau with u in the $(l+1,1)$ cell and P in the first l rows starting in column 2. By reverse induction and the fact that the elements of each row below R_u must be larger than u, we show that this u in the $(l+1,1)$ cell slides up to join row R_u with each row that it passes shifting left one cell. Using another reverse induction argument and condition (b), we show that the remaining jeu-de-taquin moves end with a u joining R_1 and leaving all lower rows unchanged. This coincides with the description of $P(wu)$ in the first paragraph, so we are done with this direction.

For the converse, we assume $w \in C(u)$ so that $P(uw) = P(wu)$. Applying Lemma 2.3 with $m = u$ and $k = 1$ immediately gives condition (a). Using ideas similar to those in the first half of the proof, we obtain condition (b). □

We now give a result that tests for w being in $C(u)$ when $|u| = 1$ by looking at the columns of $P(w)$ rather than the rows.

Theorem 3.2. *If $|u| = 1$ then*

$$C(u) = \{w \mid \textit{every column of } P = P(w) \textit{ contains a } u\}.$$

It will be useful to have a characterization of the $w \in C(1)$ which depends directly on w without having to compute $P(w)$. This will also permit us to make a connection with Yamanouchi words. To state these results, we will need some notation and definitions. Let

$$\operatorname{lwi}(w) = \text{ longest length of a weakly increasing subsequence of } w,$$

and for $a \in \mathbb{P}$

$$\operatorname{lwi}(w, a) = \text{ longest length of a weakly increasing subsequence of } w \text{ of the form } va.$$

For example, if $w = 162724534$ then $\operatorname{lwi}(w) = 5$ because of, for example, the subsequence 12244 among others. Also $\operatorname{lwi}(w, 3) = 4$ as witnessed by 1223. Note that to compute $\operatorname{lwi}(w, a)$ one needs only to know the length of a longest weakly increasing subsequence ending at the rightmost a in w: a weakly increasing sequence ending at an a further to the left can have its length increased by concatenating with the last a. Finally, recall that a word w is *Yamanouchi* if every suffix of w has at least as many i's as $(i+1)$'s for all $i \geq 1$.

For the next result, we keep the notation in the statement of Theorem 3.1.

Corollary 3.3. *The following are equivalent.*

(a) $w \in C(1)$.

(b) The entries of R_1 are all 1*'s.*

(c) $\text{lwi}(w) = \text{lwi}(w, 1)$

Furthermore, the following are equivalent.

(d) $w \in C(1) \cap [2]^n$.

(e) $w \in [2]^n$ *is Yamanouchi.*

For words u of length two or three, we will concentrate on the case when u consists of 1's and 2's. If $\#u = 2$ then $u = 11$, 21, and 22 will be taken care of by more general results in the next section. So we will content ourselves with a column characterization for $C(12)$. To state it we define a *singleton a-column* to be a column of length 1 whose entry is a.

Theorem 3.4. *We have that $w \in C(12)$ if and only if all columns C of $P(w)$ satisfy the following two conditions:*

(a) *If there is a singleton column, C, then C is a singleton 1-column or a singleton 2-column, and both types of columns must exist.*

(b) *If $\#C \geq 2$ then C must contain both 1 and 2.*

We end this section by looking at just one u of length 3.

Theorem 3.5. *We have that $w \in C(212)$ if and only if all columns C of $P(w)$ satisfy the following two conditions.*

(a) *All singleton columns are singleton 2-columns.*

(b) *If $\#C \geq 2$ then C must contain both 1 and 2.*

4 Commuting with longer u

We will now derive characterizations of $C(u)$ for certain words u of arbitrary length. We begin by considering the case where u is just the repetition of a single element. We first have a lemma. For any word u and $k \geq 1$, we let u^k be the concatenation of k copies of u.

Lemma 4.1. *For any $u \in \mathbb{P}^*$ and any $k \in \mathbb{P}$ we have*

$$C(u) \subseteq C(u^k).$$

We can show that, interestingly, when u consists of a single element a, the centralizer $C(a^k)$ does not depend on k and so can be characterized by the conditions in either Theorem 3.1 or Theorem 3.2.

