Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #146, 12 pp.

Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

The HOMFLY Polynomial of a Forest Quiver

Amanda Schwartz*[1]

[1] *Department of Mathematics, University of Michigan, Ann Arbor, U.S.A.*

Abstract. We define the HOMFLY polynomial of a forest quiver Q using a recursive definition on the underlying graph of the quiver. We then show that this polynomial is equal to the HOMFLY polynomial of any plabic link which comes from a connected plabic graph whose quiver is Q. We also prove a closed-form expression for the Alexander polynomial of a forest quiver, which is a specialization of the HOMFLY polynomial.

Keywords: plabic graph, plabic link, quiver, HOMFLY polynomial, Alexander polynomial

1 Introduction

There have been many developments in recent years relating cluster algebras and knot theory. In this work, we will define an invariant of forest quivers and relate it to certain link invariants. This gives one way to connect the study of forest quivers considered up to mutation equivalence and the study of associated links considered up to isotopy. Postnikov's plabic graphs, introduced in [18] to study a stratification of the totally nonnegative Grassmannian into positroid cells, will serve as an intermediate object in establishing this connection.

These plabic graphs and their generalizations have proved useful in establishing several connections between cluster algebras and knot theory. One can associate a plabic link to any plabic graph, as introduced in [6, 21]. In [8] Galashin and Lam studied connections between invariants of these plabic links and invariants of quivers associated to the plabic graphs. In [12], the authors defined 3D plabic graphs, generalizations of Postnikov's two-dimensional plabic graphs, and used them to construct a cluster structure on type A braid varieties. The existence of cluster structures on links or related objects such as braid varieties has been the subject of much recent work; see [4, 3, 11, 21] for several additional examples.

When a plabic graph is reduced, the associated plabic link is a positroid link and is isotopic to several other links one can obtain from objects in bijection with positroids; see [3, 8] for more details. In this setting, polynomial invariants associated to these links can

*amschw@umich.edu. The author was supported by grants DMS-1840234 and DMS-1953852 from the National Science Foundation.

provide information about the objects associated to the plabic graph G. For example, if G is a reduced plabic graph, then there is a rational function $R(Q;q)$ of the quiver Q of G which, after a normalization, yields the point count of the open positroid variety $\Pi^\circ(G)$ over a finite field $\mathbb{F}_q$; see [8, 14]. There are also many connections relating positroid varieties and the Khovanov–Rozansky homology of their associated positroid links to Catalan combinatorics [9, 10, 17, 13].

The link invariants which we will study in this abstract are the Alexander polynomial, which is the oldest knot polynomial, and a stronger invariant called the HOMFLY polynomial which specializes to the Alexander polynomial. The Alexander polynomial was discovered in 1928 by J.W. Alexander [1] while the HOMFLY polynomial was introduced in [7] and also studied independently in [19] in the 1980s. Several interesting connections between these link invariants and cluster algebras have been studied. In [2] Bazier–Matte and Schiffler described a way to associate a cluster algebra to any link diagram and related the Alexander polynomial of the link to the F-polynomial of modules associated to the cluster algebra. Lee and Schiffler related the Jones polynomial, a different specialization of the HOMFLY polynomial, of a 2-bridge link to a specialization of a cluster variable in [15].

From the cluster algebra perspective, there has also been interest in finding and studying polynomial mutation invariants of quivers. For example, in [5] Fomin and Neville studied invariants of cyclically ordered quivers, including an Alexander polynomial which they defined as the determinant of a matrix associated to the quiver. In this abstract, we will focus on forest quivers, i.e. quivers whose underlying graphs are forests. This is a subset of the class of acyclic quivers which includes all type A_n, D_n, E_6, E_7, and E_8 quivers as well as some affine type quivers. We show that given any forest quiver, there is a reduced plabic graph with that forest quiver. Therefore, the set of links which we are studying includes certain positroid links.

We will begin by defining the HOMFLY polynomial of a forest quiver using a recursive definition on the underlying graph of the quiver. This can then be specialized to the Alexander polynomial of a forest quiver, just as the Alexander polynomial of a link is a specialization of the link's HOMFLY polynomial. Since the Alexander and HOMFLY polynomials depend only on the underlying graph of a quiver, we may instead refer to the Alexander or HOMFLY polynomial of a (undirected) forest. Some examples of trees and their Alexander and HOMFLY polynomials are shown in Figure 1.

