Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #148, 11 pp.

Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

A new class of magic positive Ehrhart polynomials of reflexive polytopes

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Abstract. The magic positivity of Ehrhart polynomials is a useful tool for proving the real-rootedness of the h^*-polynomials. In this article, we provide a new class of reflexive polytopes whose Ehrhart polynomials are magic positive. First, we show that the Ehrhart polynomials of Stasheff polytopes are magic positive. Second, we also prove a partial result towards the magic positivity of the Ehrhart polynomials of the dual polytopes of the symmetric edge polytopes of cycles.

Résumé. La positivité magique des polynômes d'Ehrhart est un outil utile pour démontrer la réalité des racines des h^*-polynômes. Dans cet article, nous introduisons une nouvelle classe de polytopes réflexifs dont les polynômes d'Ehrhart sont magiquement positifs. Premièrement, nous montrons que les polynômes d'Ehrhart des polytopes de Stasheff sont magiquement positifs. Deuxièmement, nous démontrons la positivité magique des polynômes d'Ehrhart des polytopes duaux des polytopes symétriques des arêtes associés aux cycles.

Keywords: Ehrhart polynomial, Stasheff polytope, Symmetric edge polytope, Magic positive

1 Ehrhart polynomial

A *lattice polytope* is the convex hull of finitely many elements in a lattice contained in $\mathbb{R}^d$, typically $\mathbb{Z}^d$. A lattice polytope P is called *reflexive* if the dual of P

$$P^* := \{y \in \mathbb{R}^d \mid \langle x, y \rangle \leq 1 \text{ for any } x \in P\}$$

is also a lattice polytope, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product of $\mathbb{R}^d$.

By a theorem of Ehrhart [3], $|nP \cap \mathbb{Z}^d|$ is given by a polynomial $E_P(n)$ of degree $\dim P$ in n for all integers $n > 0$. The polynomial $E_P(n)$ is called the *Ehrhart polynomial* of P. The *h^*-polynomial* $h^*_P(t) = h^*_0 + h^*_1 t + \cdots + h_d t^d$ of a d-dimensional lattice polytope P encodes the Ehrhart polynomial in a particular basis consisting of binomial coefficients:

$$E_P(n) = h^*_0 \binom{n+d}{d} + h^*_1 \binom{n+d-1}{d} + \cdots + h^*_d \binom{n}{d}.$$

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A fundamental theorem by Stanley [14] states that the coefficients of the h^*-polynomial are always nonnegative integers. It was proved by Hibi [8] that a d-dimensional lattice polytope P is reflexive if and only if its h^*-polynomial is palindromic and has degree d, that is, $h^*_P(t) = t^d h^*_P\left(\frac{1}{t}\right)$ holds.

Consider the polynomial $f(n)$ of degree d expressed in a different basis:

$$f(n) = \sum_{i=0}^{d} a_i n^i (1+n)^{d-i}.$$

If $a_0, \ldots, a_d \geq 0$, then $f(n)$ is said to be *magic positive*. The term "magic positive" was introduced by Ferroni and Higashitani [4]. A common area of study in Ehrhart theory is whether h^*-polynomials are real-rooted. See Section 2 for the real-rootedness of polynomials. The motivation for studying the magic positivity comes from the following theorem.

Theorem 1.1 ([4, Theorem 4.19])**.** *Let P be a lattice polytope of dimension d. If the Ehrhart polynomial of P is magic positive, then we have simultaneously that $E_P(n)$ has positive coefficients and $h^*_P(n)$ is real-rooted.*

Therefore, the following question naturally arises.

Question 1.2. *When are the Ehrhart polynomials of lattice polytopes magic positive?*

It is known that the Ehrhart polynomials of zonotopes [1] and Pitman–Stanley polytopes [6] are magic positive, but no other example seems to be known.

A reflexive polytope P is said to be a *CL-polytope* if all of the complex roots of the Ehrhart polynomial $E_P(n)$ lie on the line $\mathrm{Re}(z) = -\frac{1}{2}$, where $\mathrm{Re}(z)$ denotes the real part of z. The CL-ness of cross polytopes, dual of the Stasheff polytopes, root polytopes of type A, and root polytopes of type C has been studied in [10]. Since the dual of these three polytopes, except for the dual of the Stasheff polytopes, are zonotopes [2], their Ehrhart polynomials are magic positive. However, since Stasheff polytopes are neither zonotopes nor Pitman–Stanley polytopes (see Remark 3.2), it is not immediately clear whether their Ehrhart polynomials are magic positive. The answer to this question is presented in the following first main theorem.