Theorem 4.2. *If $a, k \in \mathbb{P}$ then*

$$C(a^k) = C(a).$$

There is another class of words which have a particularly nice characterization of their centralizers.

Theorem 4.3. *We have $w \in C(m(m-1)\dots 1)$ if and only if $P = P(w)$ satisfies*

$$\max R_i \leq m \text{ for all } 1 \leq i \leq m$$

where R_i is the ith row of P.

5 Enumeration

We can now use the characterizations derived previously to study the integers $c_{n,m}(u)$ defined by (1.1) which count the number of $w \in C(u)$ of length n with maximum at most m. We will use Stanley's theory of $\mathfrak{P}$-partitions where $\mathfrak{P}$ is a poset [8], to show that for certain u and fixed n, these numbers are polynomials in m. For more information about this method see [6, Section 7.4] or [10, Section 3.15].

The next result shows, surprisingly, that when $|u| = 1$ the value of $c_{n,m}(u)$ depends only on the relative sizes of u and m and not on their specific values.

Theorem 5.1. *If $|u| = 1$ then*

$$c_{n,m}(u) = \begin{cases} c_{n,m}(1) & \text{if } u \leq m, \\ \delta_{n,0} & \text{if } u > m. \end{cases}$$

where $\delta_{n,m}$ is the Kronecker delta function.

Combining the previous result with Theorem 4.2, we immediately get the following.

Corollary 5.2. *If $a, k \in \mathbb{P}$ then*

$$c_{n,m}(a^k) = \begin{cases} c_{n,m}(1) & \text{if } a \leq m, \\ \delta_{n,0} & \text{if } a > m. \end{cases} \qquad \square$$

We will now show that for various u and fixed n, the quantity $c_{n,m}(u)$ is a polynomial in m and investigate its properties. Let $(\mathfrak{P}, \trianglelefteq)$ be a poset on $[n]$. Note the use of $\trianglelefteq$ to differentiate the partial order in $\mathfrak{P}$ from the total order $\leq$ on integers. A $\mathfrak{P}$-*partition* is a function $f : \mathfrak{P} \to \mathbb{N}$ satisfying

1. $i \trianglelefteq j$ implies $f(i) \geq f(j)$, and
2. $i \trianglelefteq j$ and $i > j$ implies $f(i) > f(j)$

We let

$$\operatorname{Par}_m \mathfrak{P} = \{f : \mathfrak{P} \to [m] \mid f \text{ is a } \mathfrak{P}\text{-partition}\}.$$

Now suppose $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition of n, written $\lambda \vdash n$. Partially order the cells of λ reverse component-wise so that $(i, j) \trianglelefteq (i', j')$ whenever $i \geq i'$ and $j \geq j'$. Finally, number the cells of λ with $[n]$ by numbering the first row of the Young diagram from right-to-left with $1, 2, \ldots, \lambda_1$, then the next row right-to-left with $\lambda_1 + 1, \lambda_1 + 2, \ldots, \lambda_1 + \lambda_2$, and so forth. Transferring this labeling to the poset constructed from λ we obtain a poset $\mathfrak{P}_\lambda$. It should be clear from the definitions that there is a bijection between the semistandard Young tableaux P of shape λ with maximum at most m and $\operatorname{Par}_{m-1} \mathfrak{P}_\lambda$ obtained by subtracting one from every element of P.

We now describe the generating function $\sum_{m\geq 0} |\operatorname{Par}_m \mathfrak{P}|\ x^m$. If $\mathfrak{P}$ is a poset on $[n]$ then a *linear extension* of $\mathfrak{P}$ is a permutation π in the symmetric group $\mathfrak{S}_n$ such that $i \trianglelefteq j$ in $\mathfrak{P}$ implies i is to the left of j in π. We let

$$\mathcal{L}(\mathfrak{P}) = \{\pi \mid \ \pi \text{ is a linear extension of } \mathfrak{P}\}.$$

Any $\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathfrak{S}_n$ has *descent number*

$$\operatorname{des} \pi = \#\{i \mid \pi_i > \pi_{i+1}\}.$$

We now have all the ingredients to state the $\mathfrak{P}$-partition result we will need.

