Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #15, 12 pp.

Proceedings of the 37^{th} *Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)*

Filtered RSK and matrix Schubert varieties

Abigail Price*[1], Ada Stelzer †[1], and Alexander Yong‡[1]

[1]*Department of Mathematics, University of Illinois Urbana–Champaign*

Abstract. Matrix Schubert varieties [Fulton '92] carry natural actions of Levi groups. Their coordinate rings are thereby Levi-representations; what is a combinatorial counting rule for the multiplicities of their irreducibles? When the Levi group is a torus, [Knutson–Miller '04] answers the question. We present a general solution, a common refinement of the multigraded Hilbert series, the Cauchy identity, and the Littlewood–Richardson rule. The proof introduces a "filtered" generalization of the Robinson–Schensted–Knuth correspondence and uses the operators of [Kashiwara '95] and of [Danilov–Koshevoi '05, van Leeuwen '06].

Keywords: matrix Schubert varieties, RSK, crystal graphs, Levi groups

1 Introduction

Matrix Schubert varieties, introduced by Fulton in [7], are defined as Borel-orbit closures of partial permutation matrices in the space $\mathsf{Mat}_{m,n}$ of $m \times n$ complex matrices. They have been extensively studied; see, e.g., [13, 14, 15, 16]. Since each matrix Schubert variety $\mathfrak{X}_w$ is stable under the Borel action, a torus $\mathbf{T}$ also acts on $\mathfrak{X}_w$ by restriction. Knutson–Miller studied the multigraded Hilbert series of $\mathfrak{X}_w$, i.e., the $\mathbf{T}$-character of $\mathfrak{X}_w$, in [15] and [14]. For appropriate choices of $\mathbf{I}$ and $\mathbf{J}$, $\mathfrak{X}_w$ also carries an action of a Levi group $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$. We give the first manifestly-positive combinatorial rule for computing $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-graded Hilbert series of $\mathfrak{X}_w$ for arbitrary Levi groups $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$.

Our rule simultaneously generalizes the Littlewood–Richardson rule, the classical Cauchy identity, and the Knutson–Miller formula (since $\mathbf{T}$ is a Levi group). The rule is based on a new, "filtered" generalization of the RSK algorithm. Its proof relies on Kashiwara's crystal basis theory ([12]). In fact, the rule extends to a class of varieties we call *bicrystalline*, which includes all unions and intersections of matrix Schubert varieties.

*price29@illinois.edu. Supported by an NSF RTG in Combinatorics (DMS 1937241).

†astelzer@illinois.edu. Supported by a Susan C. Morisato IGL graduate student scholarship, an NSF graduate fellowship, and an NSF RTG in Combinatorics (DMS 1937241).

‡ayong@illinois.edu. Supported by a Simons Collaboration grant and an NSF RTG in Combinatorics (DMS 1937241).

1.1 Statement of the representation-theoretic question

Let $\mathbf{GL} = GL_m(\mathbb{C}) \times GL_n(\mathbb{C})$, let $\mathbf{B} \leq \mathbf{GL}$ denote the Borel group of pairs of invertible lower triangular matrices, and let $\mathbf{T} \leq \mathbf{GL}$ denote the torus of pairs of invertible diagonal matrices. Let $\mathbf{L_{I|J}} := (GL_{i_1-i_0} \times \cdots \times GL_{i_r-i_{r-1}}) \times (GL_{j_1-j_0} \times \cdots \times GL_{j_s-j_{s-1}}) \leq \mathbf{GL}$ be a Levi subgroup, where $\mathbf{I} = \{0 = i_0 < \cdots < i_r = m\}$ and $\mathbf{J} = \{0 = j_0 < \cdots < j_s = n\}$. The finite-dimensional, irreducible polynomial representations of GL_m are the *Weyl modules* $V_\lambda(m)$, where λ is a partition with at most m parts. Analogously, the finite-dimensional, irreducible polynomial representations of $\mathbf{L_I} = GL_{i_1-i_0} \times \cdots \times GL_{i_r-i_{r-1}}$ are tensor products of Weyl modules. These *split-Weyl modules* are indexed by partition-tuples $\underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(r)})$:

$$V_{\underline{\lambda}}(m) := V_{\lambda^{(1)}}(i_1 - i_0) \boxtimes \cdots \boxtimes V_{\lambda^{(r)}}(i_r - i_{r-1}).$$

The *representation ring* $\mathrm{Rep}(G)$ of a reductive group G (such as $\mathbf{L_I}$) is the ring of (formal differences of) isomorphism classes of finite-dimensional, complex, polynomial G-representations, with ring operations given by direct sum and tensor product. We denote the isomorphism class of a G-representation V in $\mathrm{Rep}(G)$ by $[V]$. The classes of irreducible representations form a $\mathbb{Z}$-linear basis for $\mathrm{Rep}(G)$. Thus for any finite-dimensional, polynomial $\mathbf{L_{I|J}}$-representation V, there exist unique nonnegative integers $c^V_{\underline{\lambda}|\underline{\mu}}$ such that

$$[V] = \sum_{\underline{\lambda}|\underline{\mu}} c^V_{\underline{\lambda}|\underline{\mu}} [V_{\underline{\lambda}}(m) \boxtimes V_{\underline{\mu}}(n)] \text{ in } \mathrm{Rep}(\mathbf{L_{I|J}}). \tag{1.1}$$

Let $s_\lambda(\mathbf{x})$ and $s_{\underline{\lambda}}(\mathbf{x})$ denote the *Schur* and *split-Schur* polynomials, defined as the characters of $V_\lambda(m)$ and $V_{\underline{\lambda}}(m)$ respectively. Then taking characters of both sides of (1.1) yields the equivalent polynomial identity

$$\mathrm{char}(V) = \sum_{\underline{\lambda}|\underline{\mu}} c^V_{\underline{\lambda}|\underline{\mu}} s_{\underline{\lambda}}(\mathbf{x}) s_{\underline{\mu}}(\mathbf{y}).$$

What is a positive combinatorial rule for the constants $c^V_{\underline{\lambda}|\underline{\mu}}$ in the case where V is the coordinate ring of some matrix Schubert variety?[1] This extended abstract of [18] provides a complete answer to the question.

1.2 Main new combinatorial ingredient: filtered RSK

We introduce the *filtered RSK algorithm*. It takes a nonnegative integer matrix as input and produces a tuple of *semistandard Young tableaux* (not necessarily of the same shape).

