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On the Classification of Schubert Varieties in Partial Flag Varieties

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Abstract. We generalize the classification of isomorphism classes of Schubert varieties in complete flag varieties G/B (E. Richmond and W. Slofstra, 2021) to a class of partial flag varieties G/P. In particular, we classify all Schubert varieties in G/P where P is a minimal parabolic subgroup, and all Schubert surfaces which are two-dimensional Schubert varieties.

Keywords: Schubert varieties, partial flag varieties, Schubert surfaces, cohomology

1 Introduction

Schubert varieties form an extensively studied class of algebraic varieties whose properties are often characterized by combinatorics. The isomorphism problem for Schubert varieties, first raised by Develin, Martin, and Reiner in [2], asks for a classification of all Schubert varieties up to algebraic isomorphism. In the same paper, they classified a class of smooth Schubert varieties in type A partial flag varieties. Using Cartan equivalence, Richmond and Slofstra solved this problem for Schubert varieties in complete flag varieties in [8]. On the other hand, Richmond, Ţarigradschi, and Xu solved this problem for cominuscule Schubert varieties in [9] using labeled posets.

To describe our results, we set the following notations. We only consider Schubert varieties of finite types over $\mathbb{C}$ for simplicity. Let G be a complex reductive Lie group, T a maximal torus, and B a Borel subgroup containing T. Then a root system is defined, with the correspondent set of simple reflections S, Cartan matrix $A = (a_{st})_{(s,t)\in S^2}$, and the Weyl group W. The pair (W, S) forms a Coxeter system. A standard parabolic subgroup P contains B corresponding to a subset I of S. We denote the flag variety G/P by $X(A, I)$. The subset I generates a subgroup $W_I \subset W$. In every coset W/W_I, we take the elements with minimal length and form a set W^I. Then a Schubert variety is the closure of the B orbit BwP/P for some $w \in W^I$, and we write this variety as $X(w, A, I)$ since it is uniquely determined by the triple (w, A, I). We denote the *support* of w to be

$$S(w) = \{s \in S | s \leq w\}.$$

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In our notations, Richmond and Slofstra classified Schubert varieties of the form $X(w, A, \varnothing)$. Motivated by their methods, we extend their work.

Main Theorem 1.1 (See Proposition 4.5, Proposition 4.9, and Theorem 5.5)**.** *Let*

$$A = (a_{st})_{(s,t)\in S^2} \qquad \text{and} \qquad A' = (a'_{s't'})_{(s',t')\in S'^2}$$

be two Cartan matrices with associated Weyl groups W and W', and sets of simple reflections S and S', respectively. Let $I \subset S$ and $I' \subset S'$. Take $w \in W^I$ and $w' \in W'^{I'}$. Assume that $|S(w) \cap I| \leq 1$ and $|S(w') \cap I'| \leq 1$. Then the following are equivalent:

1. *the Schubert varieties $X(w, A, I)$ and $X(w', A', I')$ are algebraically isomorphic;*
2. *there exists a bijection $\tau : S(w) \to S(w')$ sending $S(w) \cap I$ to $S(w') \cap I'$, such that:*
   (a) *for some reduced word $w = s_1 \cdots s_k$, $w' = \tau(s_1) \cdots \tau(s_k)$ is also a reduced word;*
   (b) *for any $t_1, t_2 \in S(w)$, $a_{t_1,t_2} = a'_{\tau(t_1),\tau(t_2)}$ whenever $t_1 t_2 \leq w$.*

In particular, if $X(w, A, I)$ and $X(w', A', I')$ are isomorphic, then $|S(w) \cap I| = |S(w') \cap I'|$.

Since we allow the subset I to satisfy the condition $|S(w) \cap I| \leq 1$, a larger class of Schubert varieties is classified. It also completely solves the isomorphism problem of Schubert varieties in flag varieties corresponding to minimal parabolic subgroups, in which case, $|I| = 1$.

Schubert varieties of dimension zero and one are isomorphic to a point and the projective line $\mathbb{P}^1$, respectively. Applying the main theorem, we can easily classify the Schubert varieties of dimension two without explicit computation.