Theorem 1.3. *The Ehrhart polynomials of Stasheff polytopes are magic positive.*

We also study the magic positivity of the Ehrhart polynomials of the dual of symmetric edge polytopes. The result of this study is presented in the following second main theorem.

Theorem 1.4. *We transform $E_{P^*_{C_{d+1}}}(n)$ into the form $E_{P^*_{C_{d+1}}}(n) = \sum_{j=0}^{d} a_j n^j (1+n)^{d-j}$. Then, the coefficients a_i and a_{d-i} for $i = 0, 1, 2$ are positive. Here, $P_{C_{d+1}}$ represents symmetric edge polytopes of cycles of length $d+1$.*

2 Real-rootedness and magic positivity

First, we introduce the concept of real-rootedness of polynomials.

A polynomial $f = \sum_{i=0}^{d} a_i t^i$ of degree d with real coefficients is said to be *real-rooted*, if all its roots are real. If all the coefficients of a real-rooted polynomial are nonnegative, or equivalently, if all its roots are nonpositive, then $a_i^2 \geq a_{i-1}a_{i+1}$ for all i [15]. A sequence a_i of coefficients satisfying these inequalities is called *log-concave*. An immediate consequence is that the nonnegative, log-concave sequence is *unimodal*, meaning $a_0 \leq a_1 \leq \cdots \leq a_k \geq \cdots \geq a_d$ for some k.

We use the following proposition to prove Theorem 1.4.

Proposition 2.1 ([1, Proposition 4.11])**.** *Let P be a d-dimensional lattice polytope and let*

$$E_P(n) = \sum_{i=0}^{d} a_i n^i (1+n)^{d-i}$$

be its Ehrhart polynomial. Then P is reflexive if and only if $a_j = a_{d-j}$ for all j.

We explain why the dual of cross polytopes, and root polytopes of type A and type C have Ehrhart polynomials that are magic positive. Let $\mathbf{e}_i$ denote the ith coordinate unit vector of $\mathbb{R}^d$.

Example 2.2 (Dual of the Cross Polytope)**.** Let Cr_d be the convex hull of $\{\pm\mathbf{e}_i \mid 1 \leq i \leq d\}$. Then Cr_d is a reflexive polytope of dimension d, called the *cross polytope*. Since the dual polytopes of Cr_d and $[-1,1]^d$ are equal, the Ehrhart polynomial of the dual polytopes of Cr_d can be computed as follows:

$$E_{\mathrm{Cr}_d^*}(n) = (2n+1)^d = \sum_{i=0}^{d} \binom{d}{i} (n+1)^{d-i} n^i.$$

Therefore $E_{\mathrm{Cr}_d^*}(n)$ is magic positive.

Example 2.3 (Dual of the Root Polytope of Type A)**.** Let $\mathbf{A}_d$ be the convex hull of $\{\pm\mathbf{e}_i \mid 1 \leq i \leq d\} \cup \{\pm(\mathbf{e}_i + \cdots + \mathbf{e}_j) \mid 1 \leq i < j \leq d\}$. Then $\mathbf{A}_d$ is a reflexive polytope of dimension d, called the *root polytope of type A*. The Ehrhart polynomial of dual polytope of $\mathbf{A}_d$ is calculated in [10, Lemma 5.3] as follows:

$$E_{\mathbf{A}_d^*}(n) = \sum_{k=0}^{d} \binom{d+1}{k} n^k = \sum_{i=0}^{d} (n+1)^{d-i} n^i.$$

Therefore $E_{\mathbf{A}_d^*}(n)$ is magic positive.

Example 2.4 (Dual of the Root Polytope of Type C)**.** Let $\mathbf{C}_d$ be the convex hull of $\{\pm\mathbf{e}_i \mid 1 \leq i \leq d\} \cup \{\pm(\mathbf{e}_i + \cdots + \mathbf{e}_{j-1}) \mid 1 \leq i < j \leq d\} \cup \{\pm(2\mathbf{e}_i + \cdots + 2\mathbf{e}_{d-1} + \mathbf{e}_d) \mid 1 \leq i \leq d-1\}$. Then $\mathbf{C}_d$ is a reflexive polytope of dimension d, called the *root polytope of type C*. The Ehrhart polynomial of the dual polytope of $\mathbf{C}_d$ is calculated in [11, Theorem 1.1] as follows:

$$E_{\mathbf{C}_d^*}(n) = (n+1)^d + n^d.$$

From this indication, it is clear that $E_{\mathbf{C}_d^*}(n)$ is magic positive.