[1] Technically, such V are usually infinite-dimensional, so $[V]$ is only an element of the completion $\widehat{\mathrm{Rep}(\mathbf{L_{I|J}})}$ of $\mathrm{Rep}(\mathbf{L_{I|J}})$. This causes no concern since coordinate rings of matrix Schubert varieties are graded, with each graded piece V_i a finite-dimensional $\mathbf{L_{I|J}}$-representation. All copies of the split-Weyl module $V_{\underline{\lambda}}(m) \boxtimes V_{\underline{\mu}}(n)$ in V lie in a single V_i, so each $c^V_{\underline{\lambda}|\underline{\mu}}$ in (1.1) is finite.

Definition 1.1. The *row word* of a matrix $M = [m_{ij}] \in \mathsf{Mat}_{m,n}(\mathbb{Z}_{\geq 0})$, denoted $\mathsf{row}(M)$, records m_{ij} copies of the row index i for each entry m_{ij} of M. The entries are read along columns top-to-bottom, from left to right. The *column word* $\mathsf{col}(M)$ is the row word of the transpose matrix M^t.

Example 1.2. If $M = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$ then $\mathsf{row}(M) = 221222$ and $\mathsf{col}(M) = 211222$.

Definition 1.3. For an integer sequence $\mathbf{I} = \{0 = i_0 < \cdots < i_r = m\}$, the **I**-*filtration* of a word w on the alphabet $[m]$ is a tuple of words $\mathsf{filter}_{\mathbf{I}}(w) = (w^{(1)}, \ldots, w^{(r)})$. Each $w^{(k)}$ is the subword of w consisting of all letters in the interval $[i_{k-1}+1, i_k]$.

Let $\sim_K$ denote the Knuth equivalence relation on words, and let $\mathsf{word}(T)$ denote the reading word of a semistandard tableau T. Then each Knuth equivalence class contains $\mathsf{word}(T)$ for a unique tableau T. This tableau T can be constructed from any word in its equivalence class via the row-insertion or jeu-de-taquin algorithms, see, e.g., [8].

Main Definition 1.4 (Filtered RSK). *Let $M \in \mathsf{Mat}_{m,n}(\mathbb{Z}_{\geq 0})$ and fix two integer sequences*

$$\mathbf{I} = \{0 = i_0 < \cdots < i_r = m\} \ \ \textit{and} \ \ \mathbf{J} = \{0 = j_0 < \cdots < j_s = n\}. \tag{1.2}$$

Define $\mathsf{filterRSK}_{\mathbf{I}|\mathbf{J}}(M) = (\underline{P}|\underline{Q}) := (P^{(1)}, \ldots, P^{(r)}|Q^{(1)}, \ldots, Q^{(s)})$, *where $P^{(a)}$ and $Q^{(b)}$ are the unique SSYT with* $\mathsf{word}(P^{(a)}) \sim_K \mathsf{filter}_{\mathbf{I}}(\mathsf{row}(M))^{(a)}$ *and* $\mathsf{word}(Q^{(b)}) \sim_K \mathsf{filter}_{\mathbf{J}}(\mathsf{col}(M))^{(b)}$, *respectively.*

We conflate monomials $\mathsf{m} = \prod_{i,j} z_{ij}^{m_{i,j}}$ with their exponent vectors $M = [m_{i,j}] \in \mathsf{Mat}_{m,n}(\mathbb{Z}_{\geq 0})$. This makes sense of notation such as $\mathsf{filterRSK}_{\mathbf{I}|\mathbf{J}}(\mathsf{m})$ for a monomial m.

Example 1.5. Let $\mathbf{I} = \{0, 1, 3\} = \mathbf{J}$ and let $A = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$. Then $\mathsf{row}(M) = 2233112$ and $\mathsf{col}(M) = 2211211$, so

$$\mathsf{filter}_{\mathbf{I}}(\mathsf{row}(M)) = (11, 22332) \ \ \text{and} \ \ \mathsf{filter}_{\mathbf{J}}(\mathsf{col}(M)) = (1111, 222).$$

Now, consider the tuple of tableaux

$$(P^{(1)}, P^{(2)}|Q^{(1)}, Q^{(2)}) = \left(\begin{array}{|c|c|}\hline 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|c|}\hline 2 & 2 & 2 & 3 \\ \hline 3 \\ \cline{1-1} \end{array} \;\middle|\; \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 2 & 2 & 2 \\ \hline \end{array} \right).$$

The reading words of these tableaux are $(11, 32223|1111, 222)$. Since $32223 \sim_K 22332$, $\mathsf{filterRSK}(\mathsf{M}) = (P^{(1)}, P^{(2)}|Q^{(1)}, Q^{(2)})$.

Definition 1.6. The *highest-weight tableau* of *shape* λ and content $[a, b]$ is the tableau $T_{\lambda,[a,b]}$ of shape λ taking the constant value $a - 1 + i$ on each row i. For a given **I**, the *highest-weight tableau-tuple* $T_{\underline{\lambda}}$ of *shape* $\underline{\lambda}$ has components $T_{\underline{\lambda}}^{(k)} := T_{\lambda^{(k)},[i_{k-1}+1, i_k]}$.

Theorem 1.7 below foreshadows our main result, Main Theorem 1.14:

Theorem 1.7. *Let $\mathbb{C}[\mathsf{Mat}_{m,n}]$ be the ring of polynomial functions on $m \times n$ matrices, viewed as an $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-representation. A positive combinatorial rule for the coefficients $c^{\mathbb{C}[\mathsf{Mat}_{m,n}]}_{\underline{\lambda}|\underline{\mu}}$ in (1.1) is then:*

$$c^{\mathbb{C}[\mathsf{Mat}_{m,n}]}_{\underline{\lambda}|\underline{\mu}} = \#\{M \in \mathsf{Mat}_{m,n}(\mathbb{Z}_{\geq 0}) : \mathsf{filterRSK}_{\mathbf{I}|\mathbf{J}}(M) = (T_{\underline{\lambda}}|T_{\underline{\mu}})\}.$$

When $\mathbf{I} = \{0, m\}$ and $\mathbf{J} = \{0, n\}$, $\mathsf{filterRSK}_{\mathbf{I}|\mathbf{J}}$ is the classical RSK correspondence. In this special case, Theorem 1.7 is a restatement of Howe duality, and the corresponding character formula is the classical Cauchy identity:

$$\prod_{\substack{1\leq i\leq m \\ 1\leq j\leq n}} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(x_1, \ldots, x_m) s_\lambda(y_1, \ldots, y_n). \tag{1.3}$$

We sketch the proof of Theorem 1.7 in Section 2.1; it involves van Leeuwen and Danilov–Koshevoi's "pull-back" of Kashiwara's crystal operators to bicrystal operators on $\mathsf{Mat}_{m,n}(\mathbb{Z}_{\geq 0})$. In the next section, we generalize Theorem 1.7 by replacing $\mathsf{Mat}_{m,n}$ with an arbitrary matrix Schubert variety.