The last statement of our main theorem gives a necessary condition for the isomorphism of Schubert varieties, which can be slightly generalized; see Proposition 4.5.

For two Schubert varieties $X(w, A, I)$ and $X(w', A', I')$, if $|S(w)| = |S(w')|$, then we say these two Schubert varieties are *equally supported*. The way we prove our criterion enables us to attain a sufficient condition (Theorem 5.5) to determine whether the given equally supported pair is an isomorphism.

It is also interesting to consider whether our sufficient condition is necessary. It is still unknown, and we will leave it for future study.

1.1 Outline of the Abstract

In Section 2, we show some illustrative examples of Main Theorem 1.1. At the end of this section, we show the classification of Schubert surfaces; see Theorem 2.6. Then we discuss preliminaries in Section 3. The following two sections give a type-independent proof for our main theorem. In Section 4, we derive some necessary conditions of the isomorphism of Schubert varieties, and we prove that condition 1 implies condition 2 of the main theorem (Proposition 4.9). Our primary tool is the cohomology ring whose cup product can be computed by Chevalley's formula in [3, Lemma 8.1]. We need more explicit computation and discussion than what was done in [8]. Next, in Section 5, we obtain a practical sufficient condition for equally supported Schubert varieties (Theorem 5.5), and, as a corollary, we prove that condition 2 implies condition 1 of the main theorem, finishing the proof of the main theorem.

2 Classification of Schubert Surfaces

To illustrate our context, we provide several examples of Schubert varieties. In the following examples, we will use s_i to denote the simple reflection corresponding to the i-th row or column of the Cartan matrix.

Example 2.1. Considering the three Schubert varieties

$$X\left(s_4s_3s_2s_1, \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \{s_4\}\right), X\left(s_1s_2s_3s_4, \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -2 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \{s_4\}\right),$$

$$X\left(s_1s_2s_3s_4, \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \{s_4\}\right),$$

of types A_4, B_4, and F_4, respectively. The lower triangular parts of three Cartan matrices are the same, so by the main theorem, they are isomorphic.

We call a two-dimensional Schubert variety a *Schubert surface*. For $n = 1, 2, 3$, we let M_n denote the Cartan matrix

$$M_n = \begin{pmatrix} 2 & -1 \\ -n & 2 \end{pmatrix}.$$

In other words, M_1, M_2, and M_3 are the Cartan matrices of type A_2, B_2, and G_2, respectively.

Example 2.2. Consider the Schubert surfaces $X(s_1s_2, M_n, \{s_1\})$. Since the entries $a_{12} = -1$ of Cartan matrices are equal, by the main theorem, these three surfaces are isomorphic. Taking $n = 1$, $X(s_1s_2, M_1, \{s_1\})$ is a Schubert surface in the flag variety $Gr(1,3) = \mathbb{P}^2$, which is also two dimensional. It follows that we have $X(s_1s_2, M_n, \{s_1\})$ is isomorphic to $\mathbb{P}^2$. Similarly, the Schubert surfaces $X(s_1s_2, M_n, \varnothing)$ are isomorphic. By [8, Example 1.1], they are isomorphic to the *first Hirzebruch surface* Σ_1.

Example 2.3. Consider the Schubert surfaces $X(s_2s_1, M_n, \{s_2\})$. The corresponding entries $a_{21} = -n$, so these three varieties are *not* isomorphic to each other. Note that geometrically, $X(s_2s_1, M_1, \{s_2\})$ is the same as $X(s_1s_2, M_1, \{s_1\})$, so it is isomorphic to $\mathbb{P}^2$. On the other hand, it can be shown that $X(s_2s_1, M_2, \{s_2\})$ is the cone over a smooth conic.

Example 2.4. Consider the Schubert surfaces $X(s_2s_1, M_n, \varnothing)$. The corresponding entries $a_{21} = -n$, so these four surfaces are *not* isomorphic to each other. As mentioned in [8, Example 1.1], $X(s_2s_1, M_n, \varnothing)$ is isomorphic to the *n-th Hirzebruch surface* Σ_n.