3 The magic positivity of the Ehrhart polynomials of the Stasheff polytopes

Let St_d be the convex hull of $\{\pm\mathbf{e}_i : 1 \leq i \leq d\} \cup \{\mathbf{e}_i + \cdots + \mathbf{e}_j : 1 \leq i < j \leq d\}$. Then St_d is a reflexive polytope of dimension d. This polytope is the dual polytope of the so-called *Stasheff polytope* (*associahedron*). For more detailed information, see, e.g., [7].

For the proof of Theorem 1.3, we use the following theorem.

Theorem 3.1. *For $d \geq 2$, $E_{\mathrm{St}_d^*}(n)$ satisfies the following recurrence:*

$$E_{\mathrm{St}_d^*}(n) = (2n+1)E_{\mathrm{St}_{d-1}^*}(n) - \frac{1}{2}n(n+1)E_{\mathrm{St}_{d-2}^*}(n). \tag{3.1}$$

Proof. First, we define some sets. By the definition of the Stasheff polytope,

$$\mathrm{St}_d^* = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid -1 \leq x_i \leq 1, x_j + \cdots + x_k \leq 1 (1 \leq i \leq d, 1 \leq j < k \leq d)\},$$

$$\mathrm{St}_{d-1}^* \times [-1,1] = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid -1 \leq x_i \leq 1, x_j + \cdots + x_k \leq 1 (1 \leq i \leq d, 1 \leq j < k \leq d-1)\}.$$

Let

$$\mathcal{A} = (\mathrm{St}_{d-1}^* \times [-1,1]) \backslash \mathrm{St}_d^*.$$

For integers m with $-n \leq m \leq n$, let

$$A_m = \left(\bigcup_{i=1}^{d-2} (n\mathrm{St}_{d-2}^* \cap \{(x_1, \ldots, x_{d-2}) \in \mathbb{R}^{d-2} \mid x_i + \cdots + x_{d-2} = m\}) \right) \backslash \bigcup_{i=m+1}^{n} A_i,$$

$$\Delta_m = \{(x_{d-1}, x_d) \in \mathbb{R}^2 \mid x_{d-1} \leq n - \max\{m, 0\}, x_d \leq n, x_{d-1} + x_d > n - \max\{m, 0\}\}.$$

The following equation can be easily verified:

$$E_{\mathrm{St}_{d-1}^* \times [-1,1]}(n) = (2n+1)E_{\mathrm{St}_{d-1}^*}(n). \tag{3.2}$$

Since $\mathrm{St}_d^* \subset \mathrm{St}_{d-1}^* \times [-1,1]$, by (3.2)

$$\left| n\mathcal{A} \cap \mathbb{Z}^d \right| = (2n+1)E_{\mathrm{St}_{d-1}^*}(n) - E_{\mathrm{St}_d^*}(n). \tag{3.3}$$

Since Δ_m and $\{(x,y) \in \mathbb{R}^2 \mid x \geq 0, y \geq 0, x+y < n\}$ are unimodularly equivalent, we have the following equation:

$$|\Delta_m \cap \mathbb{Z}^2| = \frac{1}{2}n(n+1). \tag{3.4}$$

<ins>The first step</ins>: We prove the following equality:

$$\bigsqcup_{-n \leq m \leq n} A_m \cap \mathbb{Z}^{d-2} = n\mathrm{St}_{d-2}^* \cap \mathbb{Z}^{d-2}. \tag{3.5}$$

Since $\bigsqcup_{-n \leq m \leq n} A_m \cap \mathbb{Z}^{d-2} \subset n\mathrm{St}_{d-2}^* \cap \mathbb{Z}^{d-2}$ is clear, we will prove $\bigsqcup_{-n \leq m \leq n} A_m \cap \mathbb{Z}^{d-2} \supset n\mathrm{St}_{d-2}^* \cap \mathbb{Z}^{d-2}$. For $x = (x_1, \ldots, x_{d-2}) \in n\mathrm{St}_{d-2}^* \cap \mathbb{Z}^{d-2}$, we define the value p as follows:

$$p = \max\{x_i + \cdots + x_{d-2} \mid 1 \leq i \leq d-2\}.$$

Then, since $-n \leq p \leq n$ holds, and $x \in A_p \cap \mathbb{Z}^{d-2}$ follows from the definition of A_m. Therefore, we get (3.5).

