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Signed combinatorial interpretations in algebraic combinatorics

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Abstract. We prove the existence of signed combinatorial interpretations for several large families of structure constants. These families include standard bases of symmetric and quasisymmetric polynomials, as well as various bases in Schubert theory. The results are stated in the language of computational complexity, while the proofs are based on the effective Möbius inversion.

Keywords: symmetric functions, quasisymmetric functions, Schubert polynomials, combinatorial interpretation, structure constants, #P, GapP

1 Introduction

In this paper, we make a systematic study and prove signed combinatorial interpretations for structure constants for many families of symmetric functions, their relatives, and generalizations. We present signed combinatorial interpretations in all cases, leading to the following meta observation:

In algebraic combinatorics, all integral constants
have signed combinatorial interpretations.

This is an extended abstract of [21].

1.1 Signed combinatorial interpretations

Let $W := \{0,1\}^*$ and $W_n := \{0,1\}^n$ be sets of *words*. The length $|w|$ is called the *size* of $w \in W$. A *language* is defined as $A \subseteq W$. Denote $A_n := A \cap W_n$. We say A is in NP if the membership $[w \in^? A]$ can be decided in polynomial time (in the size $|w|$), by a nondeterministic Turing Machine. We view A_n as the set of *combinatorial objects* of size n.

Let $f : W \to \mathbb{N}$ be an integer function. We say that f has a *combinatorial interpretation* (also called a *combinatorial formula*) if $f \in$ #P. This means there is a language $B \subseteq W^2$

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in NP, such that for all $w \in W$ we have $f(w) = \#\{u : (w,u) \in B\}$. In algebraic combinatorics, all standard combinatorial interpretations are in $\#\mathsf{P}$. These include *character degrees* $f^\lambda = \chi^\lambda(1)$, *Kostka numbers* $K_{\lambda\mu}$, and the *Littlewood–Richardson* (LR) *coefficients* $c^\lambda_{\mu\nu}$, see e.g. [18]. For broader discussion, consult [13, 20].

Let $f : \{0,1\}^* \to \mathbb{Z}$ be an integer function. We say that function f has a *signed combinatorial interpretation* (also called *signed combinatorial formula*), if $f = g - h$ for some $g, h \in \#\mathsf{P}$. The set of such functions is denoted $\mathsf{GapP} := \#\mathsf{P} - \#\mathsf{P}$.

In algebraic combinatorics, there are many natural examples of signed combinatorial interpretations. As we mentioned above, these include *character values* $\chi^\lambda(\mu)$ via the *Murnaghan–Nakayama rule*. Another example is the *inverse Kostka numbers* $K^{-1}_{\lambda\mu}$ defined as the entry in the inverse Kostka matrix $(K_{\lambda\mu})^{-1}$, given by the *Eğecioğlu–Remmel rule* [5]. In both cases, the rules subtract the number of certain rim hook tableaux, some with a positive sign and some with a negative, where the sign is easily computable.

Additionally there are many examples of signed combinatorial interpretations for nonnegative functions. Famously, *Kronecker coefficients* $g(\lambda, \mu, \nu)$ are given by the large signed summation of the numbers of 3-dimensional contingency arrays. Another celebrated example is the *Schubert structure constants* $c^\gamma_{\alpha\beta}$ given by the *Postnikov–Stanley formula* in terms of the number of chains in the Bruhat order [23, Corollary 17.13].

1.2 Main results

Let R be a ring and let $\mathrm{Y} := \{\xi_\alpha\}$ be a linear basis in R, where the indices form a set A of combinatorial objects. The *structure constants* $c(\alpha, \beta, \gamma)$ for Y are defined

$$\xi_\alpha \cdot \xi_\beta = \sum_{\gamma \in A} c(\alpha, \beta, \gamma)\, \xi_\gamma \quad \text{where} \quad \alpha, \beta \in A.$$

When the structure constants are integral, one can ask whether the function $\mathbf{c} : A^3 \to \mathbb{Z}$ is in GapP, i.e., has a signed combinatorial interpretation. Additionally, when they are nonnegative, one can ask if $\mathbf{c}$ is in $\#\mathsf{P}$, i.e., has a (usual) combinatorial interpretation.

