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q-deformation of graphic arrangements

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Abstract. We first observe a mysterious similarity between the braid arrangement and the arrangement of all hyperplanes in a vector space over the finite field $\mathbb{F}_q$. These two arrangements are defined by the determinants of the Vandermonde and the Moore matrix, respectively. These two matrices are transformed to each other by replacing a natural number n with q^n (q-deformation).

In this paper, we introduce the notion of “”q-deformation of graphic arrangements” as certain subarrangements of the arrangement of all hyperplanes over $\mathbb{F}_q$. This new class of arrangements extends the relationship between the Vandermonde and Moore matrices to graphic arrangements. We show that many invariants of the “q-deformation” behave as “q-deformation” of invariants of the graphic arrangements. Such invariants include the characteristic (chromatic) polynomial, the Stirling number of the second kind, freeness, exponents, basis of logarithmic vector fields, etc.

Keywords: Graphic arrangements, chromatic polynomial, q-analogue, freeness, finite fields

1 Introduction

1.1 Mysterious similarities

A central arrangement $\mathcal{A}$ is a finite collection of linear hyperplanes in a finite dimensional vector space. Define the **intersection lattice** $L(\mathcal{A})$ and the **characteristic polynomial** $\chi(\mathcal{A}, t)$ by

$$L(\mathcal{A}) := \left\{ \bigcap_{H \in \mathcal{B}} H \;\middle|\; \mathcal{B} \subseteq \mathcal{A} \right\}, \qquad \chi(\mathcal{A}, t) := \sum_{X \in L(\mathcal{A})} \mu(X) t^{\dim X},$$

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where $L(\mathcal{A})$ is ordered by the reverse inclusion and μ denotes the **Möbius function** on the lattice $L(\mathcal{A})$.

A typical example of an arrangement is the **braid arrangement** $\mathcal{B}_\ell$ in $\mathbb{R}^\ell$ whose defining polynomial is the **Vandermonde determinant**, i.e.,

$$Q(\mathcal{B}_\ell) = \prod_{1 \le i < j \le \ell} (x_j - x_i) = \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{\ell-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{\ell-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_\ell & x_\ell^2 & \dots & x_\ell^{\ell-1} \end{vmatrix}.$$

The characteristic polynomial of the braid arrangement $\mathcal{B}_\ell$ is

$$\chi(\mathcal{B}_\ell, t) = t(t-1)(t-2)\cdots(t-\ell+1).$$

There are mysterious similarities between the braid arrangements and the arrangements consisting of all hyperplanes in vector spaces over finite fields. Let q be a prime power and $\mathbb{F}_q$ the finite field of order q. Define the arrangement $\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)$ as the set of all hyperplanes in $\mathbb{F}_q^\ell$. Its defining polynomial is the determinant of **Moore matrix**, i.e.,

$$Q\left(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)\right) = \prod_{i=1}^{\ell} \prod_{c_1,\dots,c_{i-1} \in \mathbb{F}_q} (c_1 x_1 + \dots + c_{i-1} x_{i-1} + x_i) = \begin{vmatrix} x_1 & x_1^q & x_1^{q^2} & \dots & x_1^{q^{\ell-1}} \\ x_2 & x_2^q & x_2^{q^2} & \dots & x_2^{q^{\ell-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ x_\ell & x_\ell^q & x_\ell^{q^2} & \dots & x_\ell^{q^{\ell-1}} \end{vmatrix}.$$

The characteristic polynomial of $\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)$ is

$$\chi\left(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell), t\right) = (t-1)(t-q)(t-q^2)\cdots(t-q^{\ell-1}).$$

By formally replacing q^k in the expressions of $Q\left(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)\right)$ and $\chi\left(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell), t\right)$ with k, we obtain the expressions for $Q(\mathcal{B}_\ell)$ and $\chi(\mathcal{B}_\ell, t)$. A similar phenomenon can be observed in the context of freeness of arrangements. An arrangement $\mathcal{A}$ is said to be **free** if the module of logarithmic polynomial vector fields $D(\mathcal{A})$ is a free module over the polynomial ring (See [8] for details). Both of $Q(\mathcal{B}_\ell)$ and $\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)$ are free with bases

$$\sum_{i=1}^{\ell} x_i^k \partial_i \ (0 \le k \le \ell - 1) \text{ for } D(\mathcal{B}_\ell) \text{ and } \sum_{i=1}^{\ell} x_i^{q^k} \partial_i \ (0 \le k \le \ell - 1) \text{ for } D\left(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)\right),$$

where ∂_i denotes $\frac{\partial}{\partial x_i}$ for each $i \in \{1, \dots, \ell\}$ (See [8, Example 4.22 and 4.24]).