1.3 Matrix Schubert varieties and the main theorem

The Borel group $\mathbf{B}$ acts on $\mathsf{Mat}_{m,n}$ on the right by $(b_1, b_2) \cdot M = b_1^{-1} M (b_2^{-1})^T$. This convention yields a left action of the torus $\mathbf{T} \leq \mathbf{B}$ on the coordinate ring $\mathbb{C}[\mathsf{Mat}_{m,n}]$ and ensures that the multigrading induced by this $\mathbf{T}$-action is positive. We recall the definition of matrix Schubert varieties, following Fulton's construction in [7].

Definition 1.8. A *partial permutation matrix* $M_w \in \mathsf{Mat}_{m,n}$ is a matrix with at most one 1 in each row and column and 0s everywhere else. The indexing *partial permutation* is a function $w : [m] \to [n] \cup \{\infty\}$, where $w(i) = j \in [n]$ if M_w has a 1 in position (i, j) and $w(i) = \infty$ if M_w has no 1 in row i.

Theorem 1.9 ([7, Lemma 3.1])**.** *Every $\mathbf{B}$-orbit in $\mathsf{Mat}_{m,n}$ contains a unique M_w.*

Definition 1.10 ([7, Proposition 3.3])**.** The *matrix Schubert variety* $\mathfrak{X}_w \subseteq \mathsf{Mat}_{m,n}$ is the Zariski closure of the orbit $\mathbf{B} \cdot M_w$. Each $\mathfrak{X}_w$ is irreducible.

Definition 1.11. The *row descent set* of M_w, denoted $\mathrm{Desc}_{\mathsf{row}}(w)$, consists of row indices i such that $w(i) > w(i+1)$ (where $\infty > k$ for all $k \in [n]$). The *column descent set* of M_w, denoted $\mathrm{Desc}_{\mathsf{col}}(w)$, is the row descent set of the transpose matrix M_w^t.

Remark 1.12. By [7, Equation 3.4], every $\mathfrak{X}_w \subseteq \mathsf{Mat}_{m,n}$ is equal (up to a product with affine space) to some $\mathfrak{X}_{\overline{w}} \subseteq \mathsf{Mat}_{m+n,m+n}$ indexed by a full permutation $\overline{w} \in \mathfrak{S}_{m+n}$. The row and column descent sets of w are then the descent sets of $\overline{w}$ and $\overline{w}^{-1}$ respectively.

Proposition 1.13. *Let $\mathfrak{X}_w \subseteq \mathsf{Mat}_{m,n}$ be a matrix Schubert variety. If $\mathrm{Desc}_{\mathsf{row}}(w) \subseteq \mathbf{I}$ and $\mathrm{Desc}_{\mathsf{col}}(w) \subseteq \mathbf{J}$, then $\mathfrak{X}_w$ is stable under the action of the Levi group $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$ on $\mathsf{Mat}_{m,n}$.*

Proof. The argument is straightforward; see [18, Proposition 6.9]. □

When $\mathfrak{X}_w$ is stable under the action of $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$, the coordinate ring $\mathbb{C}[\mathfrak{X}_w]$ is an $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-representation. Now, the Gröbner basis methods of Knutson–Miller [15, 14] identify a set of monomials that form a basis for $\mathbb{C}[\mathfrak{X}_w]$ as a vector space. This basis, denoted $\mathsf{Std}_{\prec}(\mathfrak{X}_w)$, is called the set of *standard monomials* of $\mathfrak{X}_w$. We recall an explicit description of $\mathsf{Std}_{\prec}(\mathfrak{X}_w)$ in Section 2.3. We now state our main theorem.

Main Theorem 1.14. *Let $\mathfrak{X}_w \subseteq \mathsf{Mat}_{m,n}$ be a matrix Schubert variety and let $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$ be a Levi group with $\mathbf{I} \supseteq \mathrm{Desc}_{\mathsf{row}}(w)$ and $\mathbf{J} \supseteq \mathrm{Desc}_{\mathsf{col}}(w)$. Then the coefficients in the expression (1.1) for the class of $\mathbb{C}[\mathfrak{X}_w]$ in $\mathrm{Rep}(\mathbf{L}_{\mathbf{I}|\mathbf{J}})$ are given by the positive combinatorial rule*

$$c^{\mathbb{C}[\mathfrak{X}_w]}_{\underline{\lambda}|\underline{\mu}} = \#\{\mathsf{m} \in \mathsf{Std}_{\prec}(\mathfrak{X}_w) : \mathsf{filterRSK}_{\mathbf{I}|\mathbf{J}}(\mathsf{m}) = (T_{\underline{\lambda}}|T_{\underline{\mu}})\}. \tag{1.4}$$

We remark that this result is strongest when $\mathbf{I}$ and $\mathbf{J}$ are chosen minimally. In Section 2, we sketch the proof of Main Theorem 1.14, showing how it follows from Theorem 1.7 and a compatibility between the Knutson–Miller description of $\mathsf{Std}_{\prec}(\mathfrak{X}_w)$ and the bicrystal operators of van Leeuwen and Danilov–Koshevoi. Our method of proof in fact extends to give an analogue of Main Theorem 1.14 for a new class of varieties we call *bicrystalline*. All unions and intersections of matrix Schubert varieties are bicrystalline; in particular, by Corollary 2.12, every $\mathbf{B}$-stable variety is bicrystalline.

Main Theorem 1.14 generalizes the study of $\mathbf{GL}$-stable varieties $\mathfrak{X} \subset \mathsf{Mat}_{m,n}$, which are exactly the *classical determinantal varieties* $\mathfrak{X}_k$ of rank $\leq k$ matrices. The $\mathbf{T}$-characters of these varieties have been long studied (see, e.g., [1, 4, 6, 9, 10, 11]).

Example 1.15. Let $\mathfrak{X} \subseteq \mathsf{Mat}_{4,4}$ be the space of 4×4 matrices of rank ≤ 3 whose northwest 1×1 submatrix has rank 0. By a result of Fulton stated below as Theorem 2.8, $\mathfrak{X}$ is the matrix Schubert variety $\mathfrak{X}_w \subseteq \mathsf{Mat}_{4,4}$ for $w = 213\infty$ (Definition 1.8 explains this notation). Now, $\mathfrak{X}$ is stable under the action of the Levi group $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$ with $\mathbf{I} = \{0, 1, 4\} = \mathbf{J}$: applying row operations to the first row (respectively, column) and the second through fourth rows (respectively, columns) of a matrix $M \in \mathfrak{X}$ returns matrices in $\mathfrak{X}$.

