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Rank-selected Segre powers of the Boolean lattice

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Abstract. Segre products of posets were defined by Björner and Welker [J. Pure Appl. Algebra (2005)]. We determine the rank-selected homology representations of the t-fold Segre power $B_n^{(t)}$ of the Boolean lattice B_n, which carries an action of the t-fold direct product $\mathfrak{S}_n^{\times t}$ of the symmetric group $\mathfrak{S}_n$. We give formulas for the decomposition into $\mathfrak{S}_n^{\times t}$-irreducibles of the homology of the full poset, as well as for the diagonal action of $\mathfrak{S}_n$. We show that the stable principal specialisation of the product Frobenius characteristic coincides with the corresponding rank-selected invariant of the t-fold Segre power of the subspace lattice.

1 Introduction

Let B_n denote the Boolean lattice of subsets of an n-element set, and let $B_{n,q}$ denote the lattice of subspaces of an n-dimensional vector space over the finite field $\mathbb{F}_q$ with q elements. Both lattices are well known to be Cohen–Macaulay [11].

The *Segre product* of posets was first defined by Björner and Welker, who showed [4, Theorem 1] that this operation preserves the property of being homotopy Cohen–Macaulay. Let $P^{(t)}$ denote the t-fold Segre power $P \circ \cdots \circ P$ (t factors) of a graded poset P. The Segre square $P \circ P$ was studied by the first author in [6], when P is the Boolean lattice B_n or the subspace lattice $B_{n,q}$. Segre powers of the subspace lattice $B_{n,q}$ appear in an early paper of Stanley [9, Example 1.2], as an example of a binomial poset. See also [11, Example 3.18.3].

The symmetric group $\mathfrak{S}_n$ acts on B_n, and hence the Cohen–Macaulay Segre power $B_n^{(t)}$ of B_n carries two actions, one for the t-fold direct product $\mathfrak{S}_n^{\times t}$ of $\mathfrak{S}_n$ with itself, and the other for $\mathfrak{S}_n$ acting diagonally.

We study both these actions on the rank-selected subposets of the Cohen–Macaulay poset $B_n^{(t)}$, giving formulas for their irreducible decomposition for the top homology of $B_n^{(t)}$. For the t-fold Segre power of the subspace lattice, a special case of a theorem of Stanley [9, Theorem 3.1] gives the rank-selected Möbius number. At $q = 1$ this interprets the dimension of the top homology of $B_n^{(t)}$ as the number $w_n^{(t)}$ of t-tuples of permutations

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in the symmetric group $\mathfrak{S}_n$ with no common ascent. The numbers $w_n^{(2)}$ first appear in work of Carlitz, Scoville and Vaughn [5]. For arbitrary t the numbers $w_n^{(t)}$ also appear in [1]. Our work lifts Stanley's enumerative connection between the Segre powers $B_n^{(t)}$ and $B_{n,q}^{(t)}$ to their rank-selected homology modules (Theorem 5.4).

We use the Whitney homology technique of [13] and an extension of the *product Frobenius characteristic* introduced in [6] to obtain explicit formulas for the (rank-selected) homology representation of $B_n^{(t)}$. A key feature of these formulas is Definition 4.6, where we introduce an injective algebra homomorphism $\Phi_t : \Lambda_n(x) \to \otimes_{j=1}^t \Lambda_n(X^j)$ from the algebra of symmetric functions of homogeneous degree n in a single set of variables, to the tensor product of the algebras of degree n-symmetric functions in t sets of variables. We show that Φ_t maps the elementary symmetric function e_n to the product Frobenius characteristic $\beta_n^{(t)}$ of the top homology of $B_n^{(t)}$. By exploiting properties of the homomorphism Φ_t, we obtain the results outlined below. See [7] for the full paper.

(1) Theorem 4.9 gives the irreducible decomposition of the top homology $\tilde{H}_{n-2}(B_n^{(t)})$ of $B_n^{(t)}$ under the action of $\mathfrak{S}_n^{\times t}$.

(2) Theorem 4.12 gives a formula for the irreducible decomposition of the diagonal $\mathfrak{S}_n$-action on $\tilde{H}_{n-2}(B_n^{(t)})$, and an explicit formula for the characters.

(3) Theorem 5.3 gives a recursive formula for the product Frobenius characteristic of the rank-selected homology, from which one can obtain explicit formulas for the irreducible decomposition.

(4) Theorem 5.4 shows that the stable principal specialisation of the product Frobenius characteristic of the rank-selected homology of $B_n^{(t)}$ gives, up to a factor, the corresponding rank-selected invariant for $B_{n,q}^{(t)}$.

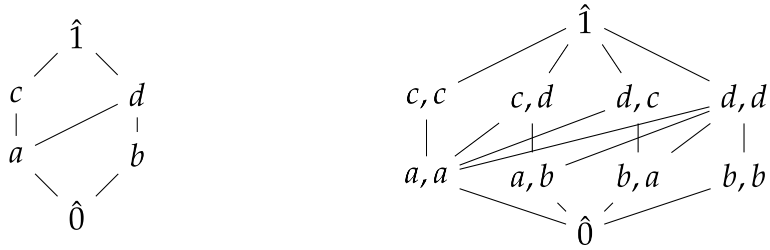
All homology in this work is reduced, and is taken with rational coefficients.

2 Segre powers and rank-selected invariants

We refer the reader to [3]and [11, Chapter 3] for background on posets and topology.

Recall [11] that the product poset $P \times Q$ of two posets P, Q has order relation defined by $(p, q) \leqslant (p', q')$ if and only if $p \leqslant_P p'$ and $q \leqslant_Q q'$. Segre products are defined in greater generality by Björner and Welker in [4]. This paper is concerned with the following special case.