From (3.4) and (3.5), we have the following equation:

$$\left| \bigsqcup_{-n \leq m \leq n} (A_m \times \Delta_m) \cap \mathbb{Z}^d \right| = \frac{1}{2}n(n+1)E_{\mathrm{St}_{d-2}^*}(n). \tag{3.6}$$

<ins>The second step</ins>: Next, we prove the following equality:

$$n\mathcal{A} \cap \mathbb{Z}^d = \bigsqcup_{-n \leq m \leq n} (A_m \times \Delta_m) \cap \mathbb{Z}^d. \tag{3.7}$$

Let $x = (x_1, \ldots, x_d) \in n\mathcal{A} \cap \mathbb{Z}^d$. Then, $x_d \leq n$ holds, and since $(x_1, \ldots, x_{d-2}) \in n\mathrm{St}_{d-2}^* \cap \mathbb{Z}^{d-2}$, that is, $(x_1, \ldots, x_{d-2}) \in \bigsqcup_{-n \leq m \leq n} A_m \cap \mathbb{Z}^{d-2}$ holds by (3.5), there exists an integer $-n \leq k \leq n$ such that $(x_1, \ldots, x_{d-2}) \in A_k$.

Now, suppose $x_{d-1} > n - \max\{k, 0\}$. From the first step, since there exists $1 \leq i' \leq d-2$ such that $x_{i'} + \cdots + x_{d-2} = \max\{k, 0\}$, we have $x_{i'} + \cdots + x_{d-1} > n$, implying $x \notin n(\mathrm{St}_{d-1}^* \times [-1,1]) \cap \mathbb{Z}^d$, which contradicts $x \in n\mathcal{A}$. Therefore, we conclude that $x_{d-1} \leq n - \max\{k, 0\}$.

Next, suppose $x_{d-1} + x_d \leq n - \max\{k, 0\}$. From $(x_1, \ldots, x_{d-2}) \in n\mathrm{St}_{d-2}^* \cap \mathbb{Z}^{d-2}$, we know that $x_i + \cdots + x_{d-2} \leq \max\{k, 0\}$ for all $1 \leq i \leq d-2$. Consequently, $x_i + \cdots + x_d \leq n$ for all $1 \leq i \leq d$, implying $x \in n\mathrm{St}_d^* \cap \mathbb{Z}^d$, which contradicts $x \in n\mathrm{St}_d^* \cap \mathbb{Z}^d$. Thus, we deduce $x_{d-1} + x_d > n - \max\{k, 0\}$. Therefore, we have $x \in \bigsqcup_{-n \leq m \leq n}(A_m \times \Delta_m) \cap \mathbb{Z}^d$.

Now, let $x = (x_1, \dots, x_d) \in \bigsqcup_{-n \le m \le n} (A_m \times \Delta_m) \cap \mathbb{Z}^d$. Then there exists $-n \le k \le n$ such that $(x_1, \dots, x_d) \in A_k \times \Delta_k$. Since $x_i + \cdots + x_{d-2} \le \max\{k, 0\}$ holds for all $1 \le i \le d-2$ by $(x_1, \dots, x_{d-2}) \in A_k$, we have $x_i + \cdots + x_{d-1} \le n$. Additionally, since there exists $1 \le i' \le d-2$ such that $x_{i'} + \cdots + x_{d-2} = \max\{k, 0\}$ and $x_{d-1} + x_d > n - \max\{k, 0\}$ by $(x_{d-1}, x_d) \in \Delta_k$, we conclude $x_{i'} + \cdots + x_d > n$. Therefore, $x \in n\mathcal{A} \cap \mathbb{Z}^d$.

The third step: From (3.3) and (3.7), we have the following equation:

$$\left| \bigsqcup_{-n \le m \le n} (A_m \times \Delta_m) \cap \mathbb{Z}^d \right| = (2n+1) E_{\mathrm{St}^*_{d-1}}(n) - E_{\mathrm{St}^*_d}(n).$$

From this equation and (3.6), we get (3.1). □

By using Theorem 3.1, we can prove Theorem 1.3 without explicitly determining $E_{\mathrm{St}^*_d}(n)$.