The Knutson–Miller Gröbner basis theorem [15, Theorem B] restated here as Theorem 2.9 shows that

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

are (exponent vectors of) standard monomials of $\mathbb{C}[\mathfrak{X}]$. Since $\mathsf{row}(A) = \mathsf{col}(A) = 3221$ and $\mathsf{row}(B) = \mathsf{col}(B) = 2132$, applying $\mathsf{filterRSK}_{\mathbf{I}|\mathbf{J}}$ to either matrix yields

$$(T_{\underline{\lambda}}|T_{\underline{\mu}}) = \left(\boxed{1}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \;\middle|\; \boxed{1}, \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \right).$$

In fact, A and B are the only two such standard monomials, so $c^{\mathbb{C}[\mathfrak{X}]}_{\underline{\lambda}|\underline{\mu}} = 2$. The character formula for $\mathbb{C}[\mathfrak{X}_{213\infty}]$ implied by Main Theorem 1.14 begins:

$$\begin{aligned}
&s_{(\varnothing,\varnothing)}(\mathbf{x})s_{(\varnothing,\varnothing)}(\mathbf{y}) + s_{(\varnothing,1)}(\mathbf{x})s_{(\varnothing,1)}(\mathbf{y}) + s_{(1,\varnothing)}(\mathbf{x})s_{(\varnothing,1)}(\mathbf{y}) + s_{(\varnothing,1)}(\mathbf{x})s_{(1,\varnothing)}(\mathbf{y}) + \\
&s_{(\varnothing,11)}(\mathbf{x})s_{(\varnothing,11)}(\mathbf{y}) + s_{(1,1)}(\mathbf{x})s_{(\varnothing,11)}(\mathbf{y}) + s_{(\varnothing,2)}(\mathbf{x})s_{(\varnothing,2)}(\mathbf{y}) + s_{(1,1)}(\mathbf{x})s_{(\varnothing,2)}(\mathbf{y}) + \\
&s_{(2,\varnothing)}(\mathbf{x})s_{(\varnothing,2)}(\mathbf{y}) + s_{(\varnothing,11)}(\mathbf{x})s_{(1,1)}(\mathbf{y}) + s_{(\varnothing,2)}(\mathbf{x})s_{(1,1)}(\mathbf{y}) + s_{(1,1)}(\mathbf{x})s_{(1,1)}(\mathbf{y}) + \\
&s_{(\varnothing,2)}(\mathbf{x})s_{(2,\varnothing)}(\mathbf{y}) + s_{(\varnothing,111)}(\mathbf{x})s_{(\varnothing,111)}(\mathbf{y}) + s_{(1,11)}(\mathbf{x})s_{(\varnothing,111)}(\mathbf{y}) + s_{(\varnothing,21)}(\mathbf{x})s_{(\varnothing,21)}(\mathbf{y}) + \\
&s_{(1,11)}(\mathbf{x})s_{(\varnothing,21)}(\mathbf{y}) + s_{(1,2)}(\mathbf{x})s_{(\varnothing,21)}(\mathbf{y}) + s_{(2,1)}(\mathbf{x})s_{(\varnothing,21)}(\mathbf{y}) + s_{(\varnothing,111)}(\mathbf{x})s_{(1,11)}(\mathbf{y}) + \\
&s_{(\varnothing,21)}(\mathbf{x})s_{(1,11)}(\mathbf{y}) + s_{(1,11)}(\mathbf{x})s_{(1,11)}(\mathbf{y}) + s_{(1,2)}(\mathbf{x})s_{(1,11)}(\mathbf{y}) + s_{(\varnothing,3)}(\mathbf{x})s_{(\varnothing,3)}(\mathbf{y}) + \\
&s_{(1,2)}(\mathbf{x})s_{(\varnothing,3)}(\mathbf{y}) + s_{(2,1)}(\mathbf{x})s_{(\varnothing,3)}(\mathbf{y}) + s_{(3,\varnothing)}(\mathbf{x})s_{(\varnothing,3)}(\mathbf{y}) + s_{(\varnothing,21)}(\mathbf{x})s_{(1,2)}(\mathbf{y}) + \\
&s_{(1,11)}(\mathbf{x})s_{(1,2)}(\mathbf{y}) + s_{(\varnothing,3)}(\mathbf{x})s_{(1,2)}(\mathbf{y}) + s_{(1,2)}(\mathbf{x})s_{(1,2)}(\mathbf{y}) + s_{(2,1)}(\mathbf{x})s_{(1,2)}(\mathbf{y}) + \\
&s_{(\varnothing,21)}(\mathbf{x})s_{(2,1)}(\mathbf{y}) + s_{(\varnothing,3)}(\mathbf{x})s_{(2,1)}(\mathbf{y}) + s_{(1,2)}(\mathbf{x})s_{(2,1)}(\mathbf{y}) + s_{(\varnothing,3)}(\mathbf{x})s_{(3,\varnothing)}(\mathbf{y}) + \\
&s_{(\varnothing,211)}(\mathbf{x})s_{(\varnothing,211)}(\mathbf{y}) + s_{(1,111)}(\mathbf{x})s_{(\varnothing,211)}(\mathbf{y}) + s_{(1,21)}(\mathbf{x})s_{(\varnothing,211)}(\mathbf{y}) + s_{(2,11)}(\mathbf{x})s_{(\varnothing,211)}(\mathbf{y}) + \\
&s_{(\varnothing,211)}(\mathbf{x})s_{(1,111)}(\mathbf{y}) + s_{(1,21)}(\mathbf{x})s_{(1,111)}(\mathbf{y}) + s_{(\varnothing,22)}(\mathbf{x})s_{(\varnothing,22)}(\mathbf{y}) + s_{(1,21)}(\mathbf{x})s_{(\varnothing,22)}(\mathbf{y}) + \\
&s_{(2,2)}(\mathbf{x})s_{(\varnothing,22)}(\mathbf{y}) + s_{(\varnothing,31)}(\mathbf{x})s_{(\varnothing,31)}(\mathbf{y}) + s_{(1,21)}(\mathbf{x})s_{(\varnothing,31)}(\mathbf{y}) + s_{(2,11)}(\mathbf{x})s_{(\varnothing,31)}(\mathbf{y}) + \\
&s_{(1,3)}(\mathbf{x})s_{(\varnothing,31)}(\mathbf{y}) + s_{(2,2)}(\mathbf{x})s_{(\varnothing,31)}(\mathbf{y}) + s_{(3,1)}(\mathbf{x})s_{(\varnothing,31)}(\mathbf{y}) + s_{(\varnothing,211)}(\mathbf{x})s_{(1,21)}(\mathbf{y}) + \\
&s_{(1,111)}(\mathbf{x})s_{(1,21)}(\mathbf{y}) + s_{(\varnothing,22)}(\mathbf{x})s_{(1,21)}(\mathbf{y}) + s_{(\varnothing,31)}(\mathbf{x})s_{(1,21)}(\mathbf{y}) + 2s_{(1,21)}(\mathbf{x})s_{(1,21)}(\mathbf{y}) + \cdots .
\end{aligned}$$