Example 2.5. Consider the Schubert surface $X(s_1s_2, 2I_2, \varnothing)$, where I_2 is the 2×2 identity matrix. The corresponding entry $a_{12} = 0$, so it differs from any Schubert surfaces mentioned above. The Cartan matrix is of type $A_1 \times A_1$, so the flag variety can be identified with $\mathbb{P}^1 \times \mathbb{P}^1$. Comparing the dimensions, we conclude that $X(s_1s_2, 2I_2, \varnothing)$ is $\mathbb{P}^1 \times \mathbb{P}^1$.

In fact, the Schubert surfaces in the above four examples are all possible Schubert surfaces. To be more precise, we have the following theorem.

Theorem 2.6 (Classification of Schubert Surfaces)**.** *There are seven isomorphism classes of Schubert surfaces. Precisely, any Schubert surface is isomorphic to one and exactly one of the following varieties:*

1. *product of projective lines* $\mathbb{P}^1 \times \mathbb{P}^1$*,*
2. *projective plane* $\mathbb{P}^2$*,*
3. *the n-th Hirzebruch surface* Σ_n *for* $n = 1, 2, 3$*,*
4. *the cone over a smooth conic, and*
5. *the variety* $X(s_2s_1, M_3, \{s_2\})$*.*

Lemma 3.7 (or [7, Lemma 4.8]) implies that every Schubert surface is isomorphic to a Schubert surface lying in a partial flag variety of rank two. Since a root system of rank two have type $A_1 \times A_1$, A_2, B_2, or G_2, Examples 2.2, 2.3, 2.4, and 2.5 exhaust all possibilities of isomorphism classes of Schubert classes. It follows that any Schubert surface is isomorphic to one of the seven surfaces listed above. Applying Main Theorem 1.1, these seven surfaces are not isomorphic to each other.

3 Preliminaries

Given a complex reductive group G with a maximal torus T and a Borel subgroup B containing T, a root system R is determined. Let $R = R^+ \cup R^-$ be the choice of positive/negative roots determined by B, and Δ the set of simple roots. With a choice of ordered for Δ, the root system R defines a Cartan matrix

$$A = (a_{st})_{(s,t)\in S^2}, \qquad \text{where } a_{st} = \langle \alpha_s^\vee, \alpha_t \rangle = \frac{2\langle \alpha_s, \alpha_t \rangle}{\langle \alpha_s, \alpha_s \rangle}.$$

Any positive root α defines a reflection s_α; conversely, a reflection s defines a positive root α_s. The set of simple reflections S generates the Weyl group W, which can be identified with $W = N(T)/T$. The pair (W, S) is a Coxeter system. For any Weyl group element $w \in W$, we denote

$$S(w) = \{s \in S | s \leq w\}$$

to be the *support* of w. If $S(w) = S$, then we say that w is *fully supported*. w can be written as a product of simple reflections, and if $w = s_1 \cdots s_k$ with k minimal, then we say the sequence $(s_1, \cdots, s_k)$ (or the product $s_1 \cdots s_k$) to be a *reduced word* and denote the set of reduced words to be $RW(w)$. To obtain a reduced word from an arbitrary word, we only need to delete an even number of simple reflections. The *Coxeter length function* $\ell : W \to \mathbb{Z}_{\geq 0}$ is defined to be the length of any reduced word. There is a unique longest element, denoted by w_0. We define the *right descent set* of $w \in W$ to be

$$D_R(w) = \{s \in S | \ell(us) = \ell(u) - 1\}.$$

In other words, $D_R(w)$ is the set of simple reflections s that are the rightmost elements in some reduced words of w.

3.1 The Weyl Group and the Root System

A standard parabolic subgroup P corresponds to a subset $I \subset S$. We let the subgroup $W_I \subset W$ be generated by I, and the set W^I to be the set of representatives of W/W_I with minimal length. It is easy to see that a simple reflection $s \in D_R(w)$ if and only if $w \notin W^{\{s\}}$. For $w \in W$, We let $w^{\min} \in W^I$ be the minimal length representative in the coset wW_I. On the other hand, I defines a subset of simple roots Δ_I. We write the root subsystem spanned by Δ_I as R_I.

