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Peak algebra in noncommuting variables

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Abstract. The well-known descent-to-peak map θ for the Hopf algebra of quasisymmetric functions, QSym, and the peak algebra Π were originally defined by Stembridge in 1997. We define the labelled descent-to-peak map Θ and extend the notion of the peak algebra to noncommuting variables.

Keywords: quasisymmetric functions, peak algebra, descent-to-peak map

1 Introduction

The first comprehensive study on the combinatorics of peaks was conducted by Stembridge [21], who developed and introduced enriched (P,γ)-partitions, which is an analog to Stanley's theory of (P,γ)-partitions, with the key distinction that the notion of peaks replaces the notion of descents in the context of linear extensions of posets. Generating functions of (P,γ)-partitions, $\Gamma(P,\gamma)$, give the Hopf algebra of quasisymmetric functions QSym, and the generating functions of enriched (P,γ)-Partitions, $\Delta(P,\gamma)$, give the peak algebra Π. Stembridge also defined the descent-to-peak algebra morphism θ from QSym to Π where $\Gamma(P,\gamma)$ maps to $\Delta(P,\gamma)$, and showed that the dimension of the homogenous functions of degree n of Π, Π_n, is equal to the number of odd compositions, compositions whose all parts are odd, which is equal to Fibonacci number f_n. Moreover, he showed that restricting the map θ to symmetric functions gives the Hopf algebra of Schur's Q functions. The Hopf algebra of Schur's Q functions, whose bases are indexed by odd partitions, are introduced in [18] to study the projective representations of symmetric and alternating groups. Combinatorially, the Schur's Q functions are equipped with a theory of shifted tableaux, including RSK correspondence, Littlewood–Richardson rule, and jeu de taquin [17, 20, 22]. In [7], Bergeron et al. showed that the peak algebra is a Hopf algebra and also the map θ is a Hopf algebra morphism. Also, the main result of [18] by Schocker is that the peak algebra is a left co-ideal of QSym under internal comultiplication, a generalized dual Kronecker product.

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It is also shown that the peak algebra corresponds to the representations of the 0-Hecke–Clifford algebra [4]. Further studies revealed connections between peaks and a variety of seemingly unrelated topics, such as the generalized Dehn–Sommerville equations [2, 6, 8] and the Schubert calculus of isotropic flag manifolds [7, 9]. Notably, in [14, 15], the peak algebra is generalized to the Poirier–Reutenauer Hopf algebra of standard Young tableaux, which is introduced in [16]. Other generalizations can be found in [1, 5, 11].

The (P, γ)-partitions are the generalized chromatic functions of certain digraphs [3], and here we introduce enriched generalized chromatic functions. The enriched (P, γ)-partitions are enriched generalized chromatic functions of certain digraphs. Now, extending generalized chromatic functions and enriched generalized chromatic functions to noncommuting variables, we define the labelled peak-to-decent map Θ from the Hopf algebra of quasisymmetric functions in noncommuting variables NCQSym to the peak algebra in noncommuting variables NCΠ. This map is indeed a Hopf algebra morphism. Applying the map ρ, where it commutes the variables, we obtain theta map θ. We compute the values of the map Θ at the fundamental and monomial basis of NCQSym. Then, we define the peak algebra in noncommuting variables NCΠ and introduce Schur's Q functions in noncommuting variables. The dimension of the homogenous functions of degree n of NCΠ, NCΠ_n, is equal to the odd set compositions, set compositions whose all parts are of odd size, which is equal to a_n where a_n is the sequence A006154 in the OEIS. Moreover, we show that the restriction of the map Θ to symmetric functions in noncommuting variables is the Hopf algebra of Schur's Q functions in noncommuting variables NCSym $\cap$ NCΠ. They are indexed by odd set partitions. We present that the peak algebra in noncommuting variables is a left co-ideal of NCQSym under internal co-multiplication, extending Schocker's result in [18] that the peak algebra is a left co-ideal of QSym under internal comultiplication, a generalized dual Kronecker product.

2 Generalized chromatic functions, descent-to-peak map, and peak algebra

In this section, we present a summary of the earlier results in the language of generalized chromatic functions [3].