Definition 2.1 ([4]). *Let P be a bounded graded poset. The t-fold Segre power $P^{(t)} := P \circ \cdots \circ P$ (t factors), $t \geqslant 2$, is the induced subposet of the t-fold product poset $P \times \cdots \times P$ (t factors) consisting of t-tuples $(x_1, \ldots, x_t)$ such that* $\operatorname{rank}(x_i) = \operatorname{rank}(x_j), 1 \leqslant i, j \leqslant t$. *When $t = 1$ we set $P^{(1)}$ equal to P.*



(a) Hasse diagram of P.

(b) Hasse diagram of $P \circ P$

Figure 1: $P \circ P$ is an induced subposet of the product poset $P \times P$.

It follows that $P^{(t)}$ is also a ranked poset which inherits the rank function of P. Figure 1 shows the Segre square $P \circ P$ of a poset P, an induced subposet of $P \times P$. Missing in $P \circ P$ are these elements in the product $P \times P$: $(a,c),(a,d),(b,c),(b,d),(c,a),(c,b),$ $(d,a),(d,b)$, as well as all $(\hat{0},y),(x,\hat{0}),(x,\hat{1}),(\hat{1},y)$ for $x,y \in P$.

2.1 Rank-selected invariants

Let P be a finite graded bounded poset of rank n, and let $J \subset [n-1] = \{1,\ldots,n-1\}$ be any subset of nontrivial ranks. Let $P(J)$ denote the rank-selected bounded subposet of P consisting of elements in the rank-set J, together with $\hat{0}$ and $\hat{1}$. Stanley [11, Section 3.13] defined two rank-selected invariants $\tilde{\alpha}_P(J)$ and $\tilde{\beta}_P(J)$ as follows: $\tilde{\alpha}_P(J)$ is the number of maximal chains in the rank-selected subposet $P(J)$, and $\tilde{\beta}_P(J)$ is the integer defined by either of the equivalent equations

$$\tilde{\beta}_P(J) = \sum_{U \subseteq J} (-1)^{|J|-|U|} \tilde{\alpha}_P(U), \quad \tilde{\alpha}_P(J) = \sum_{U \subseteq J} \tilde{\beta}_P(U). \tag{2.1}$$

The Möbius number of the rank-selected subposet $P(J)$ [11, Equation (3.54)] is:

$$\tilde{\beta}_P(J) = (-1)^{|J|-1} \mu_{P(J)}(\hat{0},\hat{1}). \tag{2.2}$$

When the poset P has the recursive structure described in the lemma below, the rank-selected invariants satisfy a pleasing recurrence that we will use later in Section 5.1.

Lemma 2.2. *Let P be a graded, bounded poset of rank n, with the property that for any $x \in P$, the poset structure of the interval $(\hat{0},x)$ depends only on the rank of x; we may write $P_i = (\hat{0},x_0)$ for any x_0 of rank i. Let $wh_i(P)$ denote the number of elements of P at rank i. Then we have a recurrence for the rank-selected invariants $\tilde{\beta}_P(J)$ of P, $J = \{1 \leqslant j_1 < \cdots < j_r \leqslant n-1\} \subseteq [n-1]$.*

$$\tilde{\beta}_P(J) + \tilde{\beta}_P(J \setminus \{j_r\}) = wh_{j_r}(P) \cdot \tilde{\beta}_{P_{j_r}}(J \setminus \{j_r\}) \tag{2.3}$$

$$\mu_P(\hat{0},\hat{1}) = -\sum_{i=0}^{n-1} wh_i(P) \cdot \mu_{P_i}(\hat{0},\hat{1}). \tag{2.4}$$

The condition of the lemma is satisfied by the Boolean lattice B_n and also by the subspace lattice $B_{n,q}$. We can now derive a recurrence for the rank-selected invariants $\tilde{\beta}_{B_{n,q}^{(t)}}(J)$, $J \subseteq [n-1]$, of the t-fold Segre power $B_{n,q}^{(t)}$. From (2.2), these are also the unsigned Möbius numbers of the corresponding rank-selected subposets.

Recall that the number of i-dimensional subspaces of the n-dimensional vector space $\mathbb{F}_q^n$ [11, Proposition 1.7.2] is given by the q-binomial coefficient

$$\begin{bmatrix} n \\ i \end{bmatrix}_q := \frac{(1-q^n)(1-q^{n-1})\cdots(1-q)}{(1-q^i)(1-q^{i-1})\cdots(1-q)\,(1-q^{n-i})(1-q^{n-i-1})\cdots(1-q)}. \tag{2.5}$$

Let $(-1)^{n-2}W_n^{(t)}(q)$ be the Möbius number of the t-fold Segre power of the subspace lattice. For $q=1$, $B_{n,q}$ is just B_n; hence Proposition 2.5 gives $W_n^{(t)}(1) = w_n^{(t)}$. To avoid an excess of parentheses, for the kth power of the q-binomial coefficient we write $\left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_q^k$.

Proposition 2.3. *We have the following recurrences for the rank-selected invariants of $B_{n,q}^{(t)}$ and for the full poset $B_{n,q}^{(t)}$. Here the rank-set J is given by $J = \{1 \leqslant j_1 < \cdots < j_r \leqslant n-1\}$.*

$$\tilde{\beta}_{B_{n,q}^{(t)}}(J) + \tilde{\beta}_{B_{n,q}^{(t)}}(J \setminus \{j_r\}) = \begin{bmatrix} n \\ j_r \end{bmatrix}_q^t \tilde{\beta}_{B_{j_r,q}^{(t)}}(J \setminus \{j_r\}), \tag{2.6}$$

$$W_n^{(t)}(q) = \textstyle\sum_{i=0}^{n-1} (-1)^{n-1-i} \left[\begin{smallmatrix} n \\ i \end{smallmatrix}\right]_q^t W_i^{(t)}(q).$$

The recurrence (2.6), in conjunction with an equivariant version of the recurrence in Lemma 2.2 for the Boolean lattice derived in Section 5, will be used in Section 5.1 when we consider the stable principal specialisation.