Proof of Theorem 1.3. Transform (3.1) into the following form:

$$E_{\mathrm{St}^*_d}(n) - \frac{1}{2} n E_{\mathrm{St}^*_{d-1}}(n) = \frac{1}{2} n E_{\mathrm{St}^*_{d-1}}(n) + (n+1) \left(E_{\mathrm{St}^*_{d-1}}(n) - \frac{1}{2} n E_{\mathrm{St}^*_{d-2}}(n) \right).$$

By the recurrence relation, we observe that if both $E_{\mathrm{St}^*_{d-1}}(n) - \frac{1}{2} n E_{\mathrm{St}^*_{d-2}}(n)$ and $\frac{1}{2} n E_{\mathrm{St}^*_{d-1}}(n)$ are magic positive, so are $E_{\mathrm{St}^*_d}(n) - \frac{1}{2} n E_{\mathrm{St}^*_{d-1}}(n)$ and $E_{\mathrm{St}^*_d}(n)$. In other words, it suffices to show that $E_{\mathrm{St}^*_1}(n)$ and $E_{\mathrm{St}^*_1}(n) - \frac{1}{2} n E_{\mathrm{St}^*_0}(n)$ are magic positive. In fact, we have $E_{\mathrm{St}^*_0}(n) = 1$, $E_{\mathrm{St}^*_1}(n) = 2n+1 = (n+1) + n$ and $E_{\mathrm{St}^*_1}(n) - \frac{1}{2} n E_{\mathrm{St}^*_0}(n) = (n+1) + \frac{1}{2} n$, as required. Therefore, $E_{\mathrm{St}^*_d}(n)$ is magic positive. □

The following table shows examples of the Ehrhart polynomials of the Stasheff polytope in dimensions 2, 3, 4, and 5, where it can be confirmed that they are magic positive.

$E_{\mathrm{St}^*_2}(n)$	$=$	$\frac{7}{2}n^2 + \frac{7}{2}n + 1$
	$=$	$(n+1)^2 + \frac{3}{2}(n+1)n + n^2$
$E_{\mathrm{St}^*_3}(n)$	$=$	$6n^3 + 9n^2 + 5n + 1$
	$=$	$(n+1)^3 + 2(n+1)^2 n + 2(n+1)n^2 + n^3$
$E_{\mathrm{St}^*_4}(n)$	$=$	$\frac{41}{4}n^4 + \frac{41}{2}n^3 + \frac{67}{4}n^2 + \frac{13}{2}n + 1$
	$=$	$(n+1)^4 + \frac{5}{2}(n+1)^3 n + \frac{13}{4}(n+1)^2 n^2 + \frac{5}{2}(n+1)n^3 + n^4$
$E_{\mathrm{St}^*_5}(n)$	$=$	$\frac{35}{2}n^5 + \frac{175}{4}n^4 + 47n^3 + \frac{107}{4}n^2 + 8n + 1$
	$=$	$(n+1)^5 + 3(n+1)^4 n + \frac{19}{4}(n+1)^3 n^2 + \frac{19}{4}(n+1)^2 n^3 + 3(n+1)n^4 + n^5$

Remark 3.2. From [16, Theorem 2.2], the coefficients of the Ehrhart polynomial of a zonotope are all integers. On the other hand, since the coefficients of the Ehrhart polynomial of a Stasheff polytope are generally rational numbers, the Stasheff polytope is never a zonotope. Additionally, the number of facets of the Pitman–Stanley polytope is $2d$ [17], while the number of facets of the Stasheff polytope is $2d + \frac{1}{2}d(d-1)$. Therefore, the Stasheff polytope is never a Pitman–Stanley polytope.

4 The magic positivity of the Ehrhart polynomials of the dual of the symmetric edge polytopes

In this section, we provide the background that led us to consider Theorem 1.4.

Let G be a finite simple graph on the vertex set $[d] := \{1, \dots, d\}$ and the edge set $E(G)$. The *symmetric edge polytope* $P_G \subset \mathbb{R}^d$ is the convex hull of the set

$$\{\pm(\mathbf{e}_v - \mathbf{e}_w) \in \mathbb{R}^d \mid vw \in E(G)\}.$$

Here, the vectors $\mathbf{e}_v$ are elements that form a lattice basis of $\mathbb{Z}^d$. It is known in [13] that P_G is unimodularly equivalent to a reflexive polytope. For more contexts on symmetric edge polytopes, see e.g. [9, 12].