2.1 Edge-coloured digraphs and some operators

Stanley [19] defined P-partitions by generalizing MacMahon's work on plane partitions [13][1]. P-partitions can be identified as certain vertex-colourings of some family of edge-

[1]For a complete history of P-partitions see I. M. Gessel's paper [10].

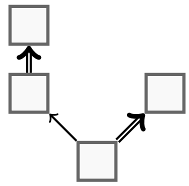
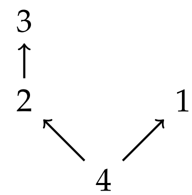


Figure 1: An edge-coloured digraph

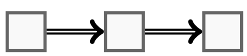
coloured digraphs. We describe this family of edge-coloured digraphs and some useful operators between them.

The Hasse diagram of a poset $P = (X, \leq)$ can be seen as a digraph whose vertices are the elements of the ground set X of the poset, and there is a directed edge from a to b if $a \leq b$ and if there is $c \in X$ such that $a \leq c \leq b$, then either $c = a$ or $c = b$. ***Throughout this paper, all digraphs are Hasse diagrams of some posets.***

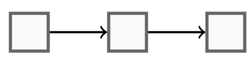


An *edge-coloured digraph* is a digraph whose edges are of the form $\rightarrow$ or $\Rightarrow$.

We now define some useful edge-coloured digraphs. Let Q_n (resp. P_n) be the edge-coloured directed path with n vertices whose all edges are of the form $\Rightarrow$ (resp. $\rightarrow$).



Q_3



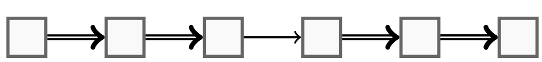
P_3

The *disjoint union* of edge-coloured digraphs G_1 and G_2 with $V(G_1) \cap V(G_2) = \emptyset$, denoted $G_1 \sqcup G_2$, is an edge-coloured digraph such that

1. The vertex set of $G_1 \sqcup G_2$ is the disjoint union of the vertex sets of G_1 and G_2.
2. The edge set of $G_1 \sqcup G_2$ is the disjoint union of the edge sets of G_1 and G_2.
3. $a \Rightarrow b$ in $G_1 \sqcup G_2$ if either $a \Rightarrow b$ in G_1 or in G_2.
4. $a \rightarrow b$ in $G_1 \sqcup G_2$ if either $a \rightarrow b$ in G_1 or in G_2.

$Q_3 \sqcup P_3$

The *solid sum* of edge-coloured directed paths G_1 and G_2, denoted by $G_1 \rightarrow G_2$, is an edge-coloured digraph obtained by connecting the last vertex of G_1 to the first vertex of G_2 by a solid edge $\rightarrow$.



$Q_3 \rightarrow Q_3$

2.2 Proper colourings and generalized chromatic functions

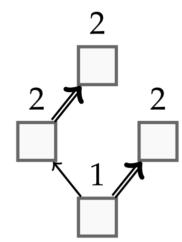
As we mentioned earlier, P-partitions can be identified as certain types of vertex--colourings of edge-coloured digraphs. We describe these types of vertex-colourings of edge-coloured digraphs, and then we construct their generating functions, which are called generalized chromatic functions.

A *proper* colouring of an edge-coloured digraph G is a function

$$\kappa : V(G) \to \mathbb{N} = \{1, 2, 3, \ldots\}$$

such that

1. If $a \Rightarrow b$, then $\kappa(a) \leq \kappa(b)$.
2. If $a \to b$, then $\kappa(a) < \kappa(b)$.

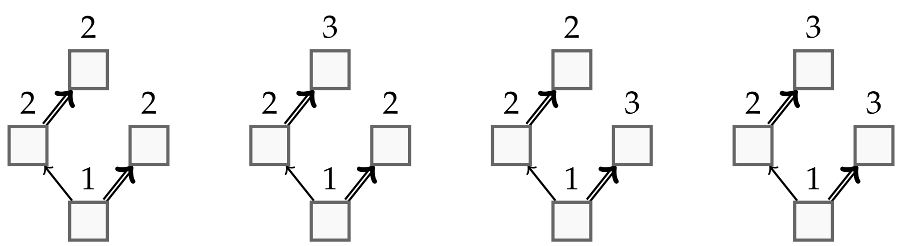


Recall that $\mathbb{Q}[[x_1, x_2, \ldots]]$ is the algebra of formal power series in infinitely many commuting variables $x = \{x_1, x_2, \ldots\}$ over $\mathbb{Q}$.