Moreover, there are mysterious similarities for subarrangements. Let G be a simple graph on $[\ell] = \{1, \ldots, \ell\}$. Define the **graphic arrangement** $\mathcal{A}_G$ in $\mathbb{R}^\ell$ by

$$\mathcal{A}_G := \left\{ \{x_i - x_j = 0\} \mid \{i, j\} \in E_G \right\}.$$

Note that every subarrangement of $\mathcal{B}_\ell$ is of the form $\mathcal{A}_G$ and it is well known that the chromatic polynomial $\chi(G, t)$ coincides with the characteristic polynomial $\chi(\mathcal{A}_G, t)$. The following proposition for the chromatic polynomial is trivial by definition.

Proposition 1.1. *Suppose $\chi(G, k) = 0$ for some $k \in \mathbb{Z}_{\geq 0}$. Then $\chi(G, j) = 0$ for $0 \leq j \leq k$.*

There is a q-version of Proposition 1.1.

Proposition 1.2 ([12, Lemma 7])**.** *Let $\mathcal{A}$ be an arrangement in $\mathbb{F}_q^\ell$. If $\chi(\mathcal{A}, q^k) = 0$ for some $k \in \mathbb{Z}_{\geq 0}$, then $\chi(\mathcal{A}, q^j) = 0$ for any $0 \leq j \leq k$.*

Proof. Let $\mathcal{A} \otimes \mathbb{F}_q^k$ denote the subspace arrangement in $(\mathbb{F}_q^k)^\ell$ defined by

$$\mathcal{A} \otimes \mathbb{F}_q^k := \left\{ H \otimes_{\mathbb{F}_q} \mathbb{F}_q^k \;\middle|\; H \in \mathcal{A} \right\}.$$

Then the intersection lattices $L(\mathcal{A})$ and $L(\mathcal{A} \otimes \mathbb{F}_q^k) = \left\{ X \otimes_{\mathbb{F}_q} \mathbb{F}_q^k \;\middle|\; X \in L(\mathcal{A}) \right\}$ are naturally isomorphic. By [3, Proposition 3.1],

$$\chi(\mathcal{A}, q^k) = \chi(\mathcal{A} \otimes \mathbb{F}_q^k, q^k) = \# \left((\mathbb{F}_q^k)^\ell \setminus \bigcup_{H \in \mathcal{A}} H \otimes_{\mathbb{F}_q} \mathbb{F}_q^k \right).$$

Thus, if $\chi(\mathcal{A}, q^k) = 0$ and $0 \leq j \leq k$, then $\chi(\mathcal{A}, q^j) = 0$. □

We define the **falling factorial** $t^{\underline{i}}$ for each $i \in \mathbb{Z}_{>0}$ by $t^{\underline{i}} := t(t-1)\cdots(t-i+1)$.

Proposition 1.3 ([10, Theorem 15])**.** *Suppose $\chi(G, t) = \sum_{i=1}^{\ell} c_i t^{\underline{i}}$. Then c_i coincides with the number of stable partitions of G into i blocks, where a stable partition of G is a set partition of the vertex set such that no edge connects vertices within the same block. In other words, c_i coincides with the number of i-dimensional subspaces in $L(\mathcal{B}_\ell)$ that are not contained in any hyperplanes in $\mathcal{A}_G$.*

Note that the falling factorial $t^{\underline{i}}$ coincides with the characteristic polynomial $\chi(\mathcal{B}_\ell, t)$. Define the polynomial $t_q^{\underline{i}}$ by

$$t_q^{\underline{i}} := \chi(\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^i), t) = (t-1)(t-q)\cdots(t-q^{i-1}).$$

The following is a q-version of Proposition 1.3.

Proposition 1.4. *Let $\mathcal{A}$ be an arrangement in $\mathbb{F}_q^\ell$ and suppose $\chi(\mathcal{A}, t) = \sum_{i=0}^{\ell} c_i t_q^{\underline{i}}$. Then c_i is the number of i-dimensional subspaces in $\mathbb{F}_q^\ell$ that are not contained in any hyperplanes in $\mathcal{A}$.*

Proof. We proceed by double induction on ℓ and $|\mathcal{A}_{\text{all}}(\mathbb{F}_q^\ell) \setminus \mathcal{A}|$. When $\ell = 1$,

$$\begin{aligned} t = (t-1) + 1 &= t_q^{\underline{1}} + t_q^{\underline{0}} && \text{(for the empty arrangement)}, \\ t - 1 &= t_q^{\underline{1}} && \text{(for the single-point arrangement)}. \end{aligned}$$

Therefore the claim is true.