Remark 1.16 (Main Theorem 1.14 and Littlewood–Richardson coefficients). Let $\mathbf{I} = \{0 < t < m\}$, $\mathbf{J} = \{0, n\}$, and $V = \mathbb{C}[\mathsf{Mat}_{m,n}]$. In this case, Theorem 1.7 shows that the coefficients $c^{V]}_{\underline{\lambda}|\underline{\mu}}$ in (1.1) are exactly the *Littlewood–Richardson coefficients* $c^{\mu}_{\lambda^{(1)},\lambda^{(2)}}$ (in their coproduct role in the Hopf algebra of symmetric functions). Thus Theorem 1.7 and Main Theorem 1.14 are generalized Littlewood–Richardson rules.

2 Proof of Main Main Theorem 1.14

2.1 Crystal operators and bicrystalline varieties

We briefly review crystal operators on words and matrices, define bicrystalline varieties, and explain how Theorem 1.7 yields combinatorial rules for the coefficients $c^{V}_{\underline{\lambda}|\underline{\mu}}$ in (1.1) whenever $V = \mathbb{C}[\mathfrak{X}]$ for a bicrystalline variety $\mathfrak{X}$. Let $w = w_1 w_2 \dots w_N$ be a word on the alphabet $[n]$ and fix $i \in [n-1]$. Kashiwara ([12]) defined crystal operators on words:

Definition 2.1. The *ith bracket operator* $\mathsf{bracket}_i$ sends w to a word on the alphabet $\{(,)\}$ by recording a ")" for each i and a "(" for each $i+1$ (maintaining order and ignoring all other letters in w).

Definition 2.2. Let $w_e = i+1$ and $w_f = i$ be the letters of w associated to the leftmost unmatched "(" and rightmost unmatched ")" of $\mathsf{bracket}_i(w)$ respectively. The *crystal raising operator* e_i sends w to the word obtained by changing w_e to i. Analogously, the *crystal lowering operator* f_i sends w to the word obtained by changing w_f to $i+1$. If no such letters w_e or w_f exist, the operators output the special symbol $\varnothing$.

Danilov–Koshevoi and van Leeuwen [5, 17] pulled these crystal operators back to commuting operators on $\mathsf{Mat}_{m,n}$ via the row and column words of Definition 1.1:

Definition 2.3. Let $M = [m_{i,j}] \in \mathsf{Mat}_{m,n}(\mathbb{Z}_{\geq 0})$, and let $(i+1, b)$ and (i, a) be the coordinates of the entries of M yielding the letters of $\mathsf{row}(M)$ altered by e_i and f_i respectively. The *row crystal raising operator* e_i^{row} sends M to the matrix $e_i^{\mathsf{row}}(M)$ obtained by subtracting one from $m_{i+1,b}$ and adding one to $m_{i,b}$. Similarly, the *row crystal lowering operator* f_i^{row} sends M to the matrix $f_i^{\mathsf{row}}(M)$ obtained by subtracting one from $m_{i,a}$ and adding one to $m_{i+1,a}$. When $e_i(\mathsf{row}(M)) = \varnothing$ or $f_i(\mathsf{row}(M)) = \varnothing$, we define the corresponding row bicrystal operators to output the special symbol $\varnothing$ instead of a matrix. The *column bicrystal operators* e_j^{col} and f_j^{col} are defined analogously using $\mathsf{col}(M)$ (or using transposes: $e_j^{\mathsf{col}}(M) = (e_j^{\mathsf{row}}(M^t))^t$ and $f_j^{\mathsf{col}}(M) = (f_j^{\mathsf{row}}(M^t)^t)$).

Identifying $\mathsf{m} = \prod_{i,j} z_{ij}^{m_{i,j}}$ with its exponent vector $M = [m_{i,j}] \in \mathsf{Mat}_{m,n}(\mathbb{Z}_{\geq 0})$, we obtain bicrystal operators on monomials. This permits a proof-sketch of Theorem 1.7 based on a non-obvious compatibility between (filtered) RSK and the bicrystal operators.

Proof of Theorem 1.7 (sketch). We first consider the case where $\mathbf{L}_{\mathbf{I}|\mathbf{J}} = \mathbf{GL}$, for which the argument appears in [5, 17]. Since the set $\mathcal{M}$ of all monomials is a $\mathbf{T}$-graded basis for $\mathbb{C}[\mathsf{Mat}_{m,n}]$ and all irreducible $\mathbf{T}$-representations are 1-dimensional, the $\mathbf{T}$-character of $\mathbb{C}[\mathsf{Mat}_{m,n}]$ is the sum of the $\mathbf{T}$-weights of all monomials. Now, we have the expression

$$\mathcal{M} = \bigsqcup_{\mathsf{m} \in \mathcal{M}} \mathcal{C}_{\mathsf{m}},$$

where the union is over monomials m such that $\mathsf{RSK}(\mathsf{m})$ is a pair of highest-weight tableaux, and $\mathcal{C}_{\mathsf{m}}$ is the set of monomials m' connected to m by a sequence of the bicrystal operators. If $\mathsf{RSK}(\mathsf{m}) = (T_\lambda | T_\lambda)$, then the sum over the $\mathbf{T}$-weights of the monomials in $\mathcal{C}_{\mathsf{m}}$ is the character of the irreducible $\mathbf{GL}$-representation $V_\lambda(m) \boxtimes V_\lambda(n)$. This fact can be proved by showing that the row and column crystal operators in Definition 2.3 commute with each other, and thus, via RSK, commute with the usual crystal operators on pairs of semistandard Young tableaux (see [3] for the definition of these operators). Our proof of Theorem 1.7 in [18] is analogous; we replace each $\mathcal{C}_{\mathsf{m}}$ with a "filtered" version $\mathcal{C}_{\mathsf{m}}^{\mathbf{I}|\mathbf{J}}$ connected by the bicrystal operators $e_i^{\mathsf{row}}, f_i^{\mathsf{row}}, e_j^{\mathsf{col}}, f_j^{\mathsf{col}}$ with $i \notin \mathbf{I}$, $j \notin \mathbf{J}$. We then show that these operators commute with the usual crystal operators on a product of tableau crystal graphs via filterRSK. Thus, the sum over the $\mathbf{T}$-weights of the monomials in any $\mathcal{C}_{\mathsf{m}}^{\mathbf{I}|\mathbf{J}}$ is the character of some irreducible representation of $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$, proving the theorem. □

Given any variety $\mathfrak{X} \subseteq \mathsf{Mat}_{m,n}$ and a choice of term order $\prec$, there exists a set $\mathsf{Std}_{\prec}(\mathfrak{X})$ of monomials called *standard monomials* which form a vector space basis for $\mathbb{C}[\mathfrak{X}]$. The set $\mathsf{Std}_{\prec}(\mathfrak{X})$ can be computed algorithmically using Gröbner bases. The next definition concerns a compatibility between standard monomials and the bicrystal operators.