In Theorem 3.8, we show that for any plabic link which arises from a connected plabic graph whose quiver Q_G is a forest, the HOMFLY polynomial of the link is the same as the HOMFLY polynomial of Q_G. Therefore, while plabic graphs serve as an intermediate object between quivers and plabic links, this result allows one to go directly from a forest quiver to an associated link invariant. While the HOMFLY polynomial of a forest quiver is defined recursively, we also prove a closed-form formula for the Alexander polynomial of a forest quiver in Theorem 4.3.

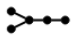
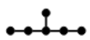
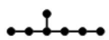
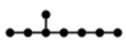
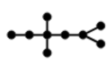
Tree	Alexander Polynomial	HOMFLY Polynomial
A_4	$t^{-2}\left(t^4 - t^3 + t^2 - t + 1\right)$	$\frac{z^4+4z^2+3}{a^4} - \frac{z^2+2}{a^6}$
D_5 	$t^{-5/2}(t^5 - t^4 + t - 1)$	$\frac{z^5+5z^3+6z+2z^{-1}}{a^5} - \frac{z^3+4z+3z^{-1}}{a^7} + \frac{z^{-1}}{a^9}$
E_6 	$t^{-3}\left(t^6 - t^5 + t^3 - t + 1\right)$	$\frac{z^6+6z^4+10z^2+5}{a^6} - \frac{z^4+5z^2+5}{a^8} + \frac{1}{a^{10}}$
E_7 	$t^{-7/2}\left(t^7 - t^6 + t^4 - t^3 + t - 1\right)$	$\frac{z^7+7z^5+15z^3+11z+2z^{-1}}{a^7} - \frac{z^5+6z^3+9z+3z^{-1}}{a^9} + \frac{z+z^{-1}}{a^{11}}$
E_8 	$t^{-4}\left(t^8 - t^7 + t^5 - t^4 + t^3 - t + 1\right)$	$\frac{z^8+8z^6+21z^4+21z^2+7}{a^8} - \frac{z^6+7z^4+14z^2+8}{a^{10}} + \frac{z^2+2}{a^{12}}$
	$t^{-9/2}(t^9 - t^8 - 3t^7 + 7t^6 - 8t^5$ $+ 8t^4 - 7t^3 + 3t^2 + t - 1)$	$\frac{z^9+9z^7+28z^5+39z^3+28z+11z^{-1}+2z^{-3}}{a^9} -$ $\frac{z^7+11z^5+36z^3+47z+28z^{-1}+7z^{-3}}{a^{11}} +$ $\frac{5z^3+20z+23z^{-1}+9z^{-3}}{a^{13}} - \frac{z+6z^{-1}+5z^{-3}}{a^{15}} + \frac{z^{-3}}{a^{17}}$

Figure 1: Examples of trees with their Alexander and HOMFLY polynomials.

2 Preliminaries

2.1 Plabic Graphs and Plabic Links

A *plabic graph* G is a planar, bicolored graph which is embedded in the disk and whose vertices are colored black and white. We assume that G has N vertices on the boundary of the disk which are labeled $1, 2, \ldots, N$ in a clockwise order, are colored black, and each have degree 1. A face in G is said to be a *boundary* (resp. *interior*) face if it is (resp. is not) adjacent to the boundary of the disk. A *strand* in G is a path which follows the edges in G, obeying the *rules of the road*. That is, the path turns maximally right at each black vertex and maximally left at each white vertex. The *strand permutation* π_G of G is a permutation on N obtained by setting $\pi_G(i) = j$ if the strand starting at boundary vertex i ends at boundary vertex j. A plabic graph without internal leaves is said to be *reduced* if it has the minimal number of faces among all such plabic graphs with the strand permutation π_G. See Figure 2 for an example of two plabic graphs, one reduced and one not reduced, which have the same strand permutation.