Suppose that $\ell \geq 2$. Since $\chi(\mathcal{A}_{\text{all}}(\mathbb{F}_q^\ell), t) = t_q^{\underline{\ell}}$, the assertion is true for $\mathcal{A} = \mathcal{A}_{\text{all}}(\mathbb{F}_q^\ell)$. Assume that $\mathcal{A} \subsetneq \mathcal{A}_{\text{all}}(\mathbb{F}_q^\ell)$ and $H \in \mathcal{A}$. Then, by the deletion-restriction formula [8, Corollary 2.57], we have

$$\chi(\mathcal{A}', t) = \chi(\mathcal{A}, t) + \chi(\mathcal{A}^H, t) = \sum_{i=0}^{\ell} c_i t_q^{\underline{i}} + \sum_{i=0}^{\ell-1} d_i t_q^{\underline{i}} = t_q^{\underline{\ell}} + \sum_{i=0}^{\ell-1} (c_i + d_i) t_q^{\underline{i-1}}.$$

By the induction hypothesis

$$\begin{aligned} c_i &= \# \left\{ X \in L(\mathcal{A}_{\text{all}}(\mathbb{F}_q^\ell)) \;\middle|\; \dim X = i, X \not\subseteq K \text{ for any } K \in \mathcal{A} \right\}, \\ d_i &= \# \{ X \in L(\mathcal{A}_{\text{all}}(H)) \mid \dim X = i, X \not\subseteq H \cap K \text{ for any } K \in \mathcal{A} \setminus \{H\} \}. \end{aligned}$$

Since

$$\begin{aligned} &\left\{ X \in L(\mathcal{A}_{\text{all}}(\mathbb{F}_q^\ell)) \;\middle|\; \dim X = i, X \not\subseteq K \text{ for any } K \in \mathcal{A} \setminus \{H\} \right\} \\ &\qquad = \left\{ X \in L(\mathcal{A}_{\text{all}}(\mathbb{F}_q^\ell)) \;\middle|\; \dim X = i, X \not\subseteq K \text{ for any } K \in \mathcal{A} \right\} \\ &\qquad\quad \sqcup \{ X \in L(\mathcal{A}_{\text{all}}(H)) \mid \dim X = i, X \not\subseteq H \cap K \text{ for any } K \in \mathcal{A} \setminus \{H\} \}, \end{aligned}$$

the claim follows. □

Example 1.5. Consider the empty arrangement in $\mathbb{F}_q^\ell$. Then we have

$$t^\ell = \sum_{i=0}^{\ell} \binom{\ell}{i}_q t_q^{\underline{i}},$$

where

$$\binom{\ell}{i}_q = \frac{[\ell]_q [\ell-1]_q \cdots [\ell-i+1]_q}{[i]_q [i-1]_q \cdots [1]_q}$$

is the q-binomial coefficient and $[k]_q = \frac{q^k - 1}{q - 1}$ denotes the q-integer.

Example 1.6. Consider the Boolean arrangement in $\mathbb{F}_q^\ell$, which consists of all coordinate hyperplanes. Then we have

$$(t-1)^\ell = \sum_{i=0}^{\ell} (q-1)^{\ell-i} S_q(\ell, i) t_q^{\underline{i}},$$

where $S_q(\ell, i)$ denotes the q-Stirling number of the second kind defined by the following recurrence formula.

$$S_q(\ell, i) = S_q(\ell-1, i-1) + [i]_q S_q(\ell-1, i), \qquad S_q(0, i) = \delta_{0,i}.$$

This is a q-version of the following well-known formula for the Stirling number of the second kind.

$$t^\ell = \sum_{i=0}^{\ell} S(\ell, i) t^{\underline{i}}, \qquad S(\ell, i) = S(\ell-1, i-1) + iS(\ell-1, i).$$

1.2 q-deformation of graphic arrangements

We will focus on specific subarrangements of $\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)$ that arise from simple graphs. Note that the braid arrangement $\mathcal{B}_\ell$ corresponds to the graphic arrangement associated with the complete graph K_ℓ. Thus, it seems natural to assign K_ℓ to $\mathcal{A}_{\mathrm{all}}(\mathbb{F}_q^\ell)$. Furthermore, it seems natural to assign every clique of G to to the set of all hyperplanes using the coordinates corresponding to the vertices in the clique.