On the Weyl group W (or, more generally, a Coxeter group), there is a partial order called the *Bruhat order*. For simplicity, we write $[u, v]^I = [u, v] \cap W^I$. [1, Proposition 2.5.1] shows that the map $w \mapsto w^{\min}$ is order-preserving.

We define the *inversion set* of $w \in W$ to be

$$I(w) = R^+ \cap w(R^-).$$

The following statement is part of [4, Lemma 1.6].

Lemma 3.1. *Let $w \in W$ and $s \in S$. Then $\ell(sw) > \ell(w)$ if and only if $w^{-1}(\alpha_s) \in R^+$. Similarly, we have $\ell(ws) > \ell(w)$ if and only if $w(\alpha_s) \in R^+$. In particular, if $w \in W^I$, then $I(w^{-1}) \subset R^+ \setminus R_I^+$.*

3.2 Lie Algebras

Denote the Lie algebra of G and T as $\mathfrak{g}$ and its Cartan subalgebra as $\mathfrak{h}$, respectively. Each coroot $\alpha^\vee = \frac{2\alpha}{\langle\alpha,\alpha\rangle}$ can be identify as an element h_α in $\mathfrak{h}$, and $\{h_\alpha\}_{\alpha\in\Delta}$ is a basis of $\mathfrak{h}$. We denote the root space corresponding to α as $\mathfrak{g}_\alpha$. Then we have the root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in R} \mathfrak{g}_\alpha$. We also denote the nilpotent Lie algebras $\mathfrak{n}^\pm = \bigoplus_{\alpha\in R^\pm} \mathfrak{g}_\alpha$, and the Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Write e_α and f_α to be nonzero elements in $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ for any positive root α, respectively. Then $\{e_\alpha\}_{\alpha\in\Delta}$ and $\{f_\alpha\}_{\alpha\in\Delta}$ generate $\mathfrak{n}^+$ and $\mathfrak{n}^-$, respectively. For $w \in W$, we denote $\mathfrak{n}^+_w = \bigoplus_{\alpha\in I(w)} \mathfrak{g}_\alpha$, which is a nilpotent Lie algebra.

Let λ be a dominant integral weight. Then there is a unique irreducible $\mathfrak{g}$-module L_λ with the highest weight λ. It can be constructed in the following way. Let $\mathbb{C}_\lambda$ be a one-dimensional complex vector space spanned by $\omega = \omega_\lambda$, whose $\mathfrak{b}$-module structure is given by

$$\begin{aligned} h\omega &= \lambda(h)\omega, && h \in \mathfrak{h}, \\ e\omega &= 0, && e \in \mathfrak{n}^+. \end{aligned}$$

Then the *Verma module* with highest weight λ is

$$M_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda,$$

where $U(\mathfrak{g})$ and $U(\mathfrak{b})$ are the universal enveloping algebras of $\mathfrak{g}$ and $\mathfrak{b}$, respectively. It has a submodule, denoted by M^1_λ, which is generated by $f_\alpha^{\lambda(h_\alpha)+1} \otimes 1$. Then by [5, Theorem 8.28],

$$L_\lambda = M_\lambda / M^1_\lambda.$$

3.3 Schubert Varieties

The variety G/P, called the *(partial) flag variety*, is determined by the Cartan matrix A and the subset I, so we denote it as $X(A, I)$. We say a weight λ is an *I-regular weight* if it is a dominant integral weight and satisfies, for a simple reflection $s \in S$, $\lambda(\alpha_s^\vee) = 0$ if and only $s \in I$. The following result is in [6, Lemma 7.1.2].

Lemma 3.2. *Suppose that λ is an I-regular weight, and V is a finite-dimensional highest weight module with the highest weight vector ω. Then there is a closed immersion*

$$G/P \hookrightarrow \mathbb{P}(V), \qquad gP \mapsto [g\omega],$$

where $[g\omega]$ denote the line through $g\omega$.

A *Schubert cell* in G/P is a subvariety BwP/P. Here, to be more precise, we need to replace w with a representative in $N(T)$, but any such choice gives the same variety. Different choices of w may give the same Schubert variety. In fact, BwP/P only depends on the coset wW_I, so we usually choose the minimal representative. If we take $w \in W^I$, then as varieties, $BwP/P \cong \mathbb{A}^{\ell(w)}$.