One can associate a plabic link L_G^{plab} to a plabic graph G as follows. For more details,

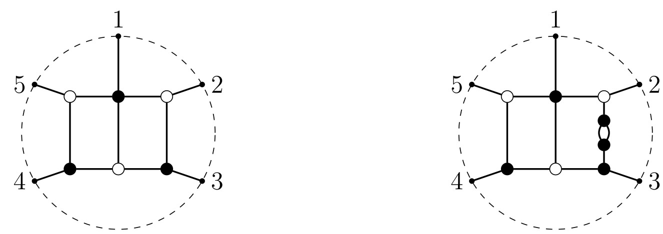


Figure 2: Two plabic graphs, one reduced (left) and one not reduced (right), which both have strand permutation $\pi = (1\,4\,2\,5\,3)$.

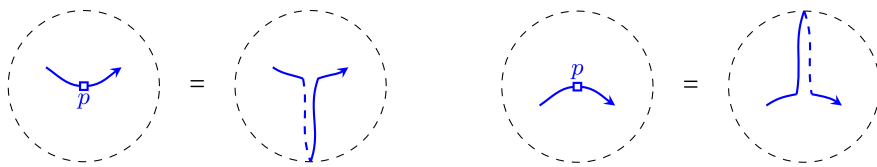


Figure 3: Modifications at a point p on a strand where the tangent vector has argument 0.

including an alternative description in terms of divides, see [6, 8]. Draw all the strands of G. When two strands S_1 and S_2 cross at a point p, consider the arguments θ_1 and θ_2 of their tangent vectors at p, respectively, when considered in the complex plane. We assume that $0 < \theta_1 \neq \theta_2 < 2\pi$. If θ_1 is greater than (resp. less than) θ_2 then S_1 goes under (resp. over) S_2. Note that we sometimes mark such points with rectangles. Taking the union of all these strands after these adjustments and connecting strands which start and end at the same boundary vertex gives a link diagram for L_G^{plab}. For an example, see Figure 4 for the plabic link of the leftmost graph in Figure 2. At any point where the tangent vector to a strand has argument 0, the strand must be adjusted as follows. If

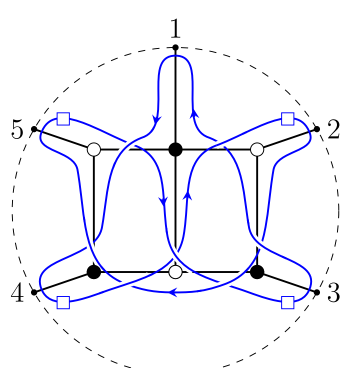


Figure 4: A plabic graph G and its plabic link L_G^{plab}.

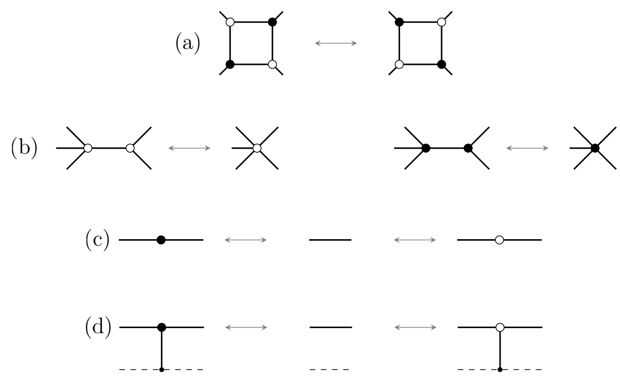


Figure 5: Local moves on plabic graphs: (a) square move, (b) contraction/uncontraction, (c) middle vertex insertion/removal, and (d) tail addition/removal.

there is a point p on a strand S where the argument changes from being just below 2π to being just above 0 as one travels through p along S, then we break S at p, sending it to the boundary just before reaching p and then back to continue along its original path after p. Along the way to the boundary, it passes under all other strands it crosses, and on the way back to the strand S, it passes over all other strands. If the argument instead changes at p from being just above 0 to being just below 2π, the procedure is the same except the strand crosses above all other strands when heading to the boundary and under all others when returning from the boundary. This is demonstrated in Figure 3. We will allow for certain local moves on plabic graphs, pictured in Figure 5. All four local moves (a) - (d) result in isotopic plabic links and therefore will not affect any of the polynomial invariants we study in this abstract. The *tail reduction* of a plabic graph G refers to the graph which results from applying tail removal to G until it is no longer possible to do so. An interior face F of G is said to be a *boundary leaf face* if in the tail reduction of G its boundary consists of two vertices of different colors which are connected by two edges, one of which separates F from a boundary face and one of which separates F from an interior face. For an example, see Example 2.2.