Theorem 1.1 (classic structure constants)**.** *Let $\Lambda_n = \mathbb{C}[x_1, \ldots, x_n]^{S_n}$ denote symmetric polynomials in n variables. The following bases in Λ_n have structure constants in $\#\mathsf{P}$: Schur polynomials $\{s_\lambda : \ell(\lambda) \le n\}$, monomial symmetric polynomials $\{m_\lambda : \ell(\lambda) \le n\}$, power sum symmetric polynomials $\{p_\lambda : \lambda_1 \le n\}$, elementary symmetric polynomials $\{e_\lambda : \lambda_1 \le n\}$, and complete homogeneous symmetric polynomials $\{h_\lambda : \lambda_1 \le n\}$.*

The last four of these items are completely straightforward and follow directly from their definition. However, the first item is highly nontrivial.

Now consider deformations of Schur polynomials. Fix $q, t, \alpha \in \mathbb{Q}$.

Theorem 1.2. *The following bases in Λ_n have structure coefficients in* $\mathsf{GapP}/\mathsf{FP}$:

- *Jack symmetric polynomials* $\{P_\lambda(x;\alpha) : \ell(\lambda) \leq n\}$*, where* $\alpha > 0$*,*
- *Hall–Littlewood polynomials* $\{P_\lambda(x;t) : \ell(\lambda) \leq n\}$*, where* $0 \leq t < 1$*, and*
- *Macdonald symmetric polynomials* $\{P_\lambda(x;q,t) : \ell(\lambda) \leq n\}$*, where* $0 \leq q, t < 1$*.*

Here $\mathsf{GapP}/\mathsf{FP}$ is a class of rational functions which can be written as f/g where $f \in \mathsf{GapP}$ and $g \in \mathsf{FP}$ is a function which can be computed in polynomial time.

Second, we consider quasisymmetric polynomials which are somewhat intermediate between symmetric and general polynomials:

Theorem 1.3 (quasisymmetric structure constants)**.** *Let* $\mathrm{QS}\textsc{ym}_n \subseteq \mathbb{C}[x_1, \ldots, x_n]$ *be the ring of quasisymmetric polynomials. The following bases have structure constants in* $\#\mathsf{P}$ *:*

- *monomial quasisymmetric polynomial* $\{M_\alpha\}$*, and*
- *fundamental quasisymmetric polynomials* $\{F_\alpha\}$*.*

The following bases have structure constants in GapP *:*

- *dual immaculate polynomials* $\{\mathfrak{S}^*_\alpha\}$*, and*
- *quasisymmetric Schur polynomials* $\{\mathcal{S}_\alpha\}$*.*

The following bases have structure constants in $\#\mathsf{P}/\mathsf{FP}$ *:*

- *type 1 and type 2 quasisymmetric power sums* $\{\Psi_\alpha\}$ *and* $\{\Phi_\alpha\}$*, as well as*
- *combinatorial quasisymmetric power sum* $\{\mathfrak{p}_\alpha\}$*.*

Here we have α *ranges over compositions into at most* n *positive parts.*

The first two items go back to Gessel [8], while the rest are new. Next, recall that Schubert polynomials mentioned above generalize Schur polynomials and form a linear basis in the ring of *all* polynomials. We further generalize Theorem 1.1.

Theorem 1.4 (polynomial structure constants)**.** *In the ring of polynomials* $\mathbb{C}[x_1, \ldots, x_n]$*, the following bases have structure constants in* $\#\mathsf{P}$ *:*

- *monomial slide polynomials* $\{\mathfrak{M}_\alpha\}$*, and*
- *fundamental slide polynomials* $\{\mathfrak{F}_\alpha\}$*.*

The following bases have structure constants in GapP *:*

- *Demazure atoms* $\{\mathsf{atom}_\alpha\}$*,*
- *key polynomials* $\{\kappa_\alpha\}$*,*

- *Schubert polynomials* $\{\mathfrak{S}_\alpha\}$,
- *Lascoux polynomials* $\{\mathfrak{L}_\alpha\}$, *and*
- *Grothendieck polynomials* $\{\mathfrak{G}_\alpha\}$.

Here we $\alpha \in \mathbb{N}^n$ *ranges over compositions into n nonnegative parts.*

The result for Schubert polynomials follows from work of Postnikov and Stanley [23]. We reprove this result in a simpler (but related) way, leading to results in other cases.

Finally, we consider *plethysm*. Let $\pi : \mathrm{GL}(V) \to \mathrm{GL}(W)$ and $\rho : \mathrm{GL}(W) \to \mathrm{GL}(U)$ be polynomial representations of the general linear group. One can define $\rho[\pi] := \rho \circ \pi$ to be the composition of these representations. At the level of characters, the composition above corresponds to *plethysm* of symmetric functions f_μ and g_ν and gives *plethysm coefficients*:

$$f_\mu[g_\nu] = \sum_\lambda a_\lambda(f_\mu, g_\nu)\, s_\lambda \quad \text{where} \quad f_\mu, g_\nu \in \Lambda,$$

and Λ is the inverse limit of Λ_n in the category of graded rings. It was shown by Fischer and Ikenmeyer [6, Section 9] that plethysm coefficients for $s_\lambda[s_\mu]$ are GapP-complete, so they are in GapP. We generalize to those bases listed in Theorem 1.1.