Theorem 5.3 ([8])**.** *For any poset $\mathfrak{P}$ on $[n]$ we have*

$$\sum_{m\geq 0} |\operatorname{Par}_m \mathfrak{P}|\ x^m = \frac{\sum_{\pi \in \mathcal{L}(\mathfrak{P})} x^{\operatorname{des} \pi}}{(1-x)^{n+1}}.$$

The theorem just stated can be used to prove the following general result about centralizer sets.

Theorem 5.4. *Let u be a word and r be a positive integer. Suppose that if $P = P(w)$ for $w \in C(u)$ then*

(a) the first r rows of P only contain elements which are at most r,

(b) the remaining rows of P can be any SSYT with elements greater than r.

Then for fixed $n \geq r$ and all $m \geq n$ we have that $c_{n,m}(u)$ is a polynomial in m of degree $n - r$ with leading coefficient $1/(n-r)!$.

Combining the previous theorem with the characterizations from Sections 3 and 4, we get the following specific cases.

Corollary 5.5. *The following are polynomials in m for n fixed and $m \geq n$.*

(a) *If* $n \geq 1$ *then* $c_{n,m}(1)$ *is a polynomial in* m *of degree* $n-1$ *with leading coefficient* $1/(n-1)!$.

(b) *If* $n \geq 2$ *then* $c_{n,m}(12)$ *is a polynomial in* m *of degree* $n-2$ *with leading coefficient* $1/(n-2)!$.

(c) *If* $n \geq k$ *then* $c_{n,m}(k(k-1)\dots 1)$ *is a polynomial in* m *of degree* $n-k$ *with leading coefficient* $1/(n-k)!$.

One can also use the method of proof in Theorem 5.4 to actually compute the polynomials involved. We illustrate this with our next result.

Theorem 5.6. *Suppose* n *is fixed and* $m \geq n$. *Then we have the following polynomial expansions.*

$$\begin{aligned}
c_{1,m}(1) &= 1,\\
c_{2,m}(1) &= \binom{m}{1},\\
c_{3,m}(1) &= \binom{m}{1} + \binom{m}{2},\\
c_{4,m}(1) &= \binom{m}{1} + 4\binom{m}{2} + \binom{m}{3},\\
c_{5,m}(1) &= \binom{m}{1} + 8\binom{m}{2} + 13\binom{m}{3} + \binom{m}{4},\\
c_{6,m}(1) &= \binom{m}{1} + 18\binom{m}{2} + 48\binom{m}{3} + 41\binom{m}{4} + \binom{m}{5},\\
c_{7,m}(1) &= \binom{m}{1} + 33\binom{m}{2} + 178\binom{m}{3} + 262\binom{m}{4} + 131\binom{m}{5} + \binom{m}{6},\\
c_{8,m}(1) &= \binom{m}{1} + 68\binom{m}{2} + 549\binom{m}{3} + 1480\binom{m}{4} + 1405\binom{m}{5} + 428\binom{m}{6} + \binom{m}{7}.
\end{aligned}$$

6 Open problems and conjectures

Although we have begun the study of the centralizer $C(u)$, we believe that there are many more interesting results to be found. Here we collect a few avenues for future research.

Lemma 2.3 is useful in the proofs of the results in Sections 3 and 4. We believe that an even stronger result is true.

Conjecture 6.1. *Given* u, *let* $m = \max u$ *and* ℓ *be the number of rows of* $P(u)$. *Suppose that* $w \in C(u)$ *and that* $P(w)$ *has rows* R_i *for* $i \geq 1$. *Then for* $1 \leq i \leq \ell$,

$$\max R_i \leq m.$$

To see why this conjecture implies Lemma 2.3, note that the existence of a subsequence of the form $m, m-1, \ldots, m-k+1$ in u implies that $\ell \geq k$ by an argument like that in the proof of the lemma. So if $\max R_i \leq m$ for $i \in [\ell]$ then certainly the inequality is true for $i \in [k]$. We have verified Conjecture 6.1 computationally[1] for $u \in [m]^n$ and $w \in [6]^l$ where $m + n \leq 10$ and $2 \leq l \leq 6$.