Definition 1.7. We define a **q-deformation of graphic arrangement** $\mathcal{A}_G^q$ in $\mathbb{F}_q^\ell$ as follows.

$$\mathcal{A}_G^q := \bigcup_{\{i_1,\dots,i_r\}} \left\{ \{a_{i_1}x_{i_1} + \cdots + a_{i_r}x_{i_r} = 0\} \;\middle|\; (a_{i_1}, \dots, a_{i_r}) \in \mathbb{F}_q^r \setminus \{0\} \right\},$$

where $\{i_1, \dots, i_r\}$ runs over all cliques of G.

Besides this definition, Athanasiadis [1], and later Postnikov and Stanley [9], also defined deformations of Coxeter arrangements as a generalization of generic arrangements, Linial arrangements, Shi arrangements, Catalan arrangements, etc. Their definition involves copies of translated hyperplanes, whereas the q-deformation focuses on the coefficients of variables and the choice of linear polynomials.

It is expected that $\mathcal{A}_G^q$ has a lot of properties similar to the graphic arrangement $\mathcal{A}_G$. The organization of this paper is as follows. In Section 2, we will show that the characteristic polynomial of $\mathcal{A}_G^q$ determines the chromatic polynomial G when q is large enough (Theorem 2.1 and Corollary 2.2) and we will describe a relationship between the characteristic polynomial and the numbers of stable partitions of G (Theorem 2.4). In Section 3, we will show that $\mathcal{A}_G^q$ is free if and only if G is chordal as in the case of graphic arrangements (Theorem 3.2) and in Section 4, we construct an explicit basis for $D(\mathcal{A}_G^q)$ for a chordal graph G (Theorem 4.2).

2 Characteristic polynomial

Theorem 2.1. *For any $k \in \mathbb{Z}_{\geq 0}$,*

$$\frac{\chi(\mathcal{A}_G^q, q^k)}{(q-1)^\ell} \equiv \chi(G,k) \pmod{q-1}.$$

Proof. Let $V = \mathbb{F}_q^k$. Define

$$\underline{\chi}_G^q(V) := \left\{ (v_1, \dots, v_\ell) \in V^\ell \;\middle|\; \begin{array}{l} \{v_{i_1}, \dots, v_{i_p}\} \text{ is linearly independent} \\ \text{over } \mathbb{F}_q \text{ if } \{i_1, \dots, i_p\} \text{ is a clique of } G. \end{array} \right\}.$$

Note that $\chi(\mathcal{A}_G^q, q^k) = \#\underline{\chi}_G^q(V)$. The group $(\mathbb{F}_q^\times)^\ell$ acts on $\underline{\chi}_G^q(V)$ by

$$(a_1, \dots, a_\ell) \cdot (v_1, \dots, v_\ell) := (a_1 v_1 \dots, a_\ell v_\ell).$$

Therefore $\#\underline{\chi}_G^q(\mathbb{P}(V)) = \frac{\chi(\mathcal{A}_G^q, q^k)}{(q-1)^\ell}$, where

$$\underline{\chi}_G^q(\mathbb{P}(V)) := \left\{ (\overline{v}_1, \dots, \overline{v}_\ell) \in \mathbb{P}(V)^\ell \;\middle|\; \begin{array}{l} \{v_{i_1}, \dots, v_{i_p}\} \text{ is linearly independent} \\ \text{over } \mathbb{F}_q \text{ if } \{i_1, \dots, i_p\} \text{ is a clique of } G. \end{array} \right\}.$$

Let $T := (\mathbb{F}_q^\times)^k / \mathbb{F}_q^\times (1, 1, \dots, 1)$. Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} \neq 0$. Then T acts on $\mathbb{P}(V)$ by

$$\overline{(a_1, \dots, a_k)} \cdot \overline{x} = \overline{\begin{pmatrix} a_1 x_1 \\ \vdots \\ a_k x_k \end{pmatrix}}.$$

Let $\nu(x) := \#\{\, i \in [k] \mid x_i \neq 0 \,\}$. Then it is easily seen that

$$\#(T \cdot \overline{x}) = (q-1)^{\nu(x)-1}.$$

In particular, $\overline{x} \in \mathbb{P}(V)$ is a fixed point if and only if $\overline{x} = \overline{e}_i$ for some i. Consider the diagonal T-action on $\mathbb{P}(V)^\ell$. This induces a T-action on $\underline{\chi}_G^q(\mathbb{P}(V))$. The fixed point set is

$$\underline{\chi}_G^q(\mathbb{P}(V))^T = \left\{ (\overline{e}_{s_1}, \dots, \overline{e}_{s_\ell}) \;\middle|\; \begin{array}{l} \{e_{s_{i_1}}, \dots, e_{s_{i_p}}\} \text{ is linearly independent} \\ \text{over } \mathbb{F}_q \text{ if } \{i_1, \dots, i_p\} \text{ is a clique of } G.. \end{array} \right\}$$