Theorem 1.5 (plethysm coefficients)**.** *Let* $\{f_\lambda\}$ *and* $\{g_\lambda\}$ *be families of symmetric polynomials from the following linear bases*:

$$\{s_\lambda\},\ \{m_\lambda\},\ \{p_\lambda\},\ \{e_\lambda\},\ \{h_\lambda\}.$$

Then the corresponding plethysm coefficients $a_\lambda(f_\mu, g_\nu)$ *are in* GapP.

1.3 Background and motivation

Combinatorial interpretations reveal structure which is a shadow of a rich but non-quantitative geometric or algebraic structure. For example, standard Young tableaux are the leading terms in a linear basis of irreducible S_n modules. On a deeper level, they are a byproduct of the *branching rule*, which comes from S_n having a long subgroup chain.

Having a combinatorial interpretation does wonders for applications. For example, for LR-coefficients, these include the *saturation theorem*, an efficient algorithm for positivity $[c^\lambda_{\mu\nu} >^? 0]$, and various lower and upper bounds. Even with combinatorial interpretations in special cases, remarkable applications follow. For example, for the Kronecker coefficients, these include unimodality and NP-hardness of positivity $[g(\lambda, \mu, \nu) >^? 0]$.

Naturally, a signed combinatorial interpretation is inherently less powerful than an unsigned one. Yet this is usually the best known tool to obtain any results. For the

Kronecker coefficients, the signed combinatorial interpretation mentioned above gives both a fast algorithm to compute the numbers and a sharp upper bound in some cases.

An interesting case study is the Murnaghan–Nakayama (MN) rule for S_n character values $\chi^\lambda(\mu)$, defined as a signed sum over certain rim hook tableaux. This gives upper bounds for character values, which in turn implies upper bounds for mixing times of random walks on S_n generated by conjugacy classes. For μ a rectangle, rim hook tableaux given by the MN rule have the same sign. This led to rich developments, including combinatorial proofs of character orthogonality, applications in probability, tilings, and LLT polynomials describing representations of Hecke algebra at roots of unity.

In a different direction, a signed combinatorial interpretation coming from the *Frobenius formula*, was used to show that deciding positivity $[\chi^\lambda(\mu) > 0]$ of the character value is PH-hard. Moreover, the character absolute value has no combinatorial interpretation, unless one believes the polynomial hierarchy collapses.

We emphasize that some natural combinatorial formulas defining the numbers above are not GapP formulas. For example, the definition of Kronecker coefficients gives:

$$g(\lambda,\mu,\nu) := \langle \chi^\lambda, \chi^\mu \cdot \chi^\nu \rangle = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^\lambda(\sigma)\, \chi^\mu(\sigma)\, \chi^\nu(\sigma),$$

for all $\lambda,\mu,\nu \vdash n$. This only shows that $\{g(\lambda,\mu,\nu)\}$ are in $\mathsf{GapP}/\mathsf{FP}$.

Indeed, while the summation above is in GapP via the MN rule, the division by $n!$ is not allowed in GapP. The same issue appears also when applying *Billey's formula* to compute Schubert coefficients, the *Féray–Śniady formula* for the characters, and Hurwitz's original formula for the *double Hurwitz numbers*. These formulas involve divisions, thus they only prove the corresponding integral functions are in $\mathsf{GapP}/\mathsf{FP}$.

There are cases when the integrality was established algebraically. For example, the integrality $\chi^\lambda(\mu) \in \mathbb{Z}$, follows from the fact that σ and σ^a are conjugate, for all $\sigma \in S_n$ and $(a, \mathrm{ord}(\sigma)) = 1$. The proof uses a Galois theoretic argument and a calculation of cyclotomic polynomials, which cannot be easily translated to a GapP formula. In other words, being in GapP can be a strong result of independent interest.

2 Basic definitions and notations

We use $\mathbb{N} = \{0,1,2,\ldots\}$ and $[n] = \{1,\ldots,n\}$. To simplify notation, for a set X and an element $x \in X$, we write $X - x := X \smallsetminus \{x\}$. Similarly, we write $X + y := X \cup \{y\}$.