A *Schubert variety* in G/P, denoted by $X(w, A, I)$, is the Zariski closure of the Schubert cell BwP/P. The triple (w, A, I) determines the Schubert variety. $X(w, A, I)$ has an affine stratification. If we take $w \in W^I$, then we have a disjoint union

$$X(w, A, I) = \bigcup_{v \in [1,w]^I} BuP/P.$$

Definition 3.3. There is a $\mathbb{Z}$-basis σ_v, indexed by $v \in [1, w]^I$, for the integral cohomology group $H^*(X(w, A, I))$. We call this basis to be *Schubert basis*, denoted by $\Sigma(w, A, I)$. The collection of Schubert basis elements of degree two is denoted by $\Sigma_1(w, A, I)$.

The following lemma is essentially the [8, Proposition 4.1], whose proof can be extended into our case with slight changes.

Lemma 3.4. *Suppose that $f : X(w, A, I) \to X(w', A', I')$ is an isomorphism of algebraic varieties. Then the induced map*

$$\varphi^* : H^*(X(w, A, I)) \to H^*(X(w', A', I'))$$

is a graded ring isomorphism that identifies Schubert bases $\Sigma(w, A, I) \to \Sigma(w', A', I')$ and $\Sigma_1(w, A, I) \to \Sigma_1(w', A', I')$.

In other words, both $\Sigma(w, A, I)$ and $\Sigma_1(w, A, I)$ are determined by the variety structure of the Schubert variety $X(w, A, I)$. From this definition, we have an identification

$$i : [1, w]^I \to \Sigma(w, A, I), \qquad v \mapsto \sigma_v.$$

Each basis element σ_u has degree $2\ell(u)$, written by $\deg \sigma_u = 2\ell(u)$, i.e.,

$$\sigma_u \in H^{2\ell(u)}(X(w, A, I)).$$

Recall that the set of simple reflections in $[1, w]^I$ is just $S(w) \setminus I$. Hence the restriction of i gives a one-one correspondence

$$i : S(w) \setminus I \to \Sigma_1(w, A, I), \qquad s \mapsto \sigma_s.$$

The cup product of $H^*(X(w, A, I))$ is given by the *Chevalley's formula* in [3, Lemma 8.1].

Lemma 3.5 (Chevalley's formula)**.** *Suppose that $\alpha \in \Delta \setminus \Delta_I$ and $w \in W^I$. Then*

$$\sigma_{s_\alpha} \sigma_w = \sum \omega_\alpha(\beta^\vee) \sigma_{(ws_\beta)^{\min}},$$

the sum over all positive roots β such that $\ell((ws_\beta)^{\min}) = \ell(w) + 1$. Here, ω_α is the fundamental weight corresponding to α.

Remark 3.6. If $\ell((ws_\beta)^{\min}) = \ell(w) + 1$, then either $ws_\beta \leq w$ or $ws_\beta > w$. If the former one holds, then we have $(ws_\beta)^{\min} \leq w^{\min} = w$, which is impossible. Hence $ws_\beta > w$.

When considering the variety structure of a Schubert variety, it is usually convenient to embed it into a smaller flag variety. This is given by [7, Lemma 4.8].

Lemma 3.7. *We write $A_w = (a_{st})_{(s,t)\in S(w)^2}$ to be the Cartan submatrix of A. Then the inclusion $X(A_w, S(w) \cap I) \hookrightarrow X(A, I)$ restricts to an isomorphism*

$$X(w, A_w, S(w) \cap I) \cong X(w, A, I).$$

Remark 3.8. The Weyl group determined by A_w is $W_{S(w)}$, and $w \in (W_{S(w)})^{S(w)\cap I}$, so the Schubert variety $X(w, A_w, S(w) \cap I)$ is well-defined.

4 The Cohomology Rings of Schubert Varieties

In this section, we will use the cohomology ring to study the intrinsic properties of a Schubert variety. In other words, we will obtain combinatorial information from an isomorphism class of Schubert variety. Some results have been proven in [8, Section 4] for the case I being an empty set, and the proofs are generalized. In the following, we fix a Schubert variety $X(w, A, I)$ with $w \in W^I$. Due to Lemma 3.7, we assume that $S = S(w)$.