We conclude this section by explaining the relevance of the numbers $w_n^{(t)}$ mentioned in the Introduction. The following expression for $\tilde{\beta}_{B_{n,q}^{(t)}}(J)$ is due to Stanley. Let $\mathrm{Asc}(\sigma)$ denote the ascent set of σ, i.e. the set $\{i : 1 \leqslant i \leqslant n-1, \sigma(i) > \sigma(i+1)\}$ of ascents of σ.

Theorem 2.4 ([9, Theorem 3.1])**.** *Let $J \subseteq [n-1]$. Write $J^c = [n-1] \setminus J$. Then for the rank-selected t-fold Segre power $B_{n,q}^{(t)}$ of the subspace lattice, one has*

$$\tilde{\beta}_{B_{n,q}^{(t)}}(J) = \textstyle\sum_{\substack{(\sigma^1,\ldots,\sigma^t) \in \mathfrak{S}_n^{\times t} \\ J^c = \cap_{i=1}^t \mathrm{Asc}(\sigma^i)}} \prod_{i=1}^t q^{\mathrm{inv}(\sigma^i)}.$$

The Möbius number of $B_{n,q}^{(t)}$ is thus $(-1)^{n-2}W_n^{(t)}(q)$, where $W_n^{(t)}(q) := \sum_{(\sigma_1,\ldots\sigma_t)} \prod_{i=1}^t q^{\mathrm{inv}(\sigma_i)}$, and the sum is over all t-tuples of permutations in $\mathfrak{S}_n$ with no common ascent.

Setting $q=1$ gives the special case of the Segre powers of the Boolean lattice, as mentioned in the Introduction. The numbers $w_n^{(t)}$ also appear in [1].

Proposition 2.5 (See [9, Equation (28) and Theorem 3.1]). *The Möbius number of $B_n^{(t)}$ is given by $(-1)^n w_n^{(t)}$, where for $n \geqslant 1$, $w_n^{(t)}$ is the number of t-tuples of permutations in $\mathfrak{S}_n$ with no common ascent. Hence, setting $w_0^{(t)} = 1$, the numbers $w_n^{(t)}$ satisfy the recurrence*

$$\sum_{i=0}^{n} (-1)^i w_i^{(t)} \binom{n}{i}^t = 0. \tag{2.7}$$

One has the generating function $\sum_{n \geqslant 0} w_n^{(t)} \frac{z^n}{n!^t} = \frac{1}{f(z)}$, where $f(z) = \sum_{n \geqslant 0} (-1)^n \frac{z^n}{n!^t}$.

More generally, for the rank selection $J \subseteq [n-1]$, the Möbius number $\mu(B_n^{(t)}(J))$ of $B_n^{(t)}(J)$ is given by $(-1)^{|J|-1} w_n^{(t)}(J)$, where $w_n^{(t)}(J)$ is the number of t-tuples of permutations in $\mathfrak{S}_n$ such that their set of common ascents coincides with the complement of J in $[n-1]$.

3 The product Frobenius characteristic

We refer to [8] for all background on symmetric functions and representations of the symmetric group $\mathfrak{S}_n$. In particular, h_n, e_n and p_n are respectively the homogeneous, elementary, and power sum symmetric functions of degree n, giving rise to basis elements h_λ, e_λ and p_λ indexed by partitions λ of n, in the algebra of symmetric functions of homogeneous degree n, while s_λ is the Schur function indexed by λ.

The action of the symmetric group $\mathfrak{S}_n$ on the Boolean lattice B_n extends naturally to an action of the t-fold direct product $\mathfrak{S}_n^{\times t} := \mathfrak{S}_n \times \cdots \times \mathfrak{S}_n$ (t factors), on the Segre power $B_n^{(t)}$. To study this action, we define a *product Frobenius map* Pch, generalizing the well-known ordinary Frobenius characteristic in [8]. See [6] for the case $t = 2$.

As in [8, Chapter 1, Section 7], let R^n denote the vector space spanned by the irreducible characters of the symmetric group $\mathfrak{S}_n$ over $\mathbb{Q}$, or equivalently the vector space spanned by the class functions of $\mathfrak{S}_n$. Let $R = \oplus_{n \geqslant 0} R^n$. Then R is equipped with the structure of a graded commutative and associative ring with identity element 1 for the group $\mathfrak{S}_0 = \{1\}$, arising from the bilinear map $R^m \times R^n \to R^{m+n}$, defined by $(f, g) \mapsto (f \times g) \uparrow_{\mathfrak{S}_m \times \mathfrak{S}_n}^{\mathfrak{S}_{m+n}}$, the induced character from f and g.

Let $\Lambda^m(X)$ be the ring of symmetric functions in the set of variables X, of homogeneous degree m, and let $\Lambda(X) = \oplus_{m \geqslant 0} \Lambda^m(X)$. Write $\mu \vdash n$ for an integer partition $\mu = (\mu_1 \geqslant \cdots \geqslant \mu_\ell)$ of the integer $n \geqslant 1$, so that $\sum_{i=1}^{\ell} \mu_i = n$, $\mu_i \geqslant 1$ for all i, and $\ell(\mu)$ for the number of parts μ_i of μ. (There is only one integer partition of 0, the empty partition with zero parts.)

Let $\underline{n} = (n_1, \ldots, n_t) \in \mathbb{Z}_{\geqslant 0}^t$ be a t-tuple of nonnegative integers, and let $\mathfrak{S}_{\underline{n}}$ be the direct product of symmetric groups $\times_{i=1}^t \mathfrak{S}_{n_i}$. The irreducible characters of $\mathfrak{S}_{\underline{n}}$ are indexed by t-tuples of partitions $\underline{\boldsymbol{\lambda}} = (\lambda^1, \ldots, \lambda^t)$ where $\lambda^i \vdash n_i$. Let $R^{\underline{n}}$ denote the vector space spanned by the irreducible characters, of the direct product of symmetric groups $\mathfrak{S}_{\underline{n}}$ over $\mathbb{Q}$. Then $R^{\underline{n}} = \otimes_i R^{n_i}$. Let $\underline{R} = \oplus_{\underline{n} \in \mathbb{Z}_{\geqslant 0}^t} R^{\underline{n}}$.