In Lemma 4.1 we noted that for any u and $k \geq 1$ we always have $C(u) \subseteq C(u^k)$. But in the particular case when $|u| = 1$ we have $C(u) = C(u^k)$ for all $k \geq 1$ by Theorem 4.2. We conjecture that such stability holds more generally.

Conjecture 6.2. *Suppose $u \in \mathbb{P}^*$.*

(a) There is a $K \in \mathbb{P}$ such that for $k \geq K$ we have

$$C(u^k) \subseteq C(u^{k+1}).$$

(b) There is an $L \in \mathbb{P}$ such that for $k \geq L$ we have

$$C(u^k) = C(u^{k+1}).$$

We have verified Conjecture 6.2(a) computationally for $u \in [m]^n$ and $w \in [5]^l$ where $m + n \leq 10$, $2 \leq l \leq 6$, and $1 \leq k \leq 10$. Note that except in the particular case that $u = 12345$ where $K = 3$, for all other words u checked, we can take $K = 1$. In support of Conjecture 6.2(b), the containments verified under these conditions become equalities for $k \geq 4$.

The expansions of $c_{n,m}(1)$ for $n \leq 8$ in Theorem 5.6 have some remarkable properties which we conjecture hold in general. Call a sequence $a_0, a_1, \ldots, a_n$ of real numbers *unimodal* if there is some index k such that

$$a_0 \leq a_1 \leq \ldots \leq a_k \geq a_{k+1} \geq \ldots a_n.$$

The sequence is said to be *log-concave* if, for all $0 < i < n$,

$$a_i^2 \geq a_{i-1}a_{i+1}.$$

Unimodal and log-concave sequences abound in combinatorics, algebra, and geometry. See the survey articles of Stanley [9], Brenti [2], or Brändén [1] for more information. It is well known that, for positive sequences, log-concavity implies unimodality.

Conjecture 6.3. *Fix n and write*

$$c_{n,m}(1) = \sum_{k=0}^{n-1} a_k \binom{m}{k}$$

for certain scalars a_k (depending on n). We have the following

[1]The code used to verify the conjectures in this section can be found at `https://github.com/wilsoa/Centralizers-in-the-Plactic-Monoid`.

(a) $a_0 = 0$, $a_1 = 1$.

(b) $a_k \in \mathbb{P}$ *for all* $k \in [n-1]$.

(c) The sequence $a_1, a_2, \ldots, a_{n-1}$ *is log-concave and hence (assuming (b)) unimodal.*

(d) The sequence $a_1, a_2, \ldots, a_{n-1}$ *has maximum at* $k = \lceil n/2 \rceil$.

It is well known that applying symmetries of the square to a permutation π (viewed as a permutation matrix) does interesting things to the output tableaux under RSK. One of these also seems to play nicely with the centralizer set. If $w = w_1 w_2 \ldots w_k$ has $\max w \leq m$ then define its *m-reverse complement* to be the word

$$\mathrm{RC}_m(w) = (m - w_k + 1)(m - w_{k-1} + 1) \ldots (m - w_1 + 1).$$

Note the dependence on the choice of m, not just on w. For example

$$\mathrm{RC}_4(31122) = 33442.$$

To extend this operation to an SSYT, T, let $\mathrm{rw}(T)$ be the row word of T obtained by reading the rows of T from left to right starting with the bottom row and moving up. It is well known that

$$P(\mathrm{rw}\, T) = T.$$

Now if $\max T \leq m$ we define the *m-evacuation* of T to be the composition

$$\epsilon_m(T) = P \circ \mathrm{RC}_m \circ \mathrm{rw}(T).$$

When T is a standard Young tableau with maximum entry m, the map ϵ_m is Schützerberger's evacuation operation, see [7] or [11, A1.2.10].