4.1 General Situation

First, we show that we can reconstruct the poset $[1, w]^I$. For any $\sigma = \sum_{v\in W^I} a_v\sigma_v \in H^*(X(w, A, I))$, we define its *support*, denoted by $\Sigma(\sigma)$, to be the collection of σ_v with $a_v \neq 0$. Let $\prec$ be the partial order on $\Sigma(w, A, I)$, the Schubert basis, generated by the relation $\sigma_u \prec \sigma_v$ if $\sigma_v \in S(\sigma_s\sigma_u)$ for some $\sigma_s \in \Sigma_1(w, A, I)$. By Lemma 3.4, the poset $\Sigma(w, A, I)$ is determined by the variety structure of $X(w, A, I)$.

Lemma 4.1. *The identification $i : [1, w]^I \to \Sigma(w, A, I)$ is a poset isomorphism.*

Given a subset J' of $\Sigma_1(w, A, I)$, let $H^{J'}$ be the subring of $H^*(X(w, A, I))$ generated by

$$\sigma_s \in \Sigma_1(w, A, I) \setminus J',$$

and let

$$\Sigma(w, A, I)^{J'} = \bigcup_{\sigma\in H^{J'}} \Sigma(\sigma).$$

Lemma 4.2. *Suppose that there is a subset $J \subset S \setminus I$. Then poset isomorphism $i : [1, w]^I \to \Sigma(w, A, I)$ restricts to another poset isomorphism*

$$i^J : [1, w]^{I\cup J} \to \Sigma(w, A, I)^{i(J)}.$$

Now we show how to reconstruct the right descent set. We define the *right descent set* of σ_v to be

$$D_R(\sigma_v) = \left\{\sigma_s \in \Sigma_1(w, A, I) | \sigma_v \notin \Sigma(w, A, I)^{\{\sigma_s\}}\right\}.$$

Lemma 4.3. *Suppose that $v \in [1, w]^I$. Then the poset isomorphism $i : [1, w]^I \to \Sigma(w, A, I)$ restricts to a bijection $i_v : D_R(v) \to D_R(\sigma_v)$.*

Next lemma will be used to reconstruct the reduced words.

Lemma 4.4. *Suppose that $\sigma_v \in \Sigma(w, A, I)$ and $\sigma_t \in D_R(\sigma_v)$. Then there exists a unique maximal element $\sigma_u \prec \sigma_v$ such that $\sigma_u \in \Sigma(w, A, I)^{\{\sigma_t\}}$.*

As a corollary, we obtain the following theorem, which gives a necessary condition.

Proposition 4.5. *If $S(w) \cap I$ is empty but $S(w') \cap I'$ is not, then $X(w, A, I)$ and $X(w', A', I')$ are not isomorphic as algebraic varieties.*

Now we compute some entries of the Cartan matrix from the cohomology ring.

Lemma 4.6. *Let $s \in S$ and $t \in S \setminus I$ with $s \neq t$. If $st \in [t, w]^I$, then the coefficient of σ_{st} in σ_t^2 is $-a_{st}$.*

Lemma 4.7. *Let $r, s \in S$ and $t \in S \setminus I$ with $rst \in [1, w]^I$ being a reduced word. Then the coefficient of σ_{rst} in $\sigma_t\sigma_{st}$ is $\delta_{rt} - a_{rt} + a_{rs}a_{st}$, where $\delta_{rt} = 1$ if $r = t$, and vanishes if $r \neq t$.*

4.2 Case $I = \{s\}$

We further assume that $I = \{s\}$. We are primarily interested in this situation.