We begin by examining the cases of tree T_d and complete graph K_d with d vertices.

Proposition 4.1. *The Ehrhart polynomials of the dual of the symmetric edge polytopes of trees and complete graphs are magic positive.*

Proof. From [12, Proposition 4.6], $P_{T_{d+1}}$ and Cr_d are unimodularly equivalent. Therefore, by Example 2.2, the Ehrhart polynomials of $P^*_{T_{d+1}}$ are magic positive.

Moreover, $P_{K_{d+1}}$ and $\mathbf{A}_d$ are unimodularly equivalent. Therefore, by Example 2.3, the Ehrhart polynomials of $P^*_{K_{d+1}}$ are magic positive. □

For any connected graph G with d vertices, the following inclusions hold:

$$P^*_{K_d} \subset P^*_G \subset P^*_{T_d}.$$

From these relations and Proposition 4.1, the following question naturally arises.

Question 4.2. *Are the Ehrhart polynomials of the dual polytopes of symmetric edge polytopes of any connected graphs magic positive?*

We expected this question to hold, but through computational experiments, we found several counterexamples.

Example 4.3. For instance, $E_{P^*_{K_{3,7}}}(n)$ is not magic positive where $K_{a,b}$ denotes the complete bipartite graph. Specifically, $E_{P^*_{K_{3,7}}}(n)$ is computed as follows:

$$\begin{aligned}
E_{P^*_{K_{3,7}}}(n) &= \frac{128}{3}n^9 + 192n^8 + 448n^7 + 672n^6 + \frac{3528}{5}n^5 + 532n^4 + \frac{820}{3}n^3 + 86n^2 + \frac{72}{5}n + 1 \\
&= (n+1)^9 + \frac{27}{5}(n+1)^8 n + \frac{34}{5}(n+1)^7 n^2 - \frac{142}{15}(n+1)^6 n^3 + \frac{88}{5}(n+1)^5 n^4 \\
&+ \frac{88}{5}(n+1)^4 n^5 - \frac{142}{15}(n+1)^3 n^6 + \frac{34}{5}(n+1)^2 n^7 + \frac{27}{5}(n+1) n^8 + n^9.
\end{aligned}$$

The following table marks $\circ$ when $E_{P^*_{K_{a,b}}}(n)$ is magic positive and $\times$ when it is not.

	2	3	4	5	6	7	8	9	10	11
2	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\times$	$\times$	$\times$
3	$\circ$	$\circ$	$\circ$	$\circ$	$\circ$	$\times$	$\times$	$\times$	$\times$	
4	$\circ$	$\circ$	$\circ$	$\circ$	$\times$	$\times$	$\times$	$\times$		
5	$\circ$	$\circ$	$\circ$	$\times$	$\times$	$\times$	$\times$			
6	$\circ$	$\circ$	$\times$	$\times$	$\times$	$\times$				
7	$\circ$	$\times$	$\times$	$\times$	$\times$					
8	$\circ$	$\times$	$\times$	$\times$						
9	$\times$	$\times$	$\times$							
10	$\times$	$\times$								
11	$\times$									

Except for the case of $K_{2,8}$, when the number of vertices in a graph exceeds 10, the Ehrhart polynomial of a dual of the symmetric edge polytopes tends not to be magic positive. However, the h^*-polynomial of a dual of the symmetric edge polytope whose Ehrhart polynomial in the above table is not magic is real-rooted.

Example 4.4. For example, $E_{P^*_{K_{10}\setminus\{e\}}}(n)$ is not magic positive where e is an edge of K_{10}. $E_{P^*_{K_{10}\setminus\{e\}}}(n)$ is calculated as follows:

$$\begin{aligned}
E_{P^*_{K_{10}\setminus\{e\}}}(n) &= \frac{92}{9}n^9 + 46n^8 + \frac{364}{3}n^7 + 210n^6 + \frac{3766}{15}n^5 + 210n^4 + \frac{1084}{9}n^3 + 45n^2 + \frac{149}{15}n + 1 \\
&= (n+1)^9 + \frac{14}{15}(n+1)^8 n + \frac{23}{15}(n+1)^7 n^2 - \frac{19}{45}(n+1)^6 n^3 + \frac{31}{15}(n+1)^5 n^4 \\
&+ \frac{31}{15}(n+1)^4 n^5 - \frac{19}{45}(n+1)^3 n^6 + \frac{23}{15}(n+1)^2 n^7 + \frac{14}{15}(n+1) n^8 + n^9.
\end{aligned}$$

Examples 4.3 and 4.4 demonstrate that there exists an Ehrhart polynomial of the dual of the symmetric edge polytope of a graph close to a complete graph that is not magic positive. So, what happens to the magic positivity of the Ehrhart polynomial of the dual of the symmetric edge polytope for graphs that are close to trees?