Let (X^i), $i = 1, \ldots, t$ be t sets of variables. For each i we consider the ring of symmetric functions $\Lambda^{n_i}(X^i)$ in the variables (X^i), of homogeneous degree n_i. As in [8, Chapter 1, Section 5, Example 25], we identify the tensor product $\bigotimes_{i=1}^t \Lambda^{n_i}(X^i)$ with products of functions of t sets of variables $(X^i)_{i=1}^t$, symmetric in each set separately, i.e., with the vector space spanned by the set of elements $\left\{\prod_{i=1}^t f_{n_i}(X^i) : f_{n_i}(X^i) \in \Lambda^{n_i}(X^i)\right\}$. Thus $\bigotimes_{i=1}^t f_{n_i}(X^i) \mapsto \prod_{i=1}^t f_{n_i}(X^i)$.

Definition 3.1 (cf. [6, Definition 3.2])**.** *Define the product Frobenius characteristic* $\mathrm{Pch} : R^{\underline{n}} \to \bigotimes_{i=1}^t \Lambda^{n_i}(X^i)$ *as follows. Let* $f_{n_i} \in R^{n_i}$ *and define*

$$\mathrm{Pch}\left(\bigotimes_{i=1}^t f_{n_i}\right) := \prod_{i=1}^t \mathrm{ch}(f_{n_i})(X^i),$$

where ch *denotes the ordinary Frobenius characteristic map on R. This can be extended multilinearly to all of $R^{\underline{n}}$. In particular for the irreducible character $\chi^{\underline{\lambda}} = \bigotimes_{i=1}^t \chi^{\lambda^i}$ indexed by the t-tuple $\underline{\lambda} = (\lambda^1, \ldots, \lambda^t)$, we have* $\mathrm{Pch}(\chi^{\underline{\lambda}}) = \prod_{i=1}^t s_{\lambda^i}(X^i)$*, a product of Schur functions in t different sets of variables.*

Expanding in terms of power sum symmetric functions, for an arbitrary character χ of $\mathfrak{S}_{\underline{n}}$ we obtain the formula $\mathrm{Pch}(\chi) = \sum_{\underline{\mu}} \chi(\underline{\mu}) \prod_{i=1}^t z_{\mu^i}^{-1} \prod_{i=1}^t p_{\mu^i}(X^i)$*, where we have written $\chi(\underline{\mu})$ for the value of the character χ on the conjugacy class of $\mathfrak{S}_{\underline{n}}$ indexed by the t-tuple $\underline{\mu} = (\mu^1, \ldots, \mu^t)$, $\mu^i \vdash n_i$, and z_μ is the order of the centraliser in $\mathfrak{S}_n$ of an element of cycle-type $\mu \vdash n$.*

When $t = 1$, $\mathrm{Pch}(\chi) = \mathrm{ch}(\chi)$ for all characters χ of $\mathfrak{S}_n$, and the product Frobenius characteristic coincides with the ordinary characteristic map.

There is an inner product on $\bigotimes_{i=1}^t \Lambda^{n_i}(X^i)$ defined by

$$\left\langle \prod_{i=1}^t f_i, \prod_{i=1}^t g_i \right\rangle := \prod_{i=1}^t \langle f_i, g_i \rangle_{\Lambda^{n_i}(X^i)},$$

where $\langle f_i, g_i \rangle_{\Lambda^{n_i}(X^i)}$ is the usual inner product [8] in a single set of variables X^i.

Example 3.2. *Let $t = 2$. Consider the regular representation ψ of $\mathfrak{S}_2 \times \mathfrak{S}_3$. Then ψ decomposes into irreducibles as* $\chi^{((2),(3))} + \chi^{((1^2),(3))} + 2\chi^{((2),(2,1))} + 2\chi^{((1^2),(2,1))} + \chi^{((2),(1^3))} + \chi^{((1^2),(1^3))}$*. Using X^1 and X^2 for the two sets of variables, we have*

$$\begin{aligned}\mathrm{Pch}(\psi) &= s_{(2)}(X^1)s_{(3)}(X^2) + s_{(1^2)}(X^1)s_{(3)}(X^2) + 2s_{(2)}(X^1)s_{(2,1)}(X^2)\\ &\quad + 2s_{(1^2)}(X^1)s_{(2,1)}(X^2) + s_{(2)}(X^1)s_{(1^3)}(X^2) + s_{(1^2)}(X^1)s_{(1^3)}(X^2)\\ &= h_1^2(X^1)h_1^3(X^2)\end{aligned}$$

We want Pch to be a ring homomorphism with respect to an induction product akin to the usual induction product. In [6, Definition 3.6], this induction product was defined to take an ordered pair (ψ, ϕ) where ψ is a character of $\mathfrak{S}_k \times \mathfrak{S}_\ell$ and ϕ is a character of

$\mathfrak{S}_m \times \mathfrak{S}_n$, and produce a character of $\mathfrak{S}_{k+m} \times \mathfrak{S}_{\ell+n}$. For the t-fold products, we wish to take a character ψ of $\mathfrak{S}_{\underline{m}} = \bigtimes_{i=1}^t \mathfrak{S}_{m_i}$ and a character ϕ of $\mathfrak{S}_{\underline{n}} = \bigtimes_{i=1}^t \mathfrak{S}_{n_i}$, and map the pair (ψ, ϕ) to a character of $\mathfrak{S}_{\underline{m}+\underline{n}} = \bigtimes_{i=1}^t \mathfrak{S}_{m_i+n_i}$. Here $\underline{m} + \underline{n} = (m_1 + n_1, \ldots, m_t + n_t)$.