First, we define the reduced word of $\sigma_v \in \Sigma$. We let σ_s be a symbol and define

$$\widetilde{\Sigma}_1 = \Sigma_1 \cup \{\sigma_s\},$$

and extend the bijection $S \setminus I \to \Sigma_1$ to a bijection $S \to \widetilde{\Sigma}_1$ by sending s to the symbol σ_s. We define the *reduced words* of $\sigma_v \in \Sigma$ inductively. First, define the reduced word of $\sigma_1 \in H^0(X(w, A, \{s\}))$ to be the singleton of the empty word. Next, suppose that $\sigma_v \in \Sigma$ with $\sigma_t \in D_R(\sigma_v)$. Take σ_u to be the unique maximal element described in Lemma 4.4, and assume that $\deg \sigma_v - \deg \sigma_u = 2n$ for $n \in \mathbb{N}$. Then we define the sequence

$$(\sigma_{s_1}, \cdots, \sigma_{s_k}, \cdots, \sigma_t, \sigma_s, \sigma_t, \sigma_s)$$

is a reduced of σ_v, where $(\sigma_{s_1}, \cdots, \sigma_{s_k})$ is a reduced word of σ_u, and the remaining part $(\cdots, \sigma_t, \sigma_s, \sigma_t, \sigma_s)$ of the sequence is an alternating sequence of σ_t and σ_s of length n. We also denote $RW(\sigma_v)$ as the set of reduced words of σ_v, which is obviously nonempty. The following lemma is clear.

Lemma 4.8. *Suppose that $\sigma_v \in \Sigma$. Then the bijection $S \to \widetilde{\Sigma}_1$ induces an inclusion*

$$RW(\sigma_u) \hookrightarrow RW(u), \qquad (\sigma_{s_1}, \cdots, \sigma_{s_k}) \mapsto (s_1, \cdots, s_k).$$

The above lemmas allow us to prove the following proposition.

Proposition 4.9. *Let $A = (a_{st})_{(s,t)\in S^2}$ and $A' = (a'_{s't'})_{(s',t')\in S'^2}$ be two Cartan matrices with associated Weyl groups W and W', and sets of simple reflections S and S', respectively. Let $s \in S$ and $s' \in S'$. Take two fully supported elements $w \in W^{\{s\}}$ and $w' \in W'^{\{s'\}}$. If $X(w, A, \{s\})$ and $X(w', A', \{s'\})$ are algebraically isomorphic, then there is a bijection $\tau : S \to S'$ sending s to s' such that*

(a) for some reduced word $w = s_1 \cdots s_k$, $w' = \tau(s_1)\cdots\tau(s_k)$ is also a reduced word, and

(b) for any $t_1, t_2 \in S$, $a_{t_1t_2} = a'_{\tau(t_1)\tau(t_2)}$ whenever $t_1t_2 \leq w$.

5 Constructing Isomorphism

In this section, we will study the sufficient conditions and obtain Theorem 5.5 and the main theorem.

Let $A = (a_{st})_{(s,t)\in S^2}$ and $A' = (a'_{s't'})_{(s',t')\in S'^2}$ be two Cartan matrices with associated Weyl groups W and W', and sets of simple reflections S and S', respectively. Let $I \subset S$ and $I' \subset S'$. Take two fully supported elements $w \in W^I$ and $w' \in W'^{I'}$. We assume that there is a bijection $\tau : S \to S'$ sending I to I' such that

(a) for some reduced word $w = s_1 \cdots s_k$, $w' = \tau(s_1)\cdots\tau(s_k)$, and

(b) for any $t_1, t_2 \in S$, $a_{t_1t_2} = a'_{\tau(t_1)\tau(t_2)}$ whenever $t_1t_2 \leq w$.

We want to show that the varieties $X(w, A, \{s\})$ and $X(w', A', \{s'\})$ are algebraically isomorphic. First, we have the following result.

Lemma 5.1. *The bijection $\tau : S \to S'$ induces a poset isomorphism $[1, w]^I \to [1, w']^{I'}$.*

In the following lemmas, we add the additional assumption that $a_{st} \leq a'_{\tau(s)\tau(t)}$ for all simple reflections $s, t \in S$.

The bijection τ induces a bijection $\tau : \Delta \mapsto \Delta'$ between simple roots, given by $\alpha_s \mapsto \alpha_{\tau(s)}$. Let $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ and $\mathfrak{g}' = \mathfrak{n}'^- \oplus \mathfrak{h}' \oplus \mathfrak{n}'^+$ be two complex reductive Lie algebras corresponding to A and A', respectively. The lemma below is [8, Lemma 3.5 (a) and 3.8].