Let $\mathcal{P} = (X, \prec)$ be a poset on the ground set X with a partial order "$\prec$". Function $h : X^2 \to \mathbb{R}$ is called *triangular* w.r.t. $\mathcal{P}$ if $h(x,y) = 0$ unless $x \preccurlyeq y$ for all $x,y \in X$, and $h(x,x) \neq 0$ for all $x \in X$. Similarly, function $h : X^2 \to \mathbb{R}$ is called *unitriangular* w.r.t. $\mathcal{P}$ if it is triangular and $h(x,x) = 1$ for all $x \in X$.

Fix n. An *integer partition* λ of k, denoted $\lambda \vdash k$, is a sequence of weakly decreasing nonnegative integers $(\lambda_1,\ldots,\lambda_n)$ which sum up to k. Let $\mathrm{U}_n = \cup_k \{\lambda \in \mathbb{N}^n : \lambda \vdash k\}$.

Similarly a *composition* (sometimes called *weak composition*) α of k, denoted $\alpha \vDash k$, is a sequence of nonnegative integers $(\alpha_1, \ldots, \alpha_n)$ which sum up to k. Let $\mathrm{V}_{n,k} := \{\alpha \in \mathbb{N}^n : \alpha \vDash k\}$, and let $\mathrm{V}_n := \cup_k \mathrm{V}_{n,k}$ be sets of compositions. A *strong composition* $\alpha \vDash k$ has all parts strictly positive. Let $\mathrm{W}_{n,k} := \{\alpha \in \mathbb{N}^m_{\geq 1} : \alpha \vDash k, m \leq n\}$, and let $\mathrm{W}_n := \cup_k \mathrm{W}_{n,k}$ be sets of strong compositions. Let $D(\alpha) := \{(i,j) : i \leq \alpha_j\}$ denote the *diagram* of α.

We write $|\alpha| := \alpha_1 + \ldots + \alpha_n$ the size of the composition, and $\ell(\alpha)$ the number of parts in α. For two compositions $\alpha, \beta \vDash k$, the *dominance order* is defined as follows:

$$\alpha \trianglelefteq \beta \iff \alpha_1 + \ldots + \alpha_i \geq \beta_1 + \ldots + \beta_i \text{ for all } i.$$

For a permutation $\sigma \in S_n$, the *Lehmer code* is a sequence $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{N}^n$ given by $c_i(\sigma) := \#\{j > i : \sigma_i \geq \sigma_j\}$. Denote by S_∞ the set of bijections $w : \mathbb{N}_{\geq 1} \to \mathbb{N}_{\geq 1}$ which eventually stabilize: $w(m) = m$ for m large enough.

The *Young diagram* of shape λ, denoted $[\lambda]$, is $D(\lambda)$. A *semistandard Young tableau* of shape λ is a map $A : [\lambda] \to \mathbb{N}$, which is weakly increasing in rows: $A(i,j) \leq A(i,j+1)$ and strictly increasing in columns: $A(i,j) < A(i+1,j)$. The *content* of A is a sequence $(m_1, m_2, \ldots)$, where m_k is the number of k in the multiset $\{A(i,j)\}$. Let $\mathrm{SSYT}(\lambda, \mu)$ denote the set of semistandard Young tableaux of shape λ and content μ.

3 Effective Möbius inversion

Let $X = \cup X_n$, where $X_n \subseteq \{0,1\}^n$, be a family of combinatorial objects. Let $\mathcal{P} := (X, \prec)$ be a poset such that $x \prec y$ only if $x, y \in X_n$ for some n. We use $\mathcal{P}_n = (X_n, \prec)$ to denote a subposet of $\mathcal{P}$. The *height* of a poset $\mathcal{Q}$, denoted $\mathrm{height}(\mathcal{Q})$, is the size of the maximal chain in $\mathcal{Q}_n$. Then $\mathcal{P}$ has *polynomial height* if $\mathrm{height}(\mathcal{P}_n) \leq Cn^c$, for some fixed $C, c > 0$.

Let $\delta : X^2 \to \{0,1\}$ be the *delta function* defined as $\delta(x,y) = 1$ if $x = y$, and $\delta(x,y) = 0$ otherwise. Let $\xi : X^2 \to \{0,1\}$ be the *incidence function* defined as $\xi(x,y) = 1$ if $x \preccurlyeq y$ and $\xi(x,y) = 0$ otherwise. Then ξ is *poly-time computable*, if for all $x, y \in X_n$ the decision problem $[x \preccurlyeq^? y]$ can be decided in $O(n^c)$ time, for some fixed $c > 0$.