Definition 3.3. *Let $\underline{\boldsymbol{\lambda}} = (\lambda^1, \ldots, \lambda^t), \lambda^i \vdash m_i$ and $\underline{\boldsymbol{\mu}} = (\mu^1, \ldots, \mu^t), \mu^i \vdash n_i$, so that $\chi^{\underline{\boldsymbol{\lambda}}} = \bigotimes_{i=1}^t \chi^{\lambda^i}$ and $\chi^{\underline{\boldsymbol{\mu}}} = \bigotimes_{i=1}^t \chi^{\mu^i}$ are respectively irreducible characters of $\mathfrak{S}_{\underline{m}}$ and $\mathfrak{S}_{\underline{n}}$. The t-fold* induction product $\chi^{\underline{\boldsymbol{\lambda}}} \circ \chi^{\underline{\boldsymbol{\mu}}}$ *is then defined to be the induced character*

$$\chi^{\underline{\boldsymbol{\lambda}}} \circ \chi^{\underline{\boldsymbol{\mu}}} := \bigotimes_{i=1}^{t} (\chi^{\lambda^i} \otimes \chi^{\mu^i}) \uparrow_{\mathfrak{S}_{m_i} \times \mathfrak{S}_{n_i}}^{\mathfrak{S}_{m_i+n_i}}, \tag{3.1}$$

a character of the direct product $\mathfrak{S}_{\underline{m}+\underline{n}}$. Now extend this definition multilinearly to any pair of representations ψ of $\mathfrak{S}_{\underline{m}} = \bigtimes_i \mathfrak{S}_{m_i}$ and ϕ of $\mathfrak{S}_{\underline{n}} = \bigtimes_i \mathfrak{S}_{n_i}$, to produce a new representation $\psi \circ \phi$ of $\mathfrak{S}_{\underline{m}+\underline{n}}$.

Proposition 3.4. *The map* Pch *is a bijective ring homomorphism, with respect to the induction product $\circ$ in $\underline{R}$, from $\underline{R}$ to $\bigotimes_{i=1}^t \Lambda(X^i)$. Explicitly, if $\underline{\boldsymbol{m}}, \underline{\boldsymbol{n}} \in \mathbb{Z}_{\geqslant 0}^t$ and ψ and ϕ are characters of $\mathfrak{S}_{\underline{m}}$ and $\mathfrak{S}_{\underline{n}}$ respectively, then* $\mathrm{Pch}(\psi \circ \phi) = \mathrm{Pch}(\psi) \cdot \mathrm{Pch}(\phi)$.

We will use the following important special case in the next section.

Corollary 3.5. *Let ψ be a character of the t-fold direct product $\mathfrak{S}_r^{\times t}$ and let ϕ be a character of the t-fold direct product $\mathfrak{S}_{n-r}^{\times t}$. Then*

$$(\psi \otimes \phi) \uparrow_{\mathfrak{S}_r^{\times t} \times \mathfrak{S}_{n-r}^{\times t}}^{\mathfrak{S}_n^{\times t}} = \psi \circ \phi, \text{ and hence } \mathrm{Pch}\left((\psi \otimes \phi) \uparrow_{\mathfrak{S}_r^{\times t} \times \mathfrak{S}_{n-r}^{\times t}}^{\mathfrak{S}_n^{\times t}} \right) = \mathrm{Pch}(\psi)\,\mathrm{Pch}(\phi).$$

4 The actions of $\mathfrak{S}_n^{\times t}$ and $\mathfrak{S}_n$ on the homology of $B_n^{(t)}$

We begin by giving a recurrence for the homology representation of $\mathfrak{S}_n^{\times t}$ for the t-fold Segre power of Boolean lattices $B_n^{(t)}$, a Cohen–Macaulay poset. The case $t = 2$ of the recurrence appears in [6, Theorem 4.1]. Recall the Whitney homology technique in [13].

Theorem 4.1 ([13, Lemma 1.1 and Theorem 1.2])**.** *Let Q be a bounded and ranked Cohen–Macaulay poset of rank n, and let G be a finite group of automorphisms of Q. Let $WH_r(Q)$ denote its rth Whitney homology, defined by $WH_r(Q) = \bigoplus_{x \in Q,\, \mathrm{rank}(x)=r} \tilde{H}_{r-2}(\hat{0}, x)$. Then $WH_r(Q)$ is a G-module, and as virtual G-modules one has the identity*

$$\tilde{H}_{n-2}(Q) = \textstyle\sum_{r=0}^{n-1} (-1)^{n-r+1} WH_r(Q).$$

Set $\beta_n^{(t)} := \mathrm{Pch}(\tilde{H}_{n-2}(B_n^{(t)})), n \geqslant 1$, the product Frobenius characteristic, and $\beta_0^{(t)} := 1$.

Theorem 4.2. *Fix $t \geqslant 1$. The $\beta_n^{(t)}$ satisfy the recurrence $\sum_{i=0}^{n}(-1)^i \beta_i^{(t)} \prod_{j=1}^{t} h_{n-i}(X^j) = 0$.*

This is the group-equivariant version of (2.7). Note the agreement of the next corollary with the case $t = 1$, when $B_n^{(t)}$ is simply the Boolean lattice B_n, and so its homology carries the sign representation of $\mathfrak{S}_n$.

Corollary 4.3. *In the $\mathfrak{S}_n^{\times t}$-module $\tilde{H}_{n-2}(B_n^{(t)})$, the multiplicity of*

(1) the trivial representation $\bigotimes_{j=1}^{t} \chi^{(n)}$ is zero unless $n = 1$, in which case it is 1;

(2) the sign representation $\bigotimes_{j=1}^{t} \chi^{(1^n)}$ is 1 for all $n \geqslant 1$.