Note that $\{e_{s_{i_1}}, \dots, e_{s_{i_p}}\}$ is linearly independent if and only if they are mutually different. Therefore we have

$$\#\underline{\chi}_G^q(\mathbb{P}(V))^T = \chi(G,k).$$

Any other orbits have cardinalities divisible by $q-1$. This completes the proof. □

Corollary 2.2. *If $q > \ell^\ell$, then $\chi(\mathcal{A}_G^q, t)$ determines $\chi(G,t)$.*

Proof. Note that $\chi(G,t)$ is determined by the values $\chi(G,1), \chi(G,2), \dots, \chi(G,\ell)$. For $k \in [\ell]$, we have $\chi(G,k) \leq k^\ell < q-1$. Hence $\chi(G,k)$ is the remainder of $\dfrac{\chi(\mathcal{A}_G^q, q^k)}{(q-1)^\ell}$ divided by $q-1$. □

Remark 2.3. There exists a pair of graphs (G,H) depicted in the following picture such that $\chi(\mathcal{A}_G^2, t) = \chi(\mathcal{A}_H^2, t)$ but $\chi(G,t) \neq \chi(H,t)$.

The characteristic polynomials are as follows.

$$\begin{aligned}\chi(\mathcal{A}_G^2, t) = \chi(\mathcal{A}_H^2, t) &= t^7 - 30t^6 + 376t^5 - 2545t^4 + 9934t^3 - 21880t^2 + 24384t - 10240,\\ \chi(G,t) &= t^7 - 14t^6 + 83t^5 - 265t^4 + 474t^3 - 441t^2 + 162t,\\ \chi(H,t) &= t^7 - 14t^6 + 83t^5 - 264t^4 + 468t^3 - 430t^2 + 156t.\end{aligned}$$

The characteristic polynomial $\chi(\mathcal{A}_G^q, t)$ can also determine the numbers of stable partitions of G by the following theorem.

Theorem 2.4. *Let $\chi(\mathcal{A}_G^q, t) = \sum_{i=0}^{\ell} c_i t_q^i$. Then $c_i/(q-1)^{\ell-i}$ is a non-negative integer and congruent to the number of stable partitions of G into i blocks modulo $q-1$.*

Proof. By Proposition 1.4, the coefficient c_i is equal to the number of subspaces that are not contained in any hyperplanes in $\mathcal{A}_G^q$. Since every i-dimensional subspace in $\mathbb{F}_q^\ell$ corresponds to a matrix consisting of ℓ columns with rank i in reduced row echelon form, c_i coincides with the number of reduced row echelon form $(v_1, \dots, v_\ell) \in V^\ell$ such that $\{v_{i_1}, \dots, v_{i_p}\}$ is linearly independent over $\mathbb{F}_q$ if $\{i_1, \dots, i_p\}$ is a clique of G, where $V = \mathbb{F}_q^i$. Therefore $c_i/(q-1)^{\ell-i}$ equals the number of $(\overline{v}_1, \dots, \overline{v}_\ell) \in \mathbb{P}(V)^\ell$ such that $\{v_{i_1}, \dots, v_{i_p}\}$ is linearly independent over $\mathbb{F}_q$ if $\{i_1, \dots, i_p\}$ is a clique of G.

Using the same group action in the proof of Theorem 2.1, we can show that c_i is congruent to the number of reduced row echelon form $(e_{s_1}, \dots, e_{s_\ell})$ such that $\{e_{s_{i_1}}, \dots, e_{s_{i_p}}\}$ is linearly independent over $\mathbb{F}_q$ if $\{i_1, \dots, i_p\}$ is a clique of G, modulo $q-1$. From such a matrix $(e_{s_{i_1}}, \dots, e_{s_{i_\ell}})$, we can construct a stable partition of G consisting of i blocks $\{\, j \in [\ell] \mid s_j = k \,\}$ $(1 \leq k \leq i)$. □

Remark 2.5. Some of properties of the chromatic polynomials for finite graphs have q-analogues. For example, let us denote by $G + K_m$ the join of a graph G and the complete graph K_m. Then it is easily seen that $\chi(G + K_m, t) = t(t-1)\cdots(t-m+1)\chi(G, t-m)$. As a q-analogue of this formula, we can prove

$$\chi(\mathcal{A}^q_{G+K_m}, t) = (t-1)(t-q)\cdots(t-q^{m-1})q^{m\ell}\chi(\mathcal{A}^q_G, q^{-m}t).$$