The *Möbius inverse* is a function $\mu(x,y) : X^2 \to \mathbb{Z}$, such that

$$\sum_{z \in X_n} \xi(x,z) \cdot \mu(z,y) = \delta(x,y) \quad \text{for all} \quad x, y \in X_n\,.$$

Proposition 3.1. *Let $\mathcal{P} := (X, \prec)$ be a poset with polynomial height, and suppose that the incidence function ξ is poly-time computable. Then the Möbius inverse function μ is in* GapP.

Let $\eta : X^2 \to \mathbb{Z}$ be unitriangular w.r.t. $\mathcal{P}$, i.e., $\eta(x,x) = 1$ for all $x \in X$, and $\eta(x,y) \neq 0$ implies $x \preccurlyeq y$ and $x, y \in X_n$ for some n. The *inverse* of η (in the incidence algebra), is a function $\rho(x,y) : X^2 \to \mathbb{Z}$, such that

$$\sum_{z \in X_n} \eta(x,z) \cdot \rho(z,y) = \delta(x,y) \quad \text{for all} \quad x, y \in X_n\,.$$

Proposition 3.2. *Let $\mathcal{P} := (X, \prec)$ be a poset with polynomial height, and suppose that the incidence function ξ is poly-time computable. Suppose function η is in* GapP*. Then the inverse function of η is also in* GapP*.*

Combining this with (uni)triangularity of the given bases yields our main results.

4 Symmetric polynomials

Let $\Lambda_n = \mathbb{C}[x_1, \ldots, x_n]^{S_n}$ be the ring of symmetric polynomials. See [18] for background on bases $\{p_\lambda\}$, $\{e_\lambda\}$, $\{h_\lambda\}$, and $\{m_\lambda\}$. Denote by $\mathcal{Q}_m$ the poset on partitions $\lambda \vdash m$ with dominance order $\lambda \trianglelefteq \mu$, for $\lambda, \mu \vdash m$. It is known that the dominance order is a lattice. Clearly, $\text{height}(\mathcal{Q}_m) = O(m^2)$.

4.1 Structure constants

Recall *Schur polynomials* $\{s_\lambda : \lambda \in \mathrm{U}_n\}$ can be defined as

$$s_\lambda(x_1, \ldots, x_n) := \sum_\mu K_{\lambda\mu}\, m_\mu(x_1, \ldots, x_n),$$

where the *Kostka numbers* $K_{\lambda\mu}$ compute the number of semistandard Young tableaux of shape λ and content μ. The *Littlewood–Richardson coefficients* $c^\lambda_{\mu\nu}$ are defined by

$$s_\mu \cdot s_\nu = \sum_\lambda c^\lambda_{\mu\nu}\, s_\lambda\,.$$

Recall that $c^\lambda_{\mu\nu}$ are given as the number of LR-tableaux of shape ν/λ with content μ, a subset of semistandard Young tableaux, see e.g. [18]. Theorem 1.1 follows by the definitions of $\{p_\lambda\}$, $\{e_\lambda\}$, $\{h_\lambda\}$, and $\{m_\lambda\}$ along with the LR rule for $\{s_\lambda\}$.

4.2 (q, t) deformations

Following [17], *Macdonald symmetric polynomials* P_λ can be defined combinatorially:

$$P_\lambda(\boldsymbol{x}; q, t) := \sum_\mu m_\mu(\boldsymbol{x}) \sum_{T \in \mathrm{SSYT}(\lambda,\mu)} \psi_T(q, t),$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\psi_T(q, t)$ is a explicit rational function given by a product formula. For fixed $q, t \in \mathbb{Q}$ such that $0 \le q, t < 1$, this function $\psi : T \to \mathbb{Q}$ is in FP/FP.

The *Hall–Littlewood polynomials* $P_\lambda(\boldsymbol{x}; t)$ are defined as $P_\lambda(\boldsymbol{x}; 0, t)$, see [16]. Similarly, the *Jack symmetric polynomials* $P_\lambda(\boldsymbol{x}; \alpha)$, see [14], specialize Macdonald symmetric polynomials in another direction, $P_\lambda(\boldsymbol{x}; \alpha) := \lim_{t \to 1} P_\lambda(\boldsymbol{x}; t^\alpha, t)$.

By the definition above and the explicit form of ψ_T, Macdonald symmetric polynomials are unitriangular in $m_\mu(\boldsymbol{x})$, see [18, Theorem 2.3]. Using Proposition 3.2 we obtain the GapP result. By the specialization to Hall–Littlewood polynomials and Jack polynomials, we obtain Theorem 1.2.