Example 4.4. *Let $t = 2$. We use the recurrence to compute some of the symmetric functions $\beta_n^{(2)}$ in two sets of variables X^1, X^2. We have $\beta_0^{(2)} = 1$ and $\beta_1^{(2)} = h_1(X^1)h_1(X^2)$, the product characteristic of the trivial representation of $\mathfrak{S}_1 \times \mathfrak{S}_1$. Then Theorem 4.2 gives*

$$\begin{aligned}
\beta_2^{(2)} &= \beta_1^{(2)}(h_1(X^1)h_1(X^2)) - h_2(X^1)h_2(X^2) \\
&= e_2(X^1)h_2(X^2) + h_2(X^1)e_2(X^2) + e_2(X^1)e_2(X^2). \\
\beta_3^{(2)} &= \beta_2^{(2)} h_1(X^1)h_1(X^2) - \beta_1^{(2)} h_2(X^1)h_2(X^2) + \beta_0^{(2)} h_3(X^1)h_3(X^2) \\
&= (h_3(X^1)e_3(X^2) + e_3(X^1)h_3(X^2)) + e_3(X^1)e_3(X^2) + s_{(2,1)}(X^1)s_{(2,1)}(X^2) \\
&\quad + 2\left(s_{(2,1)}(X^1)e_3(X^2) + e_3(X^1)s_{(2,1)}(X^2)\right).
\end{aligned}$$

By definition of the product Frobenius characteristic and the fact that $\tilde{H}_{n-1}(B_n^{(2)})$ is a true $(\mathfrak{S}_n \times \mathfrak{S}_n)$-module, it follows that $\beta_n^{(2)}$ must have a positive expansion in the basis $\{s_\lambda(X^1)s_\mu(X^2) : \lambda, \mu \vdash n\}$. This is confirmed by the above examples.

Definition 4.5. *Define $Z_i^{(t)} := \prod_{j=1}^{t} h_i(X^j)$, and define the degree of $Z_i^{(t)}$ to be i.*

Also define, for each $\lambda \vdash n$, $Z_\lambda^{(t)} = \prod_j Z_{\lambda_j}^{(t)}$. Thus $Z_\lambda^{(t)} = \prod_{j=1}^{t} h_\lambda(X^j)$.

Definition 4.6. *Define a map $\Phi_t : \Lambda(X) \to \bigotimes_{j=1}^{t} \Lambda(X^j)$ by setting $\Phi_t(h_n) := \prod_{j=1}^{t} h_n(X^j) = Z_n^{(t)}$, and extending multiplicatively and linearly to all of $\Lambda(X)$.*

For an integer partition $\lambda \vdash n$ of n, let λ' denote the conjugate partition of λ. Write $\ell(\lambda)$ for the total number of parts of λ, and $m_i(\lambda)$ for the number of parts equal to i. Also let $K_{\mu,\nu}$ be the Kostka number, i.e., the number of semistandard Young tableaux of shape $\mu \vdash n$ and weight $\nu \vdash n$ [8, Equation (5.12)]. In particular, $f^\mu = K_{\mu,(1^n)}$ is the number of standard Young tableaux of shape μ.

Proposition 4.7. *The map $\Phi_t : \Lambda(X) \to \bigotimes_{j=1}^{t} \Lambda(X^j)$ is an injective degree-preserving algebra homomorphism such that:*

(1) $\Phi_t(e_n) = \beta_n^{(t)}$.

(2) $\Phi_t(s_\lambda) = \det(Z_{\lambda_i - i + j}^{(t)})_{1 \leqslant i,j \leqslant \ell(\lambda)}$, where $Z_0^{(t)} = 1$ and we set $Z_m^{(t)} = 0$ if $m < 0$.

(3) $\Phi_t(s_{\lambda'}) = \det(\beta_{\lambda_i - i + j}^{(t)})_{1 \leqslant i,j \leqslant \ell(\lambda)}$, where $\beta_0^{(t)} = 1$ and we set $\beta_m^{(t)} = 0$ if $m < 0$.

Definition 4.8. *For $\lambda \vdash n$ with $m_i(\lambda)$ parts of size i and number of parts $\ell(\lambda)$, define c_λ to be the signed multinomial coefficient*

$$c_\lambda = (-1)^{n-\ell(\lambda)} \binom{\ell(\lambda)}{m_1(\lambda), m_2(\lambda), \ldots}.$$

The integers c_λ play an important role in the irreducible decomposition of $\beta_n^{(t)}$.

Theorem 4.9. *For the product Frobenius characteristic $\beta_n^{(t)}$ of the top homology of $B_n^{(t)}$, we have:*
(1) $\beta_n^{(t)} = \sum_{\lambda \vdash n} c_\lambda Z_\lambda^{(t)}$.
(2) The multiplicity of the $\mathfrak{S}_n^{\times t}$-irreducible indexed by the t-tuple of partitions $\underline{\mu} = (\mu^1, \ldots, \mu^t)$, $\mu^j \vdash n$, $1 \leqslant j \leqslant t$, in $\tilde{H}_{n-2}(B_n^{(t)})$ equals $c^t_{\underline{\mu}} = \sum_{\lambda \vdash n} c_\lambda \prod_{j=1}^t K_{\mu^j, \lambda}$.
(3) Let $\mathcal{M}(s,h)$ denote the transition matrix from the basis of Schur functions to the basis of homogeneous symmetric functions. The multiplicity of the $\mathfrak{S}_n^{\times t}$-irreducible indexed by the t-tuple of partitions $(\mu^1, \ldots, \mu^t)$, $\mu^i \vdash n$, $1 \leqslant i \leqslant t$, in the (possibly virtual) module with product Frobenius characteristic $\Phi_t(s_\lambda)$ is

$$\langle \Phi_t(s_\lambda), \prod_{j=1}^t s_{\mu^j}(X^j) \rangle = \sum_{\nu \vdash n} \mathcal{M}(s,h)_{\lambda,\nu} \prod_{j=1}^t K_{\mu^j,\nu}. \tag{4.1}$$