3 Freeness

An arrangement $\mathcal{A}$ is **supersolvable** if and only if there exists a filtration

$$\varnothing = \mathcal{A}_0 \subseteq \mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}_\ell = \mathcal{A}$$

such that, for each $i \in [\ell]$, $\operatorname{rank} \mathcal{A}_i = i$ and for any distinct hyperplanes $H, H' \in \mathcal{A}_i \setminus \mathcal{A}_{i-1}$ there exists H'' such that $H \cap H' \subseteq H''$ [2, Theorem 4.3]. Every supersolvable arrangement is inductively free by [7, Theorem 4.2]. When $\mathcal{A}$ is supersolvable with the filtration above, the characteristic polynomial decomposes as

$$\chi(\mathcal{A}, t) = \prod_{i=1}^{\ell}(t - |\mathcal{A}_i \setminus \mathcal{A}_{i-1}|).$$

A vertex v of a simple graph G is called **simplicial** if the neighborhood $N_G(v) := \{ u \in V_G \mid \{u, v\} \in E_G \}$ is a clique. An ordering $(v_1, \dots, v_\ell)$ of the vertices of G is a **perfect elimination ordering** if v_i is simplicial in the subgraph of G induced by $\{v_1, \dots, v_i\}$ for each $i \in [\ell]$.

Theorem 3.1 (Stanley (See [5, Theorem 3.3]), Dirac [4], Fulkerson–Gross [6])**.** *The following are equivalent.*

(1) G has a perfect elimination ordering.

(2) $\mathcal{A}_G$ is supersolvable.

(3) $\mathcal{A}_G$ is free.

(4) G is chordal.

Moreover, when $(v_1, \dots, v_\ell)$ is a perfect elimination ordering of G

$$\chi(\mathcal{A}_G, t) = \prod_{i=1}^{\ell}(t - |N_{G_i}(v_i)|),$$

where $G_i := G[\{v_1, \dots, v_i\}]$ and $N_{G_i}(v_i)$ is the set of adjacent vertices of v_i in G_i.

Theorem 3.2. *The following conditions are equivalent to the conditions in Theorem 3.1.*

(5) $\mathcal{A}_G^q$ *is supersolvable.*

(6) $\mathcal{A}_G^q$ *is free.*

Moreover, when $(v_1, \dots, v_\ell)$ *is a perfect elimination ordering of* G

$$\chi(\mathcal{A}_G^q, t) = \prod_{i=1}^{\ell} \left(t - q^{|N_{G_i}(v_i)|} \right),$$

where $G_i := G[\{v_1, \dots, v_i\}]$ *and* $N_{G_i}(v_i)$ *is the set of adjacent vertices of* v_i *in* G_i*.*

Proof. The proof is very similar to the proof of Theorem 3.1.

To show (1) $\Rightarrow$ (5), let $(v_1, \dots, v_\ell)$ be a perfect elimination ordering and $(x_1, \dots, x_\ell)$ the corresponding coordinates. Let $\mathcal{A}_i$ be the subarrangement of $\mathcal{A}_G^q$ consisting hyperplanes whose defining linear form contains $x_1, \dots, x_i$. Then the filtration

$$\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \cdots \subseteq \mathcal{A}_\ell = \mathcal{A}_G^q$$

guarantees that $\mathcal{A}_G^q$ is supersolvable and $|\mathcal{A}_i \setminus \mathcal{A}_{i-1}| = q^{|N_{G_i}(v_i)|}$.

The implication (5) $\Rightarrow$ (6) follows since every supersolvable arrangement is (inductively) free (Jambu–Terao). To prove (6) $\Rightarrow$ (4) It suffices to show that $\chi(\mathcal{A}_{C_\ell}^q, t)$ does not factor into the product of linear forms over $\mathbb{Z}$ when $\ell \geq 4$ by Terao's factorization theorem. Since $\chi(\mathcal{A}_{C_\ell}^q, t) = (t-q)^\ell + (-1)^\ell (q-1)^{\ell-1}(t-q)$ by the following Lemma 3.3, we have

$$\chi(\mathcal{A}_{C_\ell}, t) = (q-1)^\ell (x^\ell + (-1)^\ell) = (q-1)^\ell x (x^{\ell-1} + (-1)^\ell),$$

where $x = \frac{t-q}{q-1}$. Since $x^\ell + (-1)^\ell$ has an imaginary root, $\chi(\mathcal{A}_{C_\ell}^q, t)$ does not factor over $\mathbb{Z}$. □

There is a well-known explicit formula for the chromatic polynomial of a cycle graph for $\ell \geq 3$.

$$\chi(\mathcal{A}_{C_\ell}, t) = (t-1)^\ell + (-1)^\ell (t-1).$$

The following lemma is a q-version of this formula.