The *generalized chromatic function* of an edge-coloured digraph G is

$$\mathscr{X}_G = \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)}$$

where the sum is over all proper colourings κ of G. For example, if G is the edge-coloured digraph in Figure 1, then



$$\mathscr{X}_G = \quad x_1x_2^3 \quad + \quad x_1x_2^2x_3 \quad + \quad x_1x_2^2x_3 \quad + \quad x_1x_2x_3^2 \quad + \cdots .$$

Generalized chromatic functions can be seen as generating functions of P-partitions, and so they are quasisymmetric functions.

2.3 Enriched colourings and enriched chromatic functions

Stembridge [21] defined enriched P-partitions and used them to associate tableaux with Schur's Q-functions [18][2]. Enriched P-partitions can be identified as certain types of vertex-colourings of edge-coloured digraphs. We describe these types of vertex-

[2]For an English reference, see I. G. Macdonald's book [12, Chapter III, Section 8], where he described Schur's Q-functions in more detail.

colourings of edge-coloured digraphs, and then we construct their generating functions, which are called enriched chromatic functions.

Given an edge-coloured digraph G, an *enriched* colouring of G is a function

$$\kappa : V(G) \to \{\cdots \prec -1 \prec 1 \prec -2 \prec 2 \prec \cdots\}$$

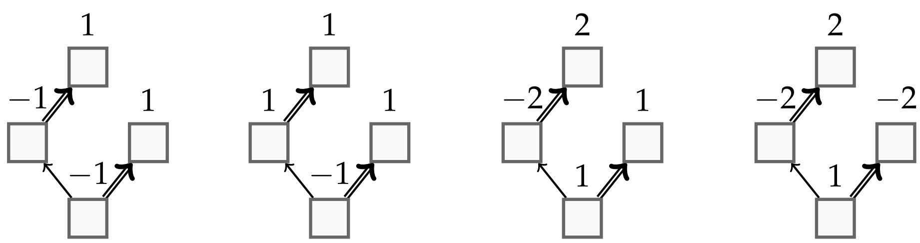
such that

1. If $a \Rightarrow b$, then either $\kappa(a) \prec \kappa(b)$ or $\kappa(a) = \kappa(b) > 0$.
2. If $a \to b$, then either $\kappa(a) \prec \kappa(b)$ or $\kappa(a) = \kappa(b) < 0$.

The *enriched chromatic function* of an edge-coloured digraph G is

$$\mathscr{E}_G = \sum_{\kappa} \prod_{v \in V(G)} x_{|\kappa(v)|}$$

where the sum is over all enriched colourings κ of G. For example, if G is the edge-coloured digraph in Figure 1, then



$$\mathscr{E}_G = x_1^4 + x_1^4 + x_1^2 x_2^2 + x_1 x_2^3 + \cdots .$$

2.4 Peak algebra and descent-to-peak map

Stembridge defined the peak algebra Π in [21] as a space spanned by the generating functions of enriched P-partitions. Since generating functions of enriched P-partitions are the enriched chromatic functions of edge-coloured digraphs and vice versa, we have that the peak algebra is spanned by the set

$$\{\mathscr{E}_G : G \text{ is an edge-coloured digraph}\}.$$

Stembridge also defined the *descent-to-peak map* Θ_{QSym}; we can write it as follows,

$$\begin{array}{rccc} \Theta_{\mathrm{QSym}} : & \mathrm{QSym} & \to & \Pi \\ & \mathscr{X}_G & \mapsto & \mathscr{E}_G . \end{array} \tag{2.1}$$

He showed the descent-to-peak map is a surjective algebra morphism [21, Theorem 3.1].