One can associate a directed, planar graph Q_G to each plabic graph G as follows. There will be one vertex v_F placed inside each interior face F of G. An edge is placed between vertices v_F and $v_{F'}$ for each edge e in G with opposite colored endpoints such that F and F' are adjacent to e. The edge in Q_G is oriented so that as one travels along it, the white vertex in e is to the left. If this graph Q_G contains no loops or directed 2-cycles, then G is said to be *simple*. We will refer to Q_G as the quiver associated to G. Throughout the rest of this abstract, we will work only with simple plabic graphs G since we will assume that Q_G is a forest quiver. Note that local moves (b) - (d) will not change the quiver of a plabic graph. If G is a connected, simple plabic graph, then G can be assumed to be trivalent, i.e. we can assume that all internal vertices have degree

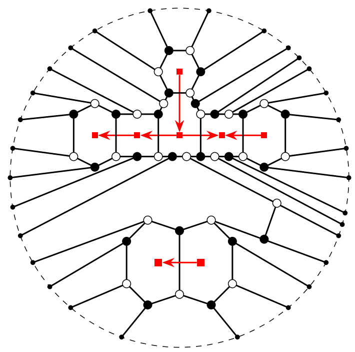


Figure 6: A reduced plabic graph with the pictured forest quiver.

3 after applying these local moves.

Our focus in this abstract will be on plabic links arising from simple plabic graphs G whose quivers Q_G are orientations of forests. Given any forest quiver Q, it is possible to find a plabic graph whose quiver is Q. In fact, one can choose such a graph to be reduced so that the resulting link is a positroid link. It was known to Lam and Speyer that one can find a reduced plabic graph whose quiver is Q for any tree quiver Q, as mentioned in [8], but their proof has not been published. We have an algorithm for constructing a reduced plabic graph G with $Q_G = Q$ for any forest quiver Q. The algorithm proceeds by first constructing reduced plabic graphs G_i for each connected component Q_i of $Q_G = Q_1 \sqcup Q_2 \sqcup \cdots \sqcup Q_k$, then placing each of these graphs G_i inside one larger disk, and finally adding an edge to connect the graph G_i to the graph G_{i+1} for each $i = 1, 2, \ldots, k-1$.

Proposition 2.1. *Let Q be a forest quiver. Then there exists a connected, reduced plabic graph G with $Q_G = Q$.*

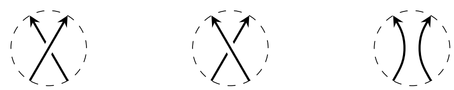
Example 2.2. See Figure 6 for an example of a reduced plabic graph G whose quiver is the pictured orientation of $E_6 \sqcup A_2$. We also make the following observation. Consider, for example, the face corresponding to the topmost leaf in the quiver as pictured in Figure 6. We note that tail removal can be applied four times at this face, resulting in a face whose boundary consists of two vertices joined by two edges. In particular, the face corresponding to this leaf is a boundary leaf face.

2.2 The HOMFLY and Alexander Polynomials

The Alexander polynomial, named after its discoverer J.W. Alexander [1], was the first knot polynomial invariant to be discovered. John Conway showed that the Alexander polynomial satisfies a skein relation

$$\Delta(L_+) - \Delta(L_-) = \left(t^{1/2} - t^{-1/2}\right) \Delta(L_0) \tag{2.1}$$

where L_+, L_-, and L_0 are links whose diagrams are the same except locally at one location where they are related as follows:



L_+ L_- L_0

Setting $\Delta(\text{unknot}) = 1$ fixes a specific choice of the Alexander polynomial for each oriented link, although typically the polynomial is defined up to multiplication by $\pm t^k$ for some k.

The Alexander polynomial is also a specialization of a stronger invariant called the HOMFLY polynomial, introduced in [7] and also studied independently in [19]. The HOMFLY polynomial is a Laurent polynomial in a and z defined by the skein relation

$$aP(L_+) - a^{-1}P(L_-) = zP(L_0) \tag{2.2}$$

and setting $P(\text{unknot}) = 1$. Setting $a = 1$ and $z = t^{1/2} - t^{-1/2}$ in the HOMFLY polynomial recovers the Alexander polynomial. The following proposition is a well-known fact about the HOMFLY polynomial of a connected sum of two links which we will use to prove our first main result. See [16] for a proof of this fact.