Lemma 3.3. *Let* $\ell \geq 4$*. Then*

$$\chi(\mathcal{A}_{C_\ell}^q, t) = (t-q)^\ell + (-1)^\ell (q-1)^{\ell-1}(t-q).$$

Proof. First consider the arrangement $\mathcal{B}$ in $\mathbb{F}_q^3$ consisting of the following hyperplanes.

$$x_1 = ax_2,\ x_2 = ax_3,\ x_3 = ax_1 \quad (a \in \mathbb{F}_q).$$

Then $\mathcal{B}$ is supersolvable with exponents $(1, q, 2q-1)$. Using the deletion-restriction theorem [8, Corollary 2.57] $q-1$ times, we have

$$\begin{aligned}
\chi(\mathcal{A}^q_{C_4}, t) &= \chi(\mathcal{A}^q_{P_4}, t) - (q-1)\chi(\mathcal{B}, t) \\
&= (t-1)(t-q)^3 - (q-1)(t-1)(t-q)(t-2q+1) \\
&= (t-1)(t-q)\left((t-q)^2 - (q-1)(t-2q+1)\right) \\
&= (t-1)(t-q)\left((t-q)^2 - (q-1)(t-q) + (q-1)^2\right) \\
&= (t-1)(t-q)\frac{(t-q)^3 + (q-1)^3}{(t-q)+(q-1)} \\
&= (t-q)^4 + (-1)^4(q-1)^3(t-1).
\end{aligned}$$

Now suppose that $\ell \geq 5$. Using the deletion-restriction theorem $q-1$ times, we have

$$\begin{aligned}
\chi(\mathcal{A}^q_{C_\ell}, t) &= \chi(\mathcal{A}^q_{P_\ell}) - (q-1)\chi(\mathcal{A}^q_{C_{\ell-1}}, t) \\
&= (t-1)(t-q)^{\ell-1} - (q-1)(t-q)^{\ell-1} - (-1)^{\ell-1}(q-1)^{\ell-1}(t-q) \\
&= (t-q)^{\ell} + (-1)^{\ell}(q-1)^{\ell-1}(t-q).
\end{aligned}$$

□

4 Basis construction

Let G be a chordal graph with a perfect elimination ordering $(v_1, \ldots, v_\ell)$ and $(x_1, \ldots, x_\ell)$ the corresponding coordinates. Define the sets $C_{\geq k}$ and $E_{<k}$ by

$$C_{\geq k} := \{k\} \cup \left\{\, i \in [\ell] \mid \text{there exists a path } v_k v_{j_1} \cdots v_{j_n} v_i \text{ such that } k < j_1 < \cdots < j_n < i \,\right\},$$
$$E_{<k} := \left\{\, j \in [\ell] \;\middle|\; j < k \text{ and } \{v_j, v_k\} \in E_G \,\right\}.$$

Let $\Delta(x_1, \ldots, x_k)$ denote the Vandermonde determinant:

$$\Delta(x_1, \ldots, x_k) := \begin{vmatrix} 1 & x_1 & x_1^2 & \ldots & x_1^{k-1} \\ 1 & x_2 & x_2^2 & \ldots & x_k^{k-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_k & x_k^2 & \ldots & x_k^{k-1} \end{vmatrix} = \prod_{1 \leq i < j \leq k} (x_j - x_i).$$

When $E_{<k} = \{j_1, \ldots, j_m\}$ with $j_1 < \cdots < j_m$,

$$\Delta(E_{<k}) := \Delta(x_{j_1}, \ldots, x_{j_m}) \quad \text{and} \quad \Delta(E_{<k}, x_i) := \Delta(x_{j_1}, \ldots, x_{j_m}, x_i).$$

Theorem 4.1 ([11, Theorem 4.1]). *Let*

$$\theta_k := \sum_{i \in C_{\geq k}} \frac{\Delta(E_{<k}, x_i)}{\Delta(E_{<k})} \partial_i \qquad (1 \leq k \leq \ell).$$

Then $\{\theta_1, \ldots, \theta_\ell\}$ *forms a basis for* $D(\mathcal{A}_G)$.