4.3 (q,t) analogues

One can also view (q,t) as variables and extend Theorem 1.2. For Hall–Littlewood polynomials $P_\lambda(\boldsymbol{x};t) \in \Lambda[t]$, the corresponding Kostka polynomials $K_{\lambda,\mu}(t) \in \mathbb{N}[t]$ are the coefficients of their Schur expansion. They have a known combinatorial interpretation by Lascoux and Schützenberger (see e.g. [18, Section III.6]). Using the LR rule, we have:

Proposition 4.1. *Hall–Littlewood polynomials $\{P_\lambda(t)\}$ have structure constants in* $\#\mathsf{P}$.

Here structure constants form the polynomials $c^\lambda_{\mu\nu}(t) \in \mathbb{N}[t]$. The proposition states there is a $\#\mathsf{P}$ function $f : \{(\lambda,\mu,k)\} \to \mathbb{N}$, such that $c^\lambda_{\mu\nu}(t) = \sum_{k\in\mathbb{N}} f(\lambda,\mu,k,\ell)\, t^k$.

Recall the *modified Macdonald polynomials* $\widetilde{H}_\mu(\boldsymbol{x};q,t) \in \Lambda[q,t]$, see e.g. [11, Theorem 2.8]. They are defined so that the corresponding *(q,t)-Kostka polynomials* $\widetilde{K}_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$ become the coefficients of their expansion in Schur polynomials. The problem of finding a combinatorial interpretation for the (q,t)-Kostka polynomials remains open

On the other hand, a *signed* combinatorial interpretation of $\widetilde{K}_{\lambda\mu}(q,t) \in \mathbb{N}[q,t]$ follows immediately from *Haglund's monomial formula* [11, Appendix A], giving a combinatorial interpretation for coefficients of their expansion in Schur functions, combined with a GapP formula for the (usual) inverse Kostka numbers. Using the LR rule again, we conclude:

Proposition 4.2. *Modified Macdonald polynomials $\{\widetilde{H}_\mu(q,t)\}$ have structure constants in* GapP.

5 Quasisymmetric bases

5.1 Posets of interest

Denote by $\mathcal{Z}_{n,k} = (\mathrm{W}_{n,k}, \lhd)$ the poset on strong compositions w.r.t. the dominance order. Let $\mathcal{Z}_n = \cup_k \mathcal{Z}_{n,k}$. Clearly, we have $\operatorname{height}(\mathcal{Z}_{n,k}) = O(kn)$.

For $\alpha \in \mathbb{N}^m$, define $\mathsf{sort}(\alpha)$ as the partition formed by listing α in weakly decreasing order. For $\alpha, \beta \in \mathrm{W}_{n,k}$ we say β is a *refinement* of α if one can obtain α by adding consecutive parts of β. This defines the *refinement order* "$\preccurlyeq$" on $\mathrm{W}_{n,k}$. Denote by $\mathcal{D}_{n,k} = (\mathrm{W}_{n,k}, \unlhd')$ the poset on $\mathrm{W}_{n,k}$ where

$$\alpha \unlhd' \beta\,, \alpha,\beta \in \mathrm{W}_{n,k} \quad \Longleftrightarrow \quad \begin{cases} \mathsf{sort}(\beta) \lhd \mathsf{sort}(\alpha) \;\text{ if }\; \mathsf{sort}(\beta) \neq \mathsf{sort}(\alpha), \\ \beta \unlhd \alpha \;\text{ if }\; \mathsf{sort}(\beta) = \mathsf{sort}(\alpha). \end{cases}$$

Observe that $\mathrm{height}(\mathcal{D}_{n,k}) = O(kn^3)$.

5.2 Quasisymmetric bases

Let $\mathrm{QSym}_n \subseteq \mathbb{C}[x_1, \ldots, x_n]$ be the ring of quasisymmetric polynomials in n variables. The *monomial quasisymmetric polynomials* $\{M_\alpha : \alpha \in \mathrm{W}_n\}$ are defined as

$$M_\alpha(x_1, \ldots, x_n) := \sum_{1 \le i_1 < \ldots < i_\ell \le n} x_{i_1}^{\alpha_1} \cdots x_{i_\ell}^{\alpha_\ell},$$

where $\ell = \ell(\alpha) \le n$. Clearly, $\{M_\alpha\}_{\alpha \in \mathrm{W}_n}$ is a linear basis in QSym_n.