Proposition 2.3. *Let L and L' be two oriented links, and let $L\#L'$ be a connected sum of these two links. Then $P(L\#L') = P(L) \cdot P(L')$.*

3 The HOMFLY Polynomial of a Forest Quiver

3.1 Defining the HOMFLY polynomial of a forest quiver

Definition 3.1. Let Q be a quiver whose underlying graph is a forest. The *HOMFLY polynomial of Q*, denoted $f(Q)$, is defined recursively by setting

- $f(Q) = 1$ if Q is empty,
- $f(Q) = \frac{z+z^{-1}}{a} - \frac{z^{-1}}{a^3}$ if Q is a single vertex,
- $f(Q) = \frac{z}{a} f(Q - \{v\}) + \frac{1}{a^2} f(Q - \{v, \tilde{v}\})$ if v is a leaf which is adjacent to $\tilde{v}$, and
- $f(Q) = f(Q_1) \cdot f(Q_2)$ if $Q = Q_1 \sqcup Q_2$.

Proposition 3.2. *The function f is well-defined.*

Remark 3.3. Since the definition does not depend on the orientation of the edges in Q, we may occasionally write $f(Q)$ where Q is an undirected forest.

Example 3.4. Fix $n \geq 3$. Let S_n be the star graph on n vertices which has one vertex of degree $n-1$ connected to $n-1$ leaves. One can show using induction that the HOMFLY polynomial of S_n for $n \geq 3$ is

$$P(S_n) = \frac{z^{n-2}}{a^n} + \frac{z^{n-1}+z^{n-3}}{a^{n-1}}\left(\frac{z+z^{-1}}{a} - \frac{z^{-1}}{a^3}\right) + \sum_{k=2}^{n-2}\left(\frac{z^{k-2}}{a^k}\left(\frac{z+z^{-1}}{a} - \frac{z^{-1}}{a^3}\right)^{n-k}\right).$$

3.2 Connections to the HOMFLY polynomial of a plabic link

In [8], Galashin and Lam made the following conjecture.

Conjecture 3.5 ([8])**.** *Let G, G' be two connected simple plabic graphs. Assume that the quivers Q_G and $Q_{G'}$ are mutation equivalent. Then $P(L_G^{plab}) = P(L_{G'}^{plab})$.*

Our first main result represents partial progress towards proving this conjecture. We show that given a forest quiver Q and any connected plabic graph G with $Q_G = Q$, $P(L_G^{\text{plab}}) = f(Q)$. This implies that given any two connected plabic graphs G, G' whose quivers are orientations of the same forest Q, $P(L_G^{\text{plab}}) = f(Q) = P(L_{G'}^{\text{plab}})$. Additionally, this result gives a way to go directly from a forest quiver to a corresponding link invariant. We will sketch the proof of this result. For more details, see [20]. The proof relies on the following results.

Lemma 3.6 ([8])**.** *Let G be a simple plabic graph with a boundary leaf face F. Let x and y be the vertices on the boundary of F and e be the edge separating F from a boundary face. Let $G' = G - e$ and $G'' = G - \{x, y\}$. Then the HOMFLY polynomials of their plabic links satisfy*

$$aP(L_G^{plab}) - a^{-1}P(L_{G''}^{plab}) = zP(L_{G'}^{plab}). \tag{3.1}$$

Proposition 3.7. *Let G be a connected plabic graph whose quiver Q_G is a disjoint union of non-empty tree quivers $Q_1, Q_2, \ldots, Q_k$ for some $k \geq 2$. Then L_G^{plab} is isotopic to a connected sum of links $L_{G_1}^{plab}, \ldots L_{G_k}^{plab}$ for some choice of connected plabic graphs G_i with $Q_{G_i} = Q_i$ for $i = 1, \ldots, k$.*

Proof Sketch. The main idea of the proof is as follows. If an edge in G separates two (not necessarily distinct) boundary faces, we say that it is a *dividing edge*. Deleting such an edge divides the plabic graph G into two smaller plabic graphs G_1 and G_2. Since Q_G is disconnected, then after potentially after applying local moves one can always find a dividing edge e in G such that G_1 and G_2 have non-empty quivers whose disjoint union is Q_G. Further, one can verify that L_G^{plab} is isotopic to a connected sum of $L_{G_1}^{\text{plab}}$ and $L_{G_2}^{\text{plab}}$; see Figure 7.