Definition 2.4. $\mathfrak{X} \subseteq \mathsf{Mat}_{m,n}$ is $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$*-bicrystal closed* if there exists a term order $\prec$ such that $e_i^{\mathsf{row}}(\mathsf{m}), f_i^{\mathsf{row}}(\mathsf{m}), e_j^{\mathsf{col}}(\mathsf{m}), f_j^{\mathsf{col}}(\mathsf{m}) \in \mathsf{Std}_{\prec}(\mathfrak{X}) \cup \{\varnothing\}$ for $\mathsf{m} \in \mathsf{Std}_{\prec}(\mathfrak{X}), i \notin \mathbf{I}$, and $j \notin \mathbf{J}$.

Definition 2.5. $\mathfrak{X} \subseteq \mathsf{Mat}_{m,n}$ is $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$*-bicrystalline* if it is $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-stable and $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-bicrystal closed.

Corollary 2.6. *If $V = \mathbb{C}[\mathfrak{X}]$ for a $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-bicrystalline variety $\mathfrak{X} \subseteq \mathsf{Mat}_{m,n}$, then the coefficients $c^V_{\underline{\lambda}|\underline{\mu}}$ in (1.1) are computed by the combinatorial rule*

$$c^{\mathbb{C}[\mathfrak{X}]}_{\underline{\lambda}|\underline{\mu}} = \#\{\mathsf{m} \in \mathsf{Std}_{\prec}(\mathfrak{X}) : \mathsf{filterRSK}_{\mathbf{I}|\mathbf{J}}(\mathsf{m}) = (T_{\underline{\lambda}}|T_{\underline{\mu}})\}. \tag{2.1}$$

Corollary 2.6 is an immediate consequence of the proof of Theorem 1.7 and the definition of $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-bicrystalline varieties. Comparing (1.4) and (2.1), we see that showing $\mathfrak{X}_w$ is $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-bicrystalline will suffice to prove Main Theorem 1.14.

2.2 Matrix Schubert varieties

We now turn to proving Main Theorem 1.14. Equations for the matrix Schubert varieties $\mathfrak{X}_w \subseteq \mathsf{Mat}_{m,n}$ were given in [7]. We recall the standard permutation combinatorics needed.

The *graph* of a partial permutation matrix M_w is an $m \times n$ grid with a $\bullet$ symbol in the entries where M_w has a 1 and blank spaces elsewhere. The *Rothe diagram* of M_w, denoted $D(w)$, consists of all boxes in $[m] \times [n]$ not weakly below or right of a $\bullet$ in the graph of M_w. The *essential set* $E(w)$ of M_w is comprised of the maximally southeast boxes of each connected component of $D(w)$, i.e., $E(w) = \{(i,j) \in D(w) : (i,j+1), (i+1,j) \notin D(w)\}$. The *rank function* for M_w is denoted $r_w : [m] \times [n] \to \mathbb{Z}_{\geq 0}$; it maps (i,j) to the rank of the northwest $i \times j$ submatrix of M_w.

Example 2.7. Figure 1 depicts $M_w \in \mathsf{Mat}_{4,9}$ and $D(w)$ for $w = 61\infty7$. Here, $\mathrm{Desc}_{\mathsf{row}}(w) = \{1,3\}$ and $\mathrm{Desc}_{\mathsf{col}}(w) = \{5\}$. The associated permutation $\overline{w} \in \mathfrak{S}_{4+9}$ from Remark 1.12 is $\overline{w} = 61(10)7234589(11)(12)(13)$.

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

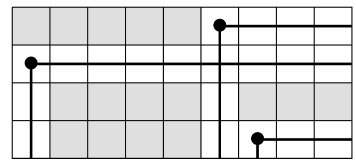

Figure 1: The partial permutation matrix M_w for $w = 61\infty7$ and its Rothe diagram; the boxes of $D(w)$ are shaded.

We now state a concrete description of matrix Schubert varieties. Write $\mathbb{C}[\mathsf{Mat}_{m,n}] = \mathbb{C}[z_{ij} : 1 \leq i \leq m, 1 \leq j \leq n]$, where z_{ij} is the (i, j)-coordinate function.

Theorem 2.8 ([7, Proposition 3.3]). *$\mathfrak{X}_w \subset \mathsf{Mat}_{m,n}$ is the set of $m \times n$ matrices M such that the rank of the northwest $i \times j$ submatrix of M has rank at most $r_w(i, j)$.*

Let $Z = [z_{ij}]_{1 \leq i \leq m, 1 \leq j \leq n}$ be the generic $m \times n$ matrix and set Z_{ij} to be the northwest $i \times j$ submatrix of Z. Then the defining ideal of $\mathfrak{X}_w \subset \mathsf{Mat}_{m,n}$ is the Schubert determinantal ideal

$$I_w := I(\mathfrak{X}_w) = \langle \text{rank } r_w(i,j) + 1 \text{ minors of } Z_{ij} : (i,j) \in E(w) \rangle. \tag{2.2}$$

By Proposition 1.13, $\mathbb{C}[\mathfrak{X}_w]$ is a $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-representation for any $\mathbf{I} \supseteq \mathrm{Desc}_{\mathsf{row}}(w)$ and $\mathbf{J} \supseteq \mathrm{Desc}_{\mathsf{col}}(w)$. We now check that $\mathfrak{X}_w$ is $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-crystal closed in order to compute the $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-graded Hilbert series of $\mathbb{C}[\mathfrak{X}_w]$.

2.3 Standard monomials of $\mathfrak{X}_w$ and proof of Main Theorem 1.14

Fix an *antidiagonal term order* $\prec$ on $S = \mathbb{C}[z_{11}, \ldots, z_{mn}]$, i.e., one that picks the antidiagonal term of any minor of Z. One example is pure lexicographic order obtained by setting $z_{ab} \succ z_{cd}$ if $a < c$, or if $a = c$ and $b > d$. This is Knutson–Miller's Gröbner basis theorem:

Theorem 2.9 ([15, Theorem B]). *Fulton's generators (2.2) for I_w form a Gröbner basis with respect to $\prec$.*

Identifying monomials with nonnegative integer matrices as before, Theorem 2.9 yields the following description of the standard monomials of $\mathfrak{X}_w$:

Corollary 2.10. *A monomial m lies in $\mathsf{Std}_\prec(\mathfrak{X}_w)$ if and only if for any Fulton generator g of I_w from (2.2), the product of the entries of m along the main antidiagonal of g is 0.*

The main result, Main Theorem 1.14, follows immediately from Theorem 2.11 below.