Question 4.5. *Are the Ehrhart polynomials of dual polytopes of symmetric edge polytopes of cycles magic positive?*

We computed the Ehrhart polynomial of the dual of the symmetric edge polytopes of cycles and partially resolved its magic positivity.

Proposition 4.6. *Let C_d denote a cycle of length $d \geq 2$. Then we have*

$$E_{P^*_{C_{d+1}}}(n) = \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} (-1)^i \binom{(d+1-2i)n + (d-i)}{d} \binom{d+1}{i}. \tag{4.1}$$

Proof. By the definition of symmetric edge polytope, $P_{C_{d+1}}$ is the convex hull of $\{\pm(e_i - e_{i+1}) \mid 1 \leq i \leq d-1\} \cup \{\pm(e_d - e_1)\}$. Since $P_{C_{d+1}}$ and the convex hull of $\{\pm e_i \mid 1 \leq i \leq d-1\} \cup \{\pm(e_1 + \cdots + e_d)\}$ are unimodularly equivalent, we obtain the following equality:

$$P^*_{C_{d+1}} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid -1 \leq x_i \leq 1, -1 \leq x_1 + \cdots + x_d \leq 1 (1 \leq i \leq d-1)\}.$$

Using [5, Proposition 2.2 and Theorem 2.5], we obtain

$$E_{P^*_{C_{d+1}}}(n) = \sum_{i=0}^{d} (-1)^i \sum_{v=0}^{d} \binom{(d+1-v)n + (d-i)}{d} \rho_{\mathbf{c},i}(v)$$

where $\mathbf{c} = (2, \ldots, 2) \in \mathbb{Z}^{d+1}_{\geq 0}$ and

$$\rho_{\mathbf{c},i}(v) := \# \left\{ I \in \binom{[d+1]}{i} : \sum_{j \in I} c_j = v \right\} = \begin{cases} \binom{d+1}{i} & \text{if } v = 2i, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we have (4.1). □

By using Proposition 4.6, we can prove Theorem 1.4.

The following table shows examples of the Ehrhart polynomials of the dual of the symmetric edge polytopes of cycles in dimensions 2, 3, 4, and 5, where it can be confirmed that they are magic positive.

$E_{P^*_{C_3}}(n)$	$=$	$3n^2 + 3n + 1$
	$=$	$(n+1)^2 + (n+1)n + n^2$
$E_{P^*_{C_4}}(n)$	$=$	$\frac{16}{3}n^3 + 8n^2 + \frac{14}{3}n + 1$
	$=$	$(n+1)^3 + \frac{5}{3}(n+1)^2 n + \frac{5}{3}(n+1)n^2 + n^3$
$E_{P^*_{C_5}}(n)$	$=$	$\frac{115}{12}n^4 + \frac{115}{6}n^3 + \frac{185}{12}n^2 + \frac{35}{6}n + 1$
	$=$	$(n+1)^4 + \frac{11}{6}(n+1)^3 n + \frac{47}{12}(n+1)^2 n^2 + \frac{11}{6}(n+1)n^3 + n^4$
$E_{P^*_{C_6}}(n)$	$=$	$\frac{88}{5}n^5 + 44n^4 + 46n^3 + 25n^2 + \frac{37}{5}n + 1$
	$=$	$(n+1)^5 + \frac{12}{5}(n+1)^4 n + \frac{27}{5}(n+1)^3 n^2 + \frac{27}{5}(n+1)^2 n^3 + \frac{12}{5}(n+1)n^4 + n^5$

By computational experiments, $E_{C_d^*}(n)$ is magic positive for all $d \leq 500$.

Remark 4.7. By a similar argument to Remark 3.2, we see that $P^*_{C_d}$ is never a zonotope. Additionally, since the number of facets of $P^*_{C_d}$ is $2d+2$, we see that it is also never a Pitman–Stanley polytope.

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