Following [8], the *fundamental quasisymmetric polynomials* $\{F_\alpha : \alpha \in \mathrm{W}_n\}$ are given by

$$F_\alpha(x_1, x_2, \ldots, x_n) := \sum_{\beta \preccurlyeq \alpha} M_\beta(x_1, x_2, \ldots, x_n).$$

The *dual immaculate polynomials* $\{\mathfrak{S}^*_\alpha : \alpha \in \mathrm{W}_n\}$ can be defined by

$$\mathfrak{S}^*_\alpha(x_1, x_2, \ldots, x_n) := \sum_\beta K^I_{\alpha,\beta} M_\beta(x_1, x_2, \ldots, x_n),$$

where $K^I_{\alpha,\beta}$ counts certain fillings of $D(\alpha)$ with content β, see [3].

The *quasisymmetric Schur polynomials* $\{\mathcal{S}_\alpha : \alpha \in \mathrm{W}_n\}$ can be defined by

$$\mathcal{S}_\alpha(x_1, x_2, \ldots, x_n) := \sum_\beta K^S_{\alpha,\beta} M_\beta(x_1, x_2, \ldots, x_n),$$

where $K^S_{\alpha,\beta}$ counts particular fillings of $D(\alpha)$ with content β, as defined in [10].

By their definition, $\{F_\alpha\}_{\alpha \in \mathrm{W}_n}$ are unitriangular w.r.t. the dominance order. The unitriangular property for $\{\mathfrak{S}^*_\alpha\}_{\alpha \in \mathrm{W}_n}$ and $\{\mathcal{S}_\alpha\}_{\alpha \in \mathrm{W}_n}$ follows by examining the combinatorial objects computing their corresponding Kostka coefficients. Then the results in Theorem 1.3 follow from this unitriangular property combined with Proposition 3.2.

Remark 5.1. Structure constants for dual immaculate and quasisymmetric Schur polynomials can be negative. Thus, Theorem 1.3 is optimal.

For brevity, we exclude the details for the combinatorial quasisymmetric power sums $\{\mathfrak{p}_\alpha : \alpha \in \mathrm{W}_n\}$ [1], as well as the type 1 quasisymmetric power sums $\{\Psi_\alpha : \alpha \in \mathrm{W}_n\}$ and type 2 quasisymmetric power sums $\{\Phi_\alpha : \alpha \in \mathrm{W}_n\}$ [2].

6 Polynomial bases

6.1 Posets of interest

For $k \in \mathbb{Z}_{>0}$, denote by $\mathcal{I}_{n,k} = (\mathrm{V}_{n,k}, \lhd)$ the poset on weak compositions w.r.t. dominance order. Clearly, $\mathrm{height}(\mathcal{I}_{n,k}) = O(kn)$. Let $\mathcal{I}_n = \cup_k \mathcal{I}_{n,k}$. Thus $\mathrm{height}(\mathcal{I}_n) = O(n^3)$.

6.2 Polynomial bases

For $\alpha \in \mathrm{V}_n$, let $\mathrm{flat}(\alpha)$ be the strong composition formed by removing 0's in α. Following [12, Section 3.4], the *monomial slide polynomials* $\{\mathfrak{M}_\alpha : \alpha \in \mathrm{V}_n\}$ and the *fundamental slide polynomials* $\{\mathfrak{F}_\alpha : \alpha \in \mathrm{V}_n\}$ are defined as

$$\mathfrak{M}_\alpha(\boldsymbol{x}) := \sum_{\substack{\beta \trianglelefteq \alpha \\ \mathrm{flat}(\beta) = \mathrm{flat}(\alpha)}} \boldsymbol{x}^\beta \quad \text{and} \quad \mathfrak{F}_\alpha(\boldsymbol{x}) := \sum_{\substack{\beta \trianglelefteq \alpha \\ \mathrm{flat}(\beta) \preccurlyeq \mathrm{flat}(\alpha)}} \boldsymbol{x}^\beta .$$

The *Demazure atoms* $\{\mathsf{atom}_\alpha : \alpha \in \mathrm{V}_n\}$ can be defined as $\mathsf{atom}_\alpha(\boldsymbol{x}) := \sum_\beta K^{\mathsf{atom}}_{\alpha,\beta}\, \boldsymbol{x}^\beta$. Here $K^{\mathsf{atom}}_{\alpha,\beta}$ counts particular fillings of $T : D(\alpha) \to \mathbb{N}_{\geq 1}$, see [19]. The *key polynomials* $\{\kappa_\alpha : \alpha \in \mathrm{V}_n\}$ and *Lascoux polynomials* $\{\mathcal{L}_\alpha : \alpha \in \mathrm{V}_n\}$ can be computed as

$$\kappa_\alpha(\boldsymbol{x}) := \sum_{S \in \mathsf{Koh}(D(\alpha))} \boldsymbol{x}^{\mathtt{wt}(S)} \quad \text{and} \quad \mathcal{L}_\alpha(\boldsymbol{x}) := \sum_{S \in \mathsf{KKoh}(D(\alpha))} (-1)^{|\alpha| - \#S}\, \boldsymbol{x}^{\mathtt{wt}(S)}.$$