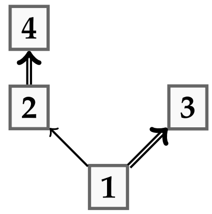


Figure 2: A labelled edge-coloured digraph

3 Generalized chromatic functions and Peak algebra in noncommuting variables, and labelled descent-to-peak map

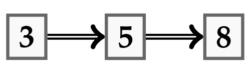
3.1 Generalized chromatic functions in noncommuting variables

A *labelled edge-coloured digraph* is an edge-coloured digraph where its vertex set is a subset of $\mathbb{N}$. We usually denote the vertices of a labelled edge-coloured digraph by bold positive integers. We usually use $\mathbf{G}$ to denote a labelled edge-coloured digraph whose underlying edge-coloured digraph is G.

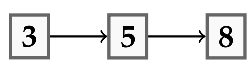
We now define some useful labelled edge-coloured digraphs. For any set

$$S = \{i_1 < i_2 < \cdots < i_k\}$$

of positive integers, let Q_S (resp. P_S) be the labelled edge-coloured directed path with vertex set S such that its coloured edges are $i_j \Rightarrow i_{j+1}$ (resp. $i_j \to i_{j+1}$) for $1 \leq j \leq k$.

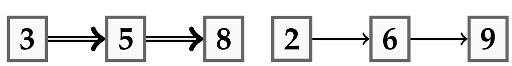


$Q_{\{3,5,8\}}$



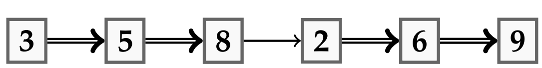
$P_{\{3,5,8\}}$

The *disjoint union* of labelled edge-coloured digraphs $\mathbf{G}_1$ and $\mathbf{G}_2$ with $V(\mathbf{G}_1) \cap V(\mathbf{G}_2) = \emptyset$, is denoted by $\mathbf{G}_1 \sqcup \mathbf{G}_2$.



$Q_{\{3,5,8\}} \sqcup P_{\{2,6,9\}}$

The *solid sum* of labelled edge-coloured directed paths $\mathbf{G}_1$ and $\mathbf{G}_2$, denoted $\mathbf{G}_1 \to \mathbf{G}_2$, is an edge-coloured digraph obtained by connecting the last vertex of $\mathbf{G}_1$ to the first vertex of $\mathbf{G}_2$ by a solid edge $\to$.



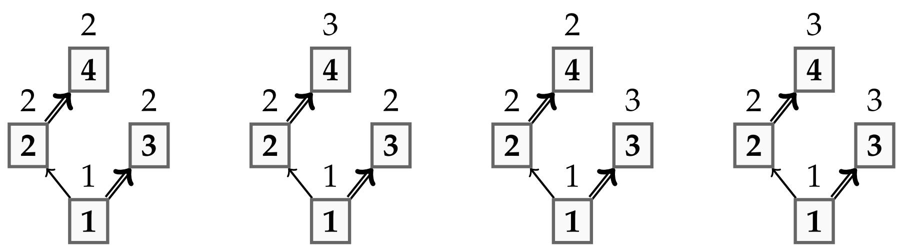
$Q_{\{3,5,8\}} \to Q_{\{2,6,9\}}$

Recall that $\mathbb{Q}\langle\langle \mathbf{x}_1, \mathbf{x}_2, \ldots \rangle\rangle$ is the algebra of formal power series in infinitely many noncommuting variables $\mathbf{x} = \mathbf{x}_1, \mathbf{x}_2, \ldots$ over $\mathbb{Q}$.