Let $\Delta_q(x_1, \ldots, x_k) \in \mathbb{F}_q[x_1, \ldots, x_k]$ denote the determinant of the Moore matrix. Namely

$$\Delta_q(x_1, \ldots, x_k) = \begin{vmatrix} x_1 & x_1^q & x_1^{q^2} & \ldots & x_1^{q^{k-1}} \\ x_2 & x_2^q & x_2^{q^2} & \ldots & x_k^{q^{k-1}} \\ \vdots & \vdots & \vdots & & \vdots \\ x_k & x_k^q & x_k^{q^2} & \ldots & x_k^{q^{k-1}} \end{vmatrix} = \prod_{i=1}^{k} \prod_{c_1, \ldots, c_{i-1} \in \mathbb{F}_q} (c_1 x_1 + \cdots + c_{i-1} x_{i-1} + x_i).$$

Theorem 4.2. *Let*

$$\theta_k := \sum_{i \in C_{\geq k}} \frac{\Delta_q(E_{<k}, x_i)}{\Delta_q(E_{<k})} \partial_i \qquad (1 \leq k \leq \ell).$$

Then $\{\theta_1, \ldots, \theta_\ell\}$ *forms a basis for* $D(\mathcal{A}_G^q)$.

Proof. Let $K = \{i_1, \ldots, i_m\}$ be a clique of G with $i_1 < \cdots < i_m$. Let $\alpha = c_1 x_{i_1} + c_2 x_{i_2} + \cdots + c_m x_{i_m}$ be a nonzero linear form over $\mathbb{F}_q$. If $K \cap C_{\geq k} = \varnothing$, then $\theta_k(\alpha) = 0$.

Suppose that $K \cap C_{\geq k} \neq \varnothing$ and take $i_s \in K \cap C_{\geq k}$ with minimal s. Then one can show that $\{i_1, \ldots, i_{s-1}\} \subseteq E_{<k}$ and $\{i_s, i_{s+1}, \ldots, i_m\} \subseteq C_{\geq k}$. Then

$$\theta_k(\alpha) = \sum_{u=s}^{k} \frac{\Delta_q(E_{<k}, x_{i_u})}{\Delta_q(E_{<k})} \cdot c_{i_u} = \frac{\Delta_q(E_{<k},\ c_{i_s} x_{i_s} + \cdots + c_{i_m} x_{i_m})}{\Delta_q(E_{<k})} = \frac{\Delta_q(E_{<k},\ \alpha)}{\Delta_q(E_{<k})} \in (\alpha).$$

Using Saito's criterion [8, Theorem 4.19], we can prove $\{\theta_1, \ldots, \theta_\ell\}$ is a basis. □

References

[1] C. A. Athanasiadis. "Characteristic Polynomials of Subspace Arrangements and Finite Fields". *Advances in Mathematics* **122**.2 (1996), pp. 193–233. DOI.

[2] A. Björner, P. H. Edelman, and G. M. Ziegler. "Hyperplane arrangements with a lattice of regions". *Discrete & Computational Geometry* **5**.3 (June 1990), pp. 263–288. DOI.

[3] A. Björner and T. Ekedahl. "Subspace Arrangements over Finite Fields: Cohomological and Enumerative Aspects". *Advances in Mathematics* **129**.2 (Aug. 1997), pp. 159–187. DOI.

[4] G. A. Dirac. "On rigid circuit graphs". *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*. Vol. 25. Springer, 1961, pp. 71–76. DOI.

[5] P. H. Edelman and V. Reiner. "Free hyperplane arrangements between A_{n-1} and B_n". *Mathematische Zeitschrift* **215**.1 (1994), pp. 347–365. DOI.

[6] D. Fulkerson and O. Gross. "Incidence matrices and interval graphs". *Pacific journal of mathematics* **15**.3 (1965), pp. 835–855. DOI.

[7] M. Jambu and H. Terao. "Free arrangements of hyperplanes and supersolvable lattices". *Advances in Mathematics* **52**.3 (June 1984), pp. 248–258. DOI.

[8] P. Orlik and H. Terao. *Arrangements of Hyperplanes*. Ed. by M. Artin et al. Vol. 300. Grundlehren der mathematischen Wissenschaften. Berlin, Heidelberg: Springer Berlin Heidelberg, 1992. DOI.

[9] A. Postnikov and R. P. Stanley. "Deformations of Coxeter Hyperplane Arrangements". *Journal of Combinatorial Theory, Series A* **91**.1 (2000), pp. 544–597. DOI.

[10] R. C. Read. "An introduction to chromatic polynomials". *Journal of Combinatorial Theory* **4**.1 (Jan. 1968), pp. 52–71. DOI.

[11] D. Suyama and S. Tsujie. "Vertex-Weighted Graphs and Freeness of ψ-Graphical Arrangements". *Discrete & Computational Geometry* **61**.1 (Jan. 2019), pp. 185–197. DOI.

[12] M. Yoshinaga. "Free arrangements over finite field". *Proceedings of the Japan Academy, Series A, Mathematical Sciences* **82**.10 (Jan. 2007), pp. 179–182. DOI.