Here $\mathsf{Koh}(D(\alpha))$ and $\mathsf{KKoh}(D(\alpha))$ are recursively generated diagrams, see [15] and [22], respectively.

Let $D \subset [n] \times [n]$ be a diagram and let $(i,j) \in D$ be a box in the diagram. The *ladder move* is a transformation $D \to D - (i,j) + (i-k, j+1)$ and the *K-ladder move* is a transformation $D \to D + (i-k, j+1)$, allowed only when the following are satisfied:

- $(i, j+1) \notin D$,
- $(i-k, j), (i-k, j+1) \notin D$ for some $0 < k < i$, and
- $(i-l, j), (i-l, j+1) \in D$ for all $0 < l < k$.

Recall that the *Lehmer code* $\mathsf{code}(w) \in \mathbb{N}^n$ uniquely determines $w \in S_\infty$. Let $\mathsf{rPipes}(w)$ denote the set of diagrams obtainable through successive ladder moves, starting from $D(\mathsf{code}(w))$, where $w \in S_n$. Similarly take $\mathsf{Pipes}(w)$ to be the set of diagrams obtainable through successive ladder and K-ladder moves, starting from $D(\mathsf{code}(w))$, where $w \in S_n$.

Using [4] and [7], the *Schubert polynomials* $\{\mathfrak{S}_w : w \in S_n\}$ and *Grothendieck polynomials* $\{\mathfrak{G}_w : w \in S_n\}$ can be defined as

$$\mathfrak{S}_w(\boldsymbol{x}) := \sum_{P \in \mathsf{rPipes}(w)} \boldsymbol{x}^{\mathtt{wt}(P)} \quad \text{and} \quad \mathfrak{G}_w(\boldsymbol{x}) = \sum_{P \in \mathsf{Pipes}(w)} (-1)^{|\alpha| - \#P}\, \boldsymbol{x}^{\mathtt{wt}(P)}.$$

Since $\mathsf{code}(w)$ uniquely determines w, we write $\mathfrak{S}_\alpha := \mathfrak{S}_{\mathsf{code}^{-1}(\alpha)}$ and $\mathfrak{G}_\alpha := \mathfrak{G}_{\mathsf{code}^{-1}(\alpha)}$.

Theorem 1.4 follows from Proposition 3.2 with the fact that each basis is unitriangular.

Remark 6.1. The poset of monomials is also unitriangular w.r.t. the reverse lexicographic order, so the Möbius inversion can also be used in this setting. However, the height of the resulting poset is exponential, so Proposition 3.2 is not applicable.

Remark 6.2. It is easy to see that the structure constants for Demazure atoms, key, and Lascoux polynomials can be negative without predictable signs. Thus, Theorem 1.3 proving their signed combinatorial interpretation is optimal in this case.

7 Plethysm and further applications

The proof of Theorem 1.5 follows by the inverse Kostka formula once we show that the coefficients in the corresponding monomial expansion are in GapP. This follows by extending the argument in [6, Section 9].

We conclude with one additional application relating structure constants.

Theorem 7.1. *Let $A = \cup A_n$ be a family of combinatorial objects, and let $\{G_w(\boldsymbol{x}) : w \in A\}$ be a family of symmetric polynomials such that $G_w(\boldsymbol{x}) = \sum_{\alpha \in \mathrm{V}_n} c_{w\alpha} F_\alpha(\boldsymbol{x})$ where the coefficients $\{c_{w\alpha}\}$ are in* GapP*. Consider the coefficients defined by $G_w(\boldsymbol{x}) = \sum_{\lambda \in \mathrm{U}_n} d_{w\lambda} s_\lambda(\boldsymbol{x})$. Then $\{d_{w\lambda}\}$ are also in* GapP*. Furthermore, the result holds when the F_α are replaced with M_α.*

Remark 7.2. Combining Theorem 7.1 with [9, Equation (82)] gives a GapP formula for the Schur expansion of *LLT polynomials*. While this expansion is proven to be Schur-positive, there is no known (unsigned) combinatorial interpretation.

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