The *generalized chromatic function in noncommuting variables* of a labelled edge-coloured digraph $\mathbf{G}$ with vertex set $[n]$ is

$$\mathscr{Y}_{\mathbf{G}} = \sum_{\kappa} \mathbf{x}_{\kappa(1)}\mathbf{x}_{\kappa(2)} \ldots \mathbf{x}_{\kappa(n)}$$

where the sum is over all proper colourings κ of G. For example, if $\mathbf{G}$ is the labelled edge-coloured digraph in Figure 2, then



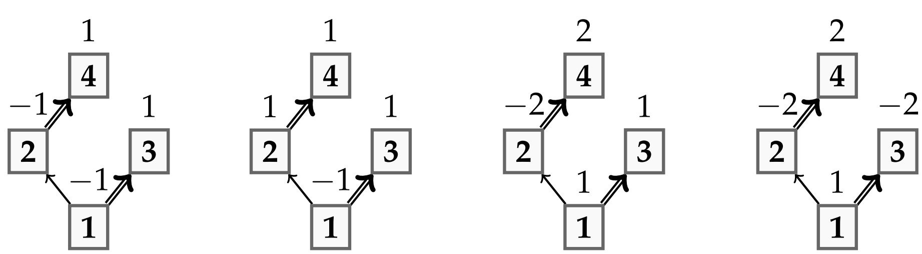
$$\mathscr{Y}_G = \mathbf{x}_1\mathbf{x}_2\mathbf{x}_2\mathbf{x}_2 + \mathbf{x}_1\mathbf{x}_2\mathbf{x}_2\mathbf{x}_3 + \mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_2 + \mathbf{x}_1\mathbf{x}_2\mathbf{x}_3\mathbf{x}_3 + \cdots .$$

3.2 Enriched chromatic functions in noncommuting variables

The *enriched chromatic function* in noncommuting variables of a labelled edge-coloured digraph $\mathbf{G}$ with vertex set $[n]$ is

$$\mathscr{F}_{\mathbf{G}} = \sum_{\kappa} \mathbf{x}_{|\kappa(1)|}\mathbf{x}_{|\kappa(2)|} \ldots \mathbf{x}_{|\kappa(n)|}$$

where the sum is over all enriched colouring κ of G. For example, if $\mathbf{G}$ is the labelled edge-coloured digraph in Figure 2, then



$$\mathscr{F}_G = \mathbf{x}_1\mathbf{x}_1\mathbf{x}_1\mathbf{x}_1 + \mathbf{x}_1\mathbf{x}_1\mathbf{x}_1\mathbf{x}_1 + \mathbf{x}_1\mathbf{x}_2\mathbf{x}_1\mathbf{x}_2 + \mathbf{x}_1\mathbf{x}_2\mathbf{x}_2\mathbf{x}_2 + \cdots .$$

3.3 Peak algebra in noncommuting variables, labelled descent-to-peak map, and new results

In this section, we demonstrate that the previous results for the peak algebra and the peak-to-descent map in the literature can be generalized to noncommuting variables. While the results in this section are natural, the proofs are intricate and complex.

The *peak algebra in noncommuting variables*, denoted $\mathrm{NC}\Pi$, is the space spanned by

$$\{\mathscr{F}_{\mathbf{G}} : \mathbf{G} \text{ is a labelled edge-coloured digraph}\}.$$

The *labelled descent-to-peak* map Θ_{NCQSym} is

$$\begin{array}{rccc} \Theta_{\mathrm{NCQSym}}: & \mathrm{NCQSym} & \to & \mathrm{NC}\Pi \\ & \mathscr{Y}_{\mathbf{G}} & \mapsto & \mathscr{F}_{\mathbf{G}}. \end{array} \tag{3.1}$$

(1) A set $B \subseteq \{2,3,\ldots,n-1\}$ is called a *peak set*[3] if $b \in B$ implies that $\{b-1, b+1\} \cap B = \emptyset$. By [21, Theorem 3.1] the dimension of the space of the homogeneous elements of degree n of Π, Π_n, is $|\{B \subseteq \{2,3,\ldots,n-1\} \text{ is a peak set}\}| = |\{\alpha \vDash [n] : \alpha \text{ is an odd composition}\}| = f_n$, the nth Fibonacci number. A *set composition* ϕ of $[n]$, denoted $\phi \vDash [n]$, is a sequence of mutually disjoint nonempty sets whose union is $[n]$. An *odd set composition* is a set composition whose all blocks have odd sizes.