□

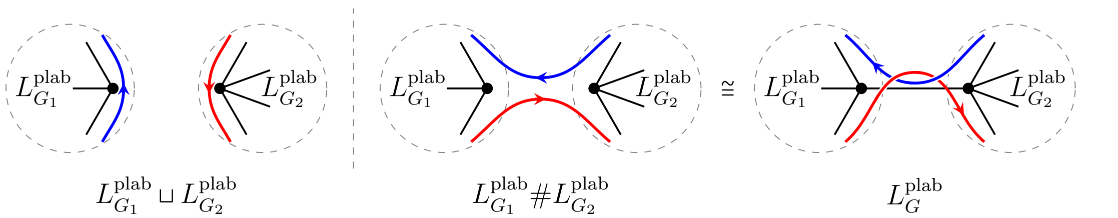


Figure 7: A dividing edge in G means L^{plab}_G is a connected sum of two other plabic links $L^{\text{plab}}_{G_1}$ and $L^{\text{plab}}_{G_2}$. The isotopy between $L^{\text{plab}}_{G_1} \# L^{\text{plab}}_{G_2}$ and L^{plab}_G corresponds to rotating the portion of the connected sum coming from $L^{\text{plab}}_{G_1}$ by $360°$ about the horizontal axis in the paper.

Theorem 3.8. *Let G be a connected plabic graph, and suppose the quiver Q_G of G is a forest quiver. Then $P(L^{plab}_G) = f(Q_G)$.*

Proof Sketch. The proof proceeds by induction on the number of vertices in Q_G. Recall that $f(Q_G)$ can be computed recursively by removing leaves using the relation

$$f(Q) = \frac{z}{a} f(Q - \{v\}) + \frac{1}{a^2} f(Q - \{v, \tilde{v}\})$$

if v is a leaf which is adjacent to $\tilde{v}$ or by taking the product over connected components using the relation

$$f(Q) = f(Q_1) \cdot f(Q_2)$$

if $Q = Q_1 \sqcup Q_2$. The inductive step relies on finding analogues of these relations between the HOMFLY polynomials of plabic links.

In particular, if Q_G is disconnected, then by Proposition 3.7 there are connected plabic graphs G_1 and G_2 with non-empty quivers such that $Q_G = Q_{G_1} \sqcup Q_{G_2}$ and $P(L^{\text{plab}}_G)$ is isotopic to a connected sum of $L^{\text{plab}}_{G_1}$ and $L^{\text{plab}}_{G_2}$. Using Proposition 2.3, it follows that

$$\begin{aligned} P(L^{\text{plab}}_G) &= P(L^{\text{plab}}_{G_1}) \cdot P(L^{\text{plab}}_{G_2}) \\ &= f(Q_{G_1}) \cdot f(Q_{G_2}) \\ &= f(Q_G). \end{aligned}$$

If Q_G is connected, a leaf v in Q_G can be assumed to correspond to a boundary leaf face in G, possibly after applying local moves. Further, $Q_{G'} = Q_G - \{v\}$ and $Q_{G''} = Q_G - \{v, \tilde{v}\}$. From the inductive hypothesis and Lemma 3.6, it follows that

$$\begin{aligned} P(L^{\text{plab}}_G) &= \frac{z}{a} P(L^{\text{plab}}_{G'}) + \frac{1}{a^2} P(L^{\text{plab}}_{G''}) \\ &= \frac{z}{a} f(Q_G - \{v\}) + \frac{1}{a^2} f(Q_G - \{v, \tilde{v}\}) \\ &= f(Q_G). \end{aligned}$$

□

4 The Alexander Polynomial of a Forest Quiver

Just as one can specialize the HOMFLY polynomial of a link to obtain the link's Alexander polynomial using the specialization $a = 1$ and $z = t^{1/2} - t^{-1/2}$, we can use the same specialization to obtain the Alexander polynomial $\Delta(Q)$ of a forest quiver Q.