Example 4.10 (Illustrating (4.1))**.** *Let $\lambda = (3,2,2)$. Then $s_\lambda = h_{322} - h_{331} - h_{421} + h_{43} + h_{511} - h_{52}$. Applying Φ_t, we obtain, for the multiplicity of the t-tuple of partitions $(\mu^1, \ldots, \mu^t)$ of 7 in the (possibly virtual) module with product Frobenius characteristic $\Phi_t(s_\lambda)$, the expression*

$$\prod_{j=1}^t K_{\mu^j,322} - \prod_{j=1}^t K_{\mu^j,331} - \prod_{j=1}^t K_{\mu^j,421} + \prod_{j=1}^t K_{\mu^j,43} + \prod_{j=1}^t K_{\mu^j,511} - \prod_{j=1}^t K_{\mu^j,52}.$$

This example also shows that $\Phi_t(s_\lambda)$ need not be a true module. Indeed, taking $t = 2$ and $(\mu^1, \mu^2) = (43, 61)$ reveals that the multiplicity of $(\chi^{\mu^1}, \chi^{\mu^2})$ in $\Phi_2(s_{322})$ is -1.

From the preceding theorem we deduce the following otherwise nonobvious fact.

Corollary 4.11. *Fix $t \geqslant 1$. For any fixed t-tuple $\underline{\mu} = (\mu^1, \ldots, \mu^t)$ of partitions of n, the following sum is a nonnegative integer.*

$$c^t_{\underline{\mu}} = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \frac{\ell(\lambda)!}{\prod_i m_i(\lambda)!} \prod_{j=1}^t K_{\mu^j,\lambda}.$$

Let $g^\lambda_{\underline{\mu}}$ denote the Kronecker coefficient $\langle \chi^\lambda, \prod_{j=1}^t \chi^{\mu^j} \rangle$, where χ^μ is the $\mathfrak{S}_n$-irreducible character indexed by $\mu \vdash n$ [8]. Recall [8] the internal product $*$ in the ring of symmetric functions $\Lambda^n(X)$ in a single set of variables X.

Theorem 4.12. *For the diagonal $\mathfrak{S}_n$-action on $\tilde{H}_{n-2}(B_n^{(t)})$:*
(1) The Frobenius characteristic is this signed sum of inner tensor powers of permutation modules:

$$\text{ch } \tilde{H}_{n-2}(B_n^{(t)}) = \sum_{\lambda \vdash n} c_\lambda \, h_\lambda * h_\lambda * \cdots * h_\lambda \ (t \text{ factors}).$$

The trace of an element $\sigma \in \mathfrak{S}_n$ is $\sum_{\lambda \vdash n} c_\lambda \, (\chi^{M^\lambda}(\sigma))^t$, where χ^{M^λ} is the character of the permutation module M^λ corresponding to h_λ. (2) The multiplicity of the $\mathfrak{S}_n$-irreducible indexed by λ in the diagonal $\mathfrak{S}_n$-action on $\tilde{H}_{n-2}(B_n^{(t)})$ is $\sum_{\underline{\mu}} c^t_{\underline{\mu}} \, g^\lambda_{\underline{\mu}}$, where the sum is over all t-tuples of partitions $\underline{\mu} = (\mu^1, \ldots, \mu^t)$, $\mu^j \vdash n$, $1 \leqslant j \leqslant t$.

See [7] for more details on the resulting character values. For small n, one can extract explicit formulas for the decompositions. In the elementary basis one has:

Proposition 4.13. *(1)* ch $\tilde{H}_0(B_2^{(t)}) = e_2 + (2^{t-1} - 1)e_1^2$, *of dimension* $w_2^{(t)} = 2^t - 1$.
(2) ch $\tilde{H}_1(B_3^{(t)}) = e_3 + (6^{t-1} - 3^{t-1})e_1^3$, *of dimension* $w_3^{(t)} = 6(6^{t-1} - 3^{t-1}) + 1$.
(See `oeis` *A248225, A127222 for the sequence* $\{6^t - 3^t\}$.*)*
(3) ch $\tilde{H}_2(B_4^{(t)}) = e_4 - (2^{t-1} - 1)e_2 e_1^2 + (2^{t-1} - 1)e_2^2 + \gamma_4 \, e_1^4$
where $\gamma_4 = \frac{(4^{t-1}-1)}{3} + \frac{3(6^{t-2})+2^{t-2}}{2} - 18(12^{t-2}) + 24^{t-1}$.

5 Rank-selection in $B_n^{(t)}$

For a fixed subset J of the nontrivial ranks $[1, n-1]$, the rank-selected subposet $B_n^{(t)}(J)$ is defined to be the bounded poset $\{x \in B_n^{(t)} : \text{rank}(x) \in J\}$ with the top and bottom elements $\hat{0}, \hat{1}$ appended. Since rank-selection preserves the property of being Cohen–Macaulay [2, Theorem 6.4] these posets have at most one nonvanishing homology module, which is in the top dimension. The study of the homology of rank-selected subposets was initiated in [10].

Denote by $\beta_n^{(t)}(J)$ and $\alpha_n^{(t)}(J)$ respectively the product Frobenius characteristics of the top homology $\tilde{H}_{k-1}(B_n(J))$ of the rank-selected subposet of $B_n^{(t)}$ for the rank-set J, and the $\mathfrak{S}_n^{\times t}$-module M_J of maximal chains in $B_n^{(t)}(J)$. By the general theory in [10], we have

$$\beta_n^{(t)}(J) = \sum_{U \subseteq J} (-1)^{|J|-|U|} \alpha_n^{(t)}(U), \quad \alpha_n^{(t)}(J) = \sum_{U \subseteq J} \beta_n^{(t)}(U).$$

In particular, the rank-selected invariants $\tilde{\beta}_P(J)$, $\tilde{\alpha}_P(J)$ of Equation (2.1), for $P = B_n^{(t)}$, are the respective dimensions of the $\mathfrak{S}_n^{\times t}$-modules corresponding to $\beta_n^{(t)}(J)$ and $\alpha_n^{(t)}(J)$.