Theorem 3.1. *The dimension of the space of homogeneous elements of degree n of* $\mathrm{NC}\Pi$, $\mathrm{NC}\Pi_n$, *is*

$$\begin{aligned} \dim(\mathrm{NC}\Pi_n) &= |\{(B,\sigma) : B \subseteq \{2,3,\ldots,n-1\} \text{ is a peak set and } \mathrm{Des}(\sigma) \subseteq \mathrm{Odd}(B)\}| \\ &= |\{\phi \vDash [n] : \phi \text{ is an odd set composition}\}| = a_n \end{aligned}$$

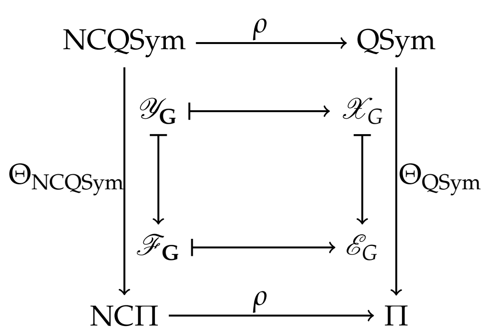
where $\mathrm{Des}(\sigma) = \{i \in [n-1] : \sigma(i) > \sigma(i+1)\}$, *and a_n is the sequence A006154 in the OEIS.*

(2) The peak algebra Π is a Hopf algebra [7, Theorem 2.2].

Theorem 3.2. $\mathrm{NC}\Pi$ *is a Hopf algebra.*

(3) The descent-to-peak map, Θ_{QSym}, is a Hopf algebra morphism [7, Section 2].

Theorem 3.3. *The labelled descent-to-peak map,* Θ_{NCQSym}, *is a surjective Hopf algebra morphism and the following diagram commutes.*



[3]The *peak set of a permutation* $\sigma \in \mathfrak{S}_n$ is the set $\mathrm{Peak}(\sigma) = \{i \in \{2,3,\ldots,n-1\} : \sigma(i-1) < \sigma(i) > \sigma(i+1)\}$. If $i \in \mathrm{Peak}(\sigma)$, then $i-1, i+1 \notin \mathrm{Peak}(\sigma)$. Thus each peak set is the peak set of a permutation.

(4) Given a subset $A = \{a_1 < a_2 < \cdots < a_k\}$ of $[n-1]$ and $\sigma \in \mathfrak{S}_n$, we say (A, σ) is *standard* if $\mathrm{Des}(\sigma) \subseteq A$. The value of the labelled descent-to-peak map at the monomial and fundamental bases elements of NCQSym are as follows.

(a) **Fundamental basis.** Let $A = \{a_1 < a_2 < \cdots < a_k\}$ of $[n-1]$ and $\sigma \in \mathfrak{S}_n$ such that (A, σ) is standard. The *fundamental* basis element $\mathbf{F}_{(A,\sigma)}$ of NCQSym is the generalized chromatic function in noncommuting variables of the labelled edge-coloured digraph

$$\mathbf{G} = Q_{\{\sigma(1),\ldots,\sigma(a_1)\}} \to Q_{\{\sigma(a_1+1),\ldots,\sigma(a_2)\}} \to \cdots \to Q_{\{\sigma(a_k+1),\ldots,\sigma(n)\}}.$$

Therefore, by the description of the labelled descent-to-peak map in (3.1)

Theorem 3.4. $\Theta_{\mathrm{NCQSym}}(\mathbf{F}_{(A,\sigma)}) = \Theta_{\mathrm{NCQSym}}(\mathscr{Y}_{\mathbf{G}}) = \mathscr{F}_{\mathbf{G}}$.