Example 4.1. One can prove by induction that for the type A_n quiver

$$\Delta(A_n) = t^{-n/2} \cdot \sum_{k=0}^{n} (-1)^{n-k} t^k.$$

Example 4.2. Similarly, one can prove via induction that

$$\Delta(D_n) = t^{-n/2} \left(t^n - t^{n-1} + (-1)^{n-1} t + (-1)^n \right) = t^{-n/2}(t-1)(t^{n-1} + (-1)^{n-1}).$$

Our second main result is a proof of a closed formula for the Alexander polynomial for any forest quiver Q.

Theorem 4.3. *Let Q be a be a forest quiver with n vertices. Then the Alexander polynomial of Q is given by*

$$\Delta(Q) = \frac{(-1)^n}{t^{n/2}} \sum_{k=0}^{\lfloor n/2 \rfloor} c_k(Q) t^k (1-t)^{n-2k} \tag{4.1}$$

where $c_k(Q)$ is the number of independent sets of size k in the line graph of Q, i.e. the number of subsets of k distinct edges in Q which do not share any endpoints.

Remark 4.4. In [14], Lam and Speyer studied the point count of acyclic cluster varieties over finite fields. They proved that for an acyclic quiver Q with n vertices the point count over $\mathbb{F}_q$ of the associated cluster variety $\mathcal{A}$ is given by

$$\#\mathcal{A}(\mathbb{F}_q) = \sum_{k \geq 0} a_k (q-1)^{n-2k} q^k \tag{4.2}$$

where a_k is the number of independent sets in the underlying graph of the quiver. It is interesting to note the similarity between this formula and the formula in (4.1). In an upcoming updated version of [20], we will prove a result about the HOMFLY polynomial which recovers both equations (4.1) and (4.2) in the case where Q is a forest quiver.

Example 4.5. Let Q be the Dynkin diagram E_6. Then there are 5 ways to pick a single edge in Q, 5 ways to pick two distinct edges which do not share a vertex, and 1 way to pick three distinct edges, none of which share any vertices; see Figure 8. Therefore,

$$\begin{aligned}\Delta(E_6) &= t^{-3} \left(1 \cdot t^0 (1-t)^6 + 5 \cdot t^1 (1-t)^4 + 5 \cdot t^2 (1-t)^2 + 1 \cdot t^3 (1-t)^0 \right) \\ &= t^{-3} \left(t^6 - t^5 + t^3 - t + 1 \right).\end{aligned}$$

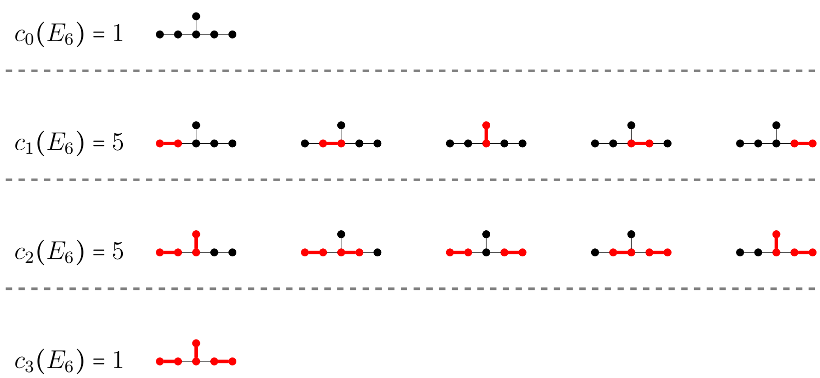


Figure 8: The sets counted by the coefficients in (4.1) when Q is E_6. The edges which are used are bolded and drawn in red.