Lemma 6.4. *If T is a SSYT with* $\max T \leq m$*, then T and $\epsilon_m(T)$ have the same shape.*

We now wish to describe a conjectural bijection between the elements of $C(u)$ and those of $C(\mathrm{RC}_m(u))$ for any $m \geq \max u$. Given an array A we let $A_{\leq m}$ be the subarray consisting of the elements of A which are at most m and similarly for $A_{>m}$. If T is an SSYT then let $\tau_m(T)$ be the result of replacing $T_{\leq m}$ with $\epsilon_m(T_{\leq m})$ and leaving $T_{>m}$ unchanged. Note that the previous lemma makes this replacement well defined since the two tableaux involved have the same shape. We similarly extend the map RC_m to all words w by letting $\mathrm{RC}_m(w)$ be the word obtained by replacing $w_{\leq m}$ with its m-reverse complement and leaving the elements of $w_{>m}$ unchanged.

We note that for any u, the centralizer $C(u)$ is a union of Knuth equivalence classes. Indeed, for $w \in C(u)$ and $w' \equiv w$, we have $uw' \equiv uw \equiv wu \equiv w'u$ so that $w' \in C(u)$. It follows that to describe $C(u)$ it suffices to describe the set $P(C(u))$ of all insertion tableaux $P(w)$ for $w \in C(u)$. Our conjecture is as follows.

Conjecture 6.5. *If u is a word with $\max u \leq m$ then*

$$P(C(\mathrm{RC}_m(u))) = \tau_m(P(C(u))).$$

Note that, since both RC_m and τ_m are involutions, it suffices to prove only one of the two set containments implied by the conjectured equality. We have verified this computationally for $u \in [m]^n$ and $w \in [6]^l$ where $m + n \leq 11$ and $2 \leq l \leq 5$.

References

[1] P. Brändén. "Unimodality, log-concavity, real-rootedness and beyond". *Handbook of enumerative combinatorics*. Discrete Math. Appl. (Boca Raton). CRC Press, Boca Raton, FL, 2015, pp. 437–483.

[2] F. Brenti. "Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update". *Jerusalem combinatorics '93*. Vol. 178. Contemp. Math. Amer. Math. Soc., Providence, RI, 1994, pp. 71–89. DOI.

[3] D. E. Knuth. "Permutations, matrices, and generalized Young tableaux". *Pacific J. Math.* **34** (1970), pp. 709–727. Link.

[4] A. Lascoux and M.-P. Schützenberger. "Le monoïde plaxique". *Noncommutative structures in algebra and geometric combinatorics (Naples, 1978)*. Vol. 109. Quad. "Ricerca Sci." CNR, Rome, 1981, pp. 129–156.

[5] B. E. Sagan. *The symmetric group*. Second. Vol. 203. Graduate Texts in Mathematics. Representations, combinatorial algorithms, and symmetric functions. Springer-Verlag, New York, 2001, pp. xvi+238. DOI.

[6] B. E. Sagan. *Combinatorics: the art of counting*. Vol. 210. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, [2020] ©2020, pp. xix+304. DOI.

[7] M. P. Schützenberger. "Quelques remarques sur une construction de Schensted". *Math. Scand.* **12** (1963), pp. 117–128. DOI.

[8] R. P. Stanley. *Ordered structures and partitions*. Thesis (Ph.D.)–Harvard University. ProQuest LLC, Ann Arbor, MI, 1971, (no paging). Link.

[9] R. P. Stanley. "Log-concave and unimodal sequences in algebra, combinatorics, and geometry". *Graph theory and its applications: East and West (Jinan, 1986)*. Vol. 576. Ann. New York Acad. Sci. New York Acad. Sci., New York, 1989, pp. 500–535. DOI.

[10] R. P. Stanley. *Enumerative combinatorics. Volume 1*. Second. Vol. 49. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2012, pp. xiv+626.

[11] R. P. Stanley. *Enumerative combinatorics. Vol. 2*. Second. Vol. 208. Cambridge Studies in Advanced Mathematics. With an appendix by Sergey Fomin. Cambridge University Press, Cambridge, [2024] ©2024, pp. xvi+783.