(b) **Monomial basis.** Let $A = \{a_1 < a_2 < \cdots < a_k\}$ of $[n-1]$ and $\sigma \in \mathfrak{S}_n$ such that (A, σ) is standard. The *monomial* basis element $\mathbf{M}_{(A,\sigma)}$ of NCQSym is

$$\mathbf{M}_{(A,\sigma)} = \sum_{C \subseteq A} (-1)^{|A|-|C|} \mathbf{F}_{(C,\sigma)}.$$

For each peak set $B \subseteq \{2, 3, \ldots, n-1\}$, the *monomial peak function in noncommuting variables* $\boldsymbol{\eta}_{(B,\sigma)}$ is

$$\boldsymbol{\eta}_{(B,\sigma)} = (-1)^{|B|} \sum_{A \subseteq \mathrm{Odd}(B)} 2^{|A|+1} \mathbf{M}_{(A,\sigma)}.$$

Theorem 3.5. *We have*

$$\Theta_{\mathrm{QSym}}(\mathbf{M}_{(A,\sigma)}) = \begin{cases} (-1)^{n-1-|B|-|A|} \boldsymbol{\eta}_{(B,\sigma)} & \text{if } n - \max(A) \text{ is odd,} \\ 0 & \text{otherwise.} \end{cases}$$

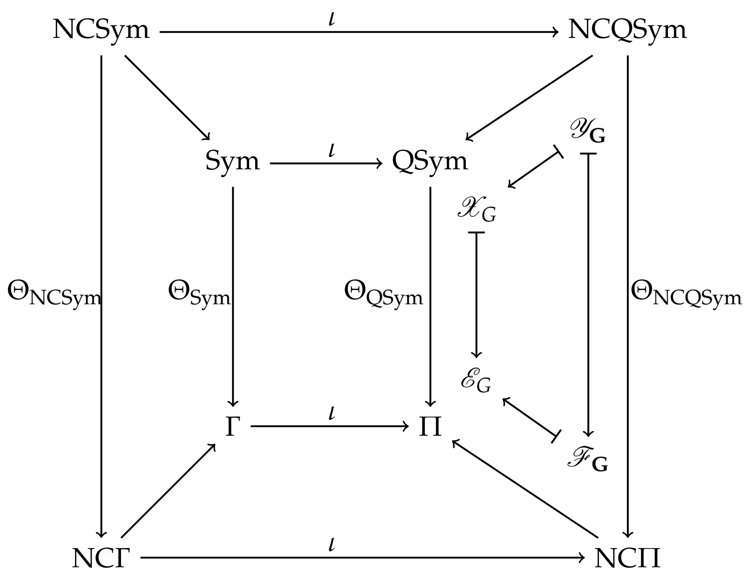
(5) The Hopf algebra of *Schur's Q-functions* introduced in [18] is denoted by Γ and by [21, Theorem 3.8], Γ is the intersection of the Peak algebra Π and the Hopf algebra of symmetric functions Sym, that is, $\Gamma = \Pi \cap \mathrm{Sym}$. Moreover, by [21, Theorem 3.1] $\Theta_{\mathrm{QSym}}(\mathrm{Sym}) = \Gamma$.

We define the Hopf algebra of *Schur's Q-functions in noncommuting variables*, denoted NCΓ, to be the intersection of the peak algebra in noncommuting variables and the Hopf algebra of symmetric functions in noncommuting variables, $\mathrm{NC}\Gamma = \mathrm{NC}\Pi \cap \mathrm{NCSym}$.

Theorem 3.6. *We have* $\Theta_{\mathrm{NCQSym}}(\mathrm{NCSym}) = \mathrm{NC}\Gamma$.

(6) The restriction of the descent-to-peak map Θ_{QSym} to Sym is denoted by Θ_{Sym}. The restriction of the labelled descent-to-peak map Θ_{NCQSym} to NCSym is denoted by Θ_{NCSym}.

Theorem 3.7. *The following diagram commutes.*



(7) A set partition $\pi = \pi_1/\pi_2/\cdots/\pi_l$ of $[n]$, denoted $\pi = \pi_1/\pi_2/\cdots/\pi_l \vdash [n]$, is the set of mutually disjoint non-empty subsets $\pi_1, \pi_2, \ldots, \pi_l$ of $[n]$ whose their union is $[n]$.

Given a labelled edge-coloured digraph $\mathbf{G}$ with vertex set S and $\sigma \in \mathfrak{S}_S$, define $\sigma \circ \mathbf{G}$ to be the labelled edge-coloured digraph with vertex set S in which

- $i \Rightarrow j$ in $\mathbf{G}$ if and only if $\sigma(i) \Rightarrow \sigma(j)$ in $\sigma \circ \mathbf{G}$.
- $i \to j$ in $\mathbf{G}$ if and only if $\sigma(i) \to \sigma(j)$ in $\sigma \circ \mathbf{G}$.