Lemma 5.2. *There are surjective Lie algebra homomorphisms*

$$\varphi^\pm : \mathfrak{n}^\pm(A) \to \mathfrak{n}^\pm(A'),$$

given by $e_\alpha \mapsto e'_{\tau(\alpha)}$ and $f_\alpha \mapsto f'_{\tau(\alpha)}$, respectively. The homomorphism φ^+ induces an isomorphism

$$\varphi^+ : \mathfrak{n}^+(A)_v \to \mathfrak{n}^+(A')_{\tau(v)}.$$

If λ and λ' are I and I'-regular weights, then we write $V = L_\lambda$ and $V' = L_{\lambda'}$ with corresponding highest weight vectors ω and ω', respectively. The lemma below is given in [8, Lemma 3.5 (b) and 3.9]

Lemma 5.3. *Let λ and λ' be I and I'-regular weights, respectively. If $\lambda(h_\alpha) = \lambda'(h'_{\tau(\alpha)})$ for each $\alpha \in \Delta$, then there is a surjective $\mathfrak{n}^-$-homomorphism $\pi : V \to V'$, sending the highest weight vector ω to ω', where V' is regarded as a $\mathfrak{n}^-$-module via the homomorphism φ^-. The homomorphism π satisfies*

$$\pi(\exp(e)v\omega) = \exp(\varphi^+(e))\tau(v)\omega'$$

for all $e \in \mathfrak{n}^+(A)$ and $v \leq w$.

Remark 5.4. The formula makes sense since the Weyl group element v can be lifted to an element in the normalizer of the maximal torus, and different liftings acting on the vector ω have the same result.

With the preparation above, we have the following sufficient condition.

Theorem 5.5. *Let*

$$A = (a_{st})_{(s,t)\in S^2} \qquad \text{and} \qquad A' = (a'_{s't'})_{(s',t')\in S'^2}$$

be two Cartan matrices with associated Weyl groups W and W', and sets of simple reflections S and S', respectively. Let $I \subset S$ and $I' \subset S'$. Take $w \in W^I$ and $w' \in W'^{I'}$. If there is a bijection $\tau : S(w) \to S(w')$ sending $S(w) \cap I$ to $S(w') \cap I'$ such that

(a) for some reduced word $w = s_1 \cdots s_k$, $w' = \tau(s_1) \cdots \tau(s_k)$ is also a reduced word, and

(b) for any $t_1, t_2 \in S$, $a_{t_1t_2} = a'_{\tau(t_1)\tau(t_2)}$ whenever $t_1t_2 \leq w$,

then $X(w, A, I)$ and $X(w', A', I')$ are isomorphic.

Finally, combining Proposition 4.9 and Theorem 5.5, we conclude that conditions 1 and 2 in our main theorem are equivalent. The last statement in the main theorem follows from Proposition 4.5.

As one may notice that the sufficient condition given in Theorem 5.5 is a generalization of the one in the main theorem, one may conjecture that the necessary condition stated in Proposition 4.9 can also be generalized to all equally supported pairs.

Conjecture 5.6. *Let*

$$A = (a_{st})_{(s,t)\in S^2} \qquad \text{and} \qquad A' = (a'_{s't'})_{(s',t')\in S'^2}$$

be two Cartan matrices with associated Weyl groups W and W', and sets of simple reflections S and S', respectively. Let $I \subset S$ and $I' \subset S'$. Take $w \in W^I$ and $w' \in W'^{I'}$. Assume that $|S(w)| = |S(w')|$. Then the following are equivalent:

1. *the Schubert varieties $X(w, A, I)$ and $X(w', A', I')$ are algebraically isomorphic;*
2. *there exists a bijection $\tau : S(w) \to S(w')$ sending $S(w) \cap I$ to $S(w') \cap I'$, such that*
   (a) *for a reduced word $w = s_1 \cdots s_k$, $w' = \tau(s_1) \cdots \tau(s_k)$ is also a reduced word, and*
   (b) *for any $t_1, t_2 \in S(w)$, $a_{t_1,t_2} = a'_{\tau(t_1),\tau(t_2)}$ whenever $t_1 t_2 \leq w$.*

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