Example 4.6. Fix $n \geq 3$. Recall from Example 3.4 that S_n is the star graph on n vertices which has one vertex of degree $n-1$ connected to $n-1$ leaves. Any edge in S_n must have the degree $n-1$ vertex as one of its endpoints, so it is impossible to pick multiple edges in S_n which do not share an endpoint. It follows that

$$\begin{aligned}\Delta(S_n) &= \frac{(-1)^n}{t^{n/2}}\left((1-t)^n + (n-1)\cdot t(1-t)^{n-2}\right)\\ &= \frac{(-1)^n}{t^{n/2}}\left(\left(1-(n-3)t+t^2\right)(1-t)^{n-2}\right).\end{aligned}$$

Acknowledgements

I thank my advisor, Thomas Lam, for his guidance and comments. I also thank Eugene Gorsky for helpful discussions and his suggestion of the formula which is proved in Theorem 4.3.

References

[1] J. W. Alexander. "Topological invariants of knots and links". *Transactions of the American Mathematical Society* **30** (1928), pp. 275–306. DOI.

[2] V. Bazier-Matte and R. Schiffler. "Knot theory and cluster algebras". *Advances in Mathematics* **408** (2022), Paper No. 108609, 45. DOI.

[3] R. Casals, E. Gorsky, M. Gorsky, and J. Simental. "Positroid Links and Braid varieties". 2024. arXiv:2105.13948.

[4] R. Casals, E. Gorsky, M. Gorsky, I. Le, L. Shen, and J. Simental. "Cluster structures on braid varieties". *Journal of the American Mathematical Society* **38**.2 (2025), pp. 369–479. DOI.

[5] S. Fomin and S. Neville. “Cyclically ordered quivers”. 2024. arXiv:2406.03604.

[6] S. Fomin, P. Pylyavskyy, E. Shustin, and D. Thurston. “Morsifications and mutations”. *Journal of the London Mathematical Society* **105**.4 (2022), 2478—2554. DOI.

[7] P. Freyd, D. Yetter, J. Hoste, W. B. R. Lickorish, K. Millett, and A. Ocneanu. “A new polynomial invariant of knots and links”. *Bulletin of the American Mathematical Society* **12**.2 (1985), pp. 239–246. DOI.

[8] P. Galashin and T. Lam. “Plabic links, quivers, and skein relations”. *Algebraic Combinatorics* **7**.2 (2024), pp. 431–474. DOI.

[9] P. Galashin and T. Lam. “Positroid Catalan numbers”. *Communications of the American Mathematical Society* **4** (2024), pp. 357–386. DOI.

[10] P. Galashin and T. Lam. “Positroids, knots, and q,t-Catalan numbers”. *Duke Mathematical Journal* **173**.11 (2024), pp. 2117–2195. DOI.

[11] P. Galashin, T. Lam, and M. Sherman-Bennett. “Braid variety cluster structures, II: general type”. 2023. arXiv:2301.07268.

[12] P. Galashin, T. Lam, M. Sherman-Bennett, and D. Speyer. “Braid variety cluster structures, I: 3D plabic graphs”. 2024. arXiv:2210.04778.

[13] M. Hogancamp and A. Mellit. “Torus link homology”. 2019. arXiv:1909.00418.

[14] T. Lam and D. Speyer. “Cohomology of cluster varieties II: Acyclic case”. *Journal of the London Mathematical Society. Second Series* **108**.6 (2023), pp. 2377–2414. DOI.

[15] K. Lee and R. Schiffler. “Cluster algebras and Jones polynomials”. *Selecta Mathematica* **25**.4 (2019), Paper No. 58, 41. DOI.

[16] W. B. R. Lickorish. *An introduction to knot theory*. Vol. 175. Graduate Texts in Mathematics. Springer-Verlag, New York, 1997, pp. x+201. DOI.

[17] A. Mellit. “Homology of torus knots”. *Geometry & Topology* **26**.1 (2022), 47—70. DOI.

[18] A. Postnikov. “Total positivity, Grassmannians, and networks”. 2006. arXiv:math/0609764.

[19] J. H. Przytycki and P. Traczyk. “Conway algebras and skein equivalence of links”. *Proceedings of the American Mathematical Society* **100**.4 (1987), pp. 744–748. DOI.

[20] A. Schwartz. “The HOMFLY Polynomial of a Forest Quiver”. 2024. arXiv:2410.00399.

[21] V. Shende, D. Treumann, H. Williams, and E. Zaslow. “Cluster varieties from Legendrian knots”. *Duke Mathematical Journal* **168**.15 (2019), pp. 2801–2871. DOI.