The values of the labelled descent-to-peak map at different bases of NCSym are as follows.

(a) For $\pi = \pi_1/\pi_2/\cdots/\pi_l \vdash [n]$, we have

$$\Theta_{\mathrm{NCSym}}(\mathbf{p}_\pi) = \begin{cases} 2^l \mathbf{p}_\pi & \text{if all blocks of } \pi \text{ have odd sizes,} \\ 0 & \text{otherwise.} \end{cases}$$

(b) For $\pi = \pi_1/\pi_2/\cdots/\pi_l \vdash [n]$, by [3, Section 10], we have that the *complete homogeneous symmetric function in noncommuting variables* $\mathbf{h}_\pi$ (resp., *elementary symmetric function in noncommuting variables* $\mathbf{e}_\pi$) is

$$\mathbf{h}_\pi = \sum_{\sigma \in \mathfrak{S}_\pi} \mathscr{Y}_{\sigma \circ Q_\pi}, \quad (\text{resp. } \mathbf{e}_\pi = \sum_{\sigma \in \mathfrak{S}_\pi} \mathscr{Y}_{\sigma \circ P_\pi}), \quad \text{where}$$

$$Q_\pi = Q_{\pi_1} \sqcup Q_{\pi_2} \sqcup \cdots \sqcup Q_{\pi_{\ell(\pi)}}, (\text{resp. } P_\pi = P_{\pi_1} \sqcup P_{\pi_2} \sqcup \cdots \sqcup P_{\pi_{\ell(\pi)}}) \text{ and}$$
$$\mathfrak{S}_\pi = \mathfrak{S}_{\pi_1} \times \mathfrak{S}_{\pi_2} \times \cdots \times \mathfrak{S}_{\pi_{\ell(\pi)}}.$$

Thus by (3.1) and the fact that $\mathscr{F}_{\sigma \circ P_\pi} = \mathscr{F}_{\sigma \circ Q_\pi}$,

$$\Theta_{\mathrm{NCSym}}(\mathbf{h}_\pi) = \sum_{\sigma \in \mathfrak{S}_\pi} \mathscr{F}_{\sigma \circ Q_\pi} = \Theta(\mathbf{e}_\pi) \quad (\text{resp. } \Theta_{\mathrm{NCSym}}(\mathbf{e}_\pi) = \sum_{\sigma \in \mathfrak{S}_\pi} \mathscr{F}_{\sigma \circ P_\pi} = \Theta(\mathbf{h}_\pi)).$$

(c) For $\pi = \pi_1/\pi_2/\cdots/\pi_l \vdash [n]$, define the permutation $\delta_\pi \in \mathfrak{S}_n$ such that if $\delta_\pi(i) \in \pi_s$ and $\delta_\pi(i+1) \in \pi_t$, then either $s = t$ and $\delta_\pi(i) < \delta_\pi(i+1)$ or $\min(\pi_s) < \min(\pi_t)$. For example, if $\pi = 156/24/36$, then $\delta_\pi = 1562436$. The *Schur function in noncommuting variable* $\mathbf{s}_\pi$ is

$$\mathbf{s}_\pi = \delta_\pi \circ \mathbf{det}\left(\mathscr{Y}_{Q_{[\lambda_i - i + j]}}\right)_{1\le i,j\le l}.$$

Thus by (3.1),

$$\Theta_{\mathrm{NCSym}}(\mathbf{s}_\pi) = \delta_\pi \circ \mathbf{det}\left(\mathscr{F}_{Q_{[\lambda_i - i + j]}}\right)_{1\le i,j\le l}.$$

(8) In [18] Schocker proves that the peak algebra is a left co-ideal of QSym under internal comultiplication, a generalized dual Kronecker product. We provide a combinatorial proof for the following theorem.

Theorem 3.8. *The peak algebra in noncommuting variables is a left co-ideal of* NCQSym *under internal comultiplication.*

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