Theorem 2.11. *Let $\mathfrak{X}_w \subseteq \mathsf{Mat}_{m,n}$ and suppose $\mathbf{I} \supseteq \mathrm{Desc}_{\mathsf{row}}(w)$ and $\mathbf{J} \supseteq \mathrm{Desc}_{\mathsf{col}}(w)$. Then $\mathfrak{X}_w$ is $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-bicrystalline.*

Proof. By Proposition 1.13, $\mathfrak{X}_w$ is $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-stable. It remains to show that $\mathfrak{X}_w$ is $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-bicrystal closed. We focus first on the row operators e_i^{row} and f_i^{row}. Since e_i^{row} and f_i^{row} are inverses whenever their outputs are not $\varnothing$, the $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-bicrystal closed claim follows from the (slightly stronger) statement that if $\mathsf{m} \notin \mathsf{Std}_{\prec}(\mathfrak{X}_w)$ (identified with a matrix in $\mathsf{Mat}_{m,n}(\mathbb{Z}_{\geq 0})$), then $e_i^{\mathsf{row}}(\mathsf{m}) \notin \mathsf{Std}_{\prec}(\mathfrak{X}_w)$ for all i and $f_i^{\mathsf{row}}(\mathsf{m}) \notin \mathsf{Std}_{\prec}(\mathfrak{X}_w)$ for $i \notin \mathrm{Desc}_{\mathsf{row}}(w)$. We prove this latter statement.

Since $\mathsf{m} \notin \mathsf{Std}_{\prec}(\mathfrak{X}_w)$, by Corollary 2.10, there exists a Fulton generator g for I_w such that the product of all entries of m along the antidiagonal A_g of g is nonzero. Let $R \subseteq [m]$ and $C \subseteq [n]$ be the row and column indices, respectively, of the minor g.

First, we argue that $e_i^{\mathsf{row}}(\mathsf{m}) \notin \mathsf{Std}_{\prec}(\mathfrak{X}_w)$ by constructing a Fulton generator g' for I_w such that the product of all entries of $e_i^{\mathsf{row}}(\mathsf{m})$ along $A_{g'}$ is positive.

Case e1: $(i, i+1 \notin R)$ Here, e_i^{row} does not affect any of the entries of A_g in m. Take $g' = g$.

Case e2: $(i \in R, i+1 \notin R)$ All entries of A_g in $e_i^{\mathsf{row}}(\mathsf{m})$ are only larger (by at most 1) in comparison to the same entry in m, so we may again take $g' = g$.

Case e3: $(i \notin R, i+1 \in R)$ If e_i^{row} does not affect the entry of A_g in row $i+1$, take $g' = g$. Otherwise, we may take g' to be the minor defined by row indices $R' = (R \setminus \{i+1\}) \cup \{i\}$ and column indices C (which is also a Fulton generator for I_w by (2.2)).

Case e4: $(i, i+1 \in R)$ Let b be as in Definition 2.3. If $m_{i+1,b}$ does not lie on the antidiagonal A_g or $m_{i+1,b} \geq 2$, then we may take $g' = g$. Otherwise, since we assume $e_i^{\mathsf{row}}(\mathsf{m}) \neq \varnothing$, the entry $\mathsf{m}_{i+1,b} = 1$ corresponds to an unmatched "(" in $\mathsf{bracket}_i(\mathsf{row}(\mathsf{m}))$. Let b' be the first column to the right of b in C, so $(i, b') \in A_g$. None of the ")" in $\mathsf{bracket}_i(\mathsf{row}(\mathsf{m}))$ associated to $\mathsf{m}_{i,b'} > 0$ match with this aforementioned "(". In particular, the leftmost ")" associated to $\mathsf{m}_{i,b'}$ matches with a "(" associated to $\mathsf{m}_{i+1,c'} > 0$ for some $b < c' < b'$. Take g' to be the minor defined by row indices R and column indices $C' = (C \setminus \{b\}) \cup \{c'\}$ (which is also a Fulton generator for I_w by (2.2)).

We show that $f_i^{\mathsf{row}}(\mathsf{m}) \notin \mathsf{Std}_{\prec}(\mathfrak{X}_w)$ when $i \notin \mathrm{Desc}_{\mathsf{row}}(w)$ by constructing a Fulton generator g'' for I_w such that the product of entries of $f_i^{\mathsf{row}}(\mathsf{m})$ along $A_{g''}$ is positive.

Case f1: $(i, i+1 \notin R)$ Same argument as Case e1, take $g'' = g$.

Case f2: $(i \in R, i+1 \notin R)$ If f_i^{row} does not affect the entry of A_g in row i, let $g'' = g$. Otherwise, let g'' be the minor that uses the rows $R'' = (R \setminus \{i\}) \cup \{i+1\}$ and columns C (which is also a Fulton generator of I_w provided $i \notin \mathrm{Desc}_{\mathsf{row}}(w)$ by (2.2)).

Case f3: $(i \notin R, i+1 \in R)$ Same argument as Case e2, take $g'' = g$.

Case f4: $(i, i+1 \in R)$ We use left-right "flipped" version of Case e4. Let a be as in Definition 2.3. If $m_{i,a}$ does not lie on A_g or $m_{i,a} \geq 2$, then we may take $g'' = g$. Otherwise, since we assume $f_i^{\mathsf{row}}(\mathsf{m}) \neq \varnothing$, the entry $\mathsf{m}_{i,a} = 1$ corresponds to an unmatched ")" in $\mathsf{bracket}_i(\mathsf{row}(\mathsf{m}))$. Let a' be the first column to the left of a in C, so $(i+1, a') \in A_g$. None of the "(" in $\mathsf{bracket}_i(\mathsf{row}(\mathsf{m}))$ associated to $\mathsf{m}_{i+1,a'} > 0$ match with this aforementioned ")".

In particular, the rightmost "(" associated to $\mathsf{m}_{i+1,a'} > 0$ matches with a ")" associated to $\mathsf{m}_{i,c''} > 0$ for some $a' < c'' < a$. Take g'' to be the minor defined by row indices R and column indices $C'' = (C \setminus \{a\}) \cup \{c''\}$ (which is a Fulton generator of I_w by (2.2)).