For a standard Young tableau τ of shape $\lambda \vdash n$ (see [12] for definitions), the descent set $\text{Des}(\tau)$ of τ is the set of entries i such that $i+1$ appears in a row strictly below i.

To determine the rank-selected representations of $B_n^{(t)}$, we use the known results [10, Section 4] for the Boolean lattice, and then apply the induction product and the homomorphism Φ_t to obtain the corresponding results for the Segre product $B_n^{(t)}$. Note that while $\beta_n^{(t)}(J)$ and $\alpha_n^{(t)}(J)$ correspond to true $\mathfrak{S}_n^{\times t}$-modules, $\Phi_t(s_\lambda)$ is in general not a true module under the image of the product Frobenius map Pch; see Example 4.10.

Theorem 5.1. *For any subset J of nontrivial ranks of $B_n^{(t)}$, the homomorphism Φ_t maps the Frobenius characteristic of the $\mathfrak{S}_n$-action on the chains of $B_n(J)$ to the product Frobenius characteristic of the $\mathfrak{S}_n^{\times t}$-action on the chains of $B_n^{(t)}(J)$. More precisely, we have:*

(1) $\alpha_n^{(t)}(J) = \Phi_t(\alpha_n(J)) = \sum_{\lambda \vdash n} \Phi_t(s_\lambda)\ |\{SYT\ \tau \text{ of shape } \lambda : \mathrm{Des}(\tau) \subseteq J\}|$.

(2) $\beta_n^{(t)}(J) = \Phi_t(\beta_n(J)) = \sum_{\lambda \vdash n} \Phi_t(s_\lambda)\ |\{SYT\ \tau \text{ of shape } \lambda : \mathrm{Des}(\tau) = J\}|$.

We can now collect some facts about the map $\Phi_t : \Lambda(X) \to \bigotimes_{j=1}^t \Lambda(X^j)$.

Proposition 5.2. *Let $t \geqslant 2$ and let f be a symmetric function of degree n. Then $\Phi_t(f)$ is the product Frobenius characteristic of a true $\mathfrak{S}_n^{\times t}$-module in the following cases:*

(1) $f = h_\lambda, e_\lambda$ for $\lambda \vdash n$.

(2) $f = s_\lambda$ and $\lambda \vdash n$ has at most two parts.

(3) $f = s_\lambda$ and $\lambda \vdash n$ is a hook $(n-k, 1^k)$.

(4) f is a ribbon Schur function (i.e. $f = s_{\lambda/\mu}$ where the skew shape λ/μ is a ribbon).

Moreover, if $\Phi_t(s_\lambda)$ gives a true $\mathfrak{S}_n^{\times t}$-module for some $t \geqslant 2$, then so does $\Phi_2(s_\lambda)$.

It would be interesting to characterise when $\Phi_t(s_\lambda)$ gives a true $\mathfrak{S}_n^{\times t}$-module.

By adapting the proof of Theorem 4.2, we also obtain the $\mathfrak{S}_n^{\times t}$-equivariant version of (2.6) of Proposition 2.3, a recurrence that will be used in the next section.

Theorem 5.3. *Let $J = \{1 \leqslant j_1 < \cdots < j_r \leqslant n-1\}$ be a subset of nontrivial ranks in $B_n^{(t)}$. The product Frobenius characteristic $\beta_n^{(t)}(J) = \mathrm{Pch}\ \tilde{H}(B_n^{(t)}(J))$ of the rank-selected subposet of $B_n^{(t)}$ satisfies the following recurrence:*

$$\beta_n^{(t)}(J) + \beta_n^{(t)}(J \setminus \{j_r\}) = \beta_{j_r}^{(t)}(J \setminus \{j_r\}) \prod_{i=1}^{t} h_{n-j_r}(X^i).$$

5.1 The stable principal specialisation and the subspace lattice

In conclusion, we show that the surprising relationship discovered in [6] between the Boolean Segre square $B_n \circ B_n$ and subspace lattice Segre square $B_{n,q} \circ B_{n,q}$ holds for all rank-selected subposets of the t-fold Segre powers in each case.

The *stable principal specialisation* [12, Chapter 7, Section 8] of a symmetric function f in variables $x_1, x_2, \ldots$ is obtained from f by substituting $x_i \to q^{i-1}, i \geqslant 1$. Define the stable principal specialisation $\mathrm{ps} f$ of f in $\otimes_{i=1}^t \Lambda(X^i)$ to be the function of q obtained by replacing each set of variables X^i by the set $\{1, q, q^2, \ldots\}$. Recall from Section 2 that for the t-fold Segre power $B_{n,q}^{(t)}$ and the rank-set J, we denoted by $\tilde{\beta}_{B_{n,q}^{(t)}}(J)$ its rank-selected Betti number. Theorem 5.3 and the recurrence (2.6) for $\tilde{\beta}_{B_{n,q}^{(t)}}(J)$ now give:

Theorem 5.4. *The stable principal specialisation of $\beta_n^{(t)}(J)$ for the rank-selected homology module of the t-fold Segre power of the Boolean lattice B_n, and the rank-selected Betti number $\tilde{\beta}_{B_{n,q}^{(t)}}(J)$ of the t-fold Segre power of the subspace lattice $B_{n,q}$, are related by the equation*

$$\mathrm{ps}\,\beta_n^{(t)}(J) = \tilde{\beta}_{B_{n,q}^{(t)}}(J)\prod_{i=1}^{n}(1-q^i)^{-t}.$$

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