The statements for f_j^{col} and e_j^{col} hold by taking transposes of all matrices involved. □

Proof of Main Theorem 1.14. Let $\mathfrak{X}_w \subseteq \mathsf{Mat}_{m,n}$, let $\mathbf{I} \supseteq \mathsf{Desc}_{\mathsf{row}}(w)$, and let $\mathbf{J} \supseteq \mathsf{Desc}_{\mathsf{col}}(w)$. Then Theorem 2.11 shows that $\mathfrak{X}_w$ is $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-bicrystalline. Thus Corollary 2.6 applies, giving the desired combinatorial rule for $c^{\mathbb{C}[\mathfrak{X}_w]}_{\underline{\lambda}|\underline{\mu}}$. □

In fact, Theorem 2.11 extends to show that any $\mathbf{B}$-stable $\mathfrak{X} \subseteq \mathsf{Mat}_{m,n}$ is bicrystalline. The following corollary of Theorem 1.9 is known.

Corollary 2.12. *Any $\mathbf{B}$-stable variety $\mathfrak{X} \subseteq \mathsf{Mat}_{m,n}$ is a finite union of matrix Schubert varieties.*

Proof. There are finitely many $M_w \in \mathsf{Mat}_{m,n}$. Apply Theorem 1.9 and Definition 1.10. □

The standard monomials of any $\mathbf{B}$-stable variety $\mathfrak{X} \subset \mathsf{Mat}_{m,n}$ are described in terms of the standard monomials of matrix Schubert varieties. This follows immediately from work of Knutson on Frobenius splittings [13], although it is not explicitly stated there.[2]

Proposition 2.13 (cf. [13])**.** *Let $\mathfrak{X} \subseteq \mathsf{Mat}_{m,n}$ be a $\mathbf{B}$-stable variety. Write $\mathfrak{X} = \bigcup_{i=1}^{k} \mathfrak{X}_{w^{(i)}}$ as a union of matrix Schubert varieties (by Corollary 2.12). Then the set of standard monomials for $\mathfrak{X}$, with respect to $\prec$, is $\mathsf{Std}_{\prec}(\mathfrak{X}) = \bigcup_{i=1}^{k} \mathsf{Std}_{\prec}(\mathfrak{X}_{w^{(i)}})$.*

When each component $\mathfrak{X}_{w^{(i)}}$ of a $\mathbf{B}$-stable variety $\mathfrak{X}$ is $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-bicrystalline for some $\mathbf{I}$ and $\mathbf{J}$, Proposition 2.13 implies that $\mathfrak{X}$ is $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-bicrystalline as well. Thus Corollary 2.6 applies to $\mathfrak{X}$, giving a combinatorial rule for its $\mathbf{L}_{\mathbf{I}|\mathbf{J}}$-graded Hilbert series via filterRSK.

Acknowledgements

We thank Andrew Hardt for a helpful conversation about [14] and Shiliang Gao for pointing us to the motivating reference [19] on standard monomials and Hilbert series.

References

[1] S. Abhyankar. "Combinatoire des tableaux de Young, variétés déterminantielles et calcul de fonctions de Hilbert". *Rend. Sem. Mat. Univ. Politec. Torino* **42**.3 (1984). Prepared by A. Galligo, pp. 65–88.

[2] A. Bertiger. "Generating the ideals defining unions of Schubert varieties". *Int. Math. Res. Not. IMRN* 21 (2015), pp. 10847–10858. DOI.

[2] See also [2] for Bertiger's construction of a Gröbner basis for arbitrary $\mathbf{B}$-stable varieties.

[3] D. Bump and A. Schilling. *Crystal bases: Representations and combinatorics*. World Scientific Publishing Co. Pte. Ltd, 2017, pp. xii+279.

[4] C. de Concini and C. Procesi. "A characteristic free approach to invariant theory". *Advances in Math.* **21**.3 (1976), pp. 330–354. DOI.

[5] V. I. Danilov and G. A. Koshevoĭ. "Massifs and the combinatorics of Young tableaux". *Uspekhi Mat. Nauk* **60**.2(362) (2005), pp. 79–142. DOI.

[6] P. Doubilet, G.-C. Rota, and J. Stein. "On the foundations of combinatorial theory. IX. Combinatorial methods in invariant theory". *Studies in Appl. Math.* **53** (1974), pp. 185–216. DOI.

[7] W. Fulton. "Flags, Schubert polynomials, degeneracy loci, and determinantal formulas". *Duke Math. J.* **65**.3 (1992), pp. 381–420. DOI.

[8] W. Fulton. *Young tableaux*. Vol. 35. London Mathematical Society Student Texts. With applications to representation theory and geometry. Cambridge University Press, Cambridge, 1997, pp. x+260.

[9] A. Galligo. "Computations of some Hilbert functions related with Schubert calculus". *Algebraic geometry, Sitges (Barcelona), 1983*. Vol. 1124. Lecture Notes in Math. Springer, Berlin, 1985, pp. 79–97. DOI.

[10] S. R. Ghorpade. "Young bitableaux, lattice paths and Hilbert functions". *J. Statist. Plann. Inference* **54**.1 (1996), pp. 55–66. DOI.

[11] J. Herzog and N. V. Trung. "Gröbner bases and multiplicity of determinantal and Pfaffian ideals". *Adv. Math.* **96**.1 (1992), pp. 1–37. DOI.

[12] M. Kashiwara. "On crystal bases". *Representations of groups (Banff, AB, 1994)*. Vol. 16. CMS Conf. Proc. Amer. Math. Soc., Providence, RI, 1995, pp. 155–197.

[13] A. Knutson. "Frobenius splitting, point-counting, and degeneration". 2009. arXiv: 0911.4941.

[14] A. Knutson and E. Miller. "Subword complexes in Coxeter groups". *Adv. Math.* **184**.1 (2004), pp. 161–176. DOI.

[15] A. Knutson and E. Miller. "Gröbner geometry of Schubert polynomials". *Ann. of Math. (2)* **161**.3 (2005), pp. 1245–1318. DOI.

[16] A. Knutson, E. Miller, and A. Yong. "Gröbner geometry of vertex decompositions and of flagged tableaux". *J. Reine Angew. Math.* **630** (2009), pp. 1–31. DOI.

[17] M. A. A. van Leeuwen. "Double crystals of binary and integral matrices". *Electron. J. Combin.* **13**.1 (2006), Research Paper 86, 93 pp. DOI.

[18] A. Price, A. Stelzer, and A. Yong. "Representations from matrix varieties, and filtered RSK". 2024. arXiv:2403.09938.

[19] B. Sturmfels. "Gröbner bases and Stanley decompositions of determinantal rings". *Math. Z.* **205**.1 (1990), pp. 137–144. DOI.