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# The Isomorphism Problem for (Co)adjoint Schubert Varieties

Linji Chen[1*], Yunhao Lai[2†], and Zhehan Xu[3‡]

[1] *School of Mathematical Science, Peking University, Beijing, China*
[2] *School of Mathematics, Sun Yat-sen University, Guangzhou, China*
[3] *School of Mathematical Science, Peking University, Beijing, China*

**Abstract.** We classify isomorphism classes of Schubert varieties coming from adjoint and coadjoint partial flag varieties across all Dynkin types via Hasse diagrams given by the Chevalley formula.

**Keywords:** Schubert variety, flag variety, adjoint, coadjoint

## 1 Introduction

Being relatively simple families of homogeneous spaces, adjoint and coadjoint flag varieties have been extensively studied [1, 3, 8, 10, 14]. There is one of each for every complex simple reductive group $G$. The adjoint flag variety is given by the unique closed $G$-orbit in $\mathbb{P}(\mathfrak{g})$, where $G$ acts on its Lie algebra $\mathfrak{g}$ via the adjoint representation. It is of the form $G/P_{\text{adjoint}}$, where $P_{\text{adjoint}} \subset G$ is the parabolic subgroup whose corresponding weight is the highest (long) root. The coajoint flag variety is $G/P_{\text{coadjoint}}$, where $P_{\text{coadjoint}} \subset G$ is the parabolic subgroup whose corresponding weight is the highest short root. In simply-laced Lie types, adjoint and codjoint flag varieties coincide. See Table 1 for a list of adjoint and coadjoint flag varieties and their corresponding Dynkin diagrams. In each case, the subgroup $P_{\text{adjoint}}$ (resp. $P_{\text{coadjoint}}$) is determined by the subset of empty nodes, which are simple roots perpendicular to the highest root (resp. highest short root). Except in type $A$, these parabolic subgroups are maximal.

The adjoint (resp. coadjoint) Schubert varieties, being the closure of Borel orbits in an adjoint (resp. coadjoint) flag variety, are naturally indexed by long roots (resp. short roots) in the root system. We give the following combinatorial criterion for distinguishing their isomorphism classes.

**Theorem 1.1.** *Let $X_\alpha \subseteq X$ and $Y_\beta \subseteq Y$ be adjoint or coadjoint Schubert varieties. Then the following are equivalent:*

*cmuuuu7@gmail.com
†laiyh36@mail2.sysu.edu.cn
‡kxu3nive@gmail.com

*(i)* $X_\alpha$ *and* $Y_\beta$ *are isomorphic as abstract algebraic varieties;*

*(ii) Their corresponding Chevalley–Hasse diagrams,* $P_\alpha$ *and* $P_\beta$*, are isomorphic.*

The Chevalley–Hasse diagram of a (co)adjoint Schubert variety $X_\alpha$ is a weighted, directed graph whose vertices index a well-chosen basis (the Schubert basis) of $\mathrm{CH}(X_\alpha)$ and whose edges represent how $\mathrm{Pic}(X_\alpha)$ acts on the basis elements via the intersection product. For a precise definition, as well as an equivalent combinatorial definition based on the root system, see Section 4.

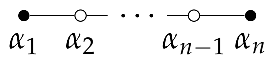
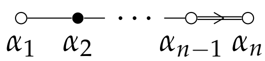
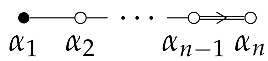
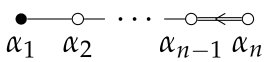
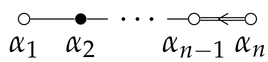
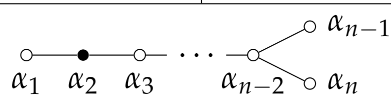
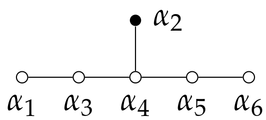
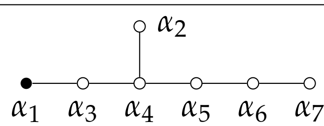
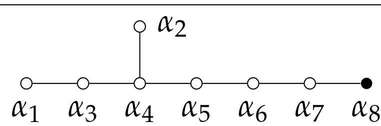
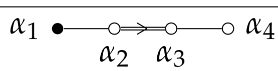
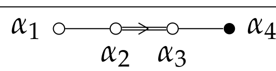
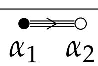
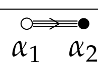
| Adjoint | Coadjoint | Description |
|---|---|---|
|  | | Type $A_n$ ($n \geq 1$): $G/P_{\text{(co)adjoint}} \simeq \mathrm{Fl}(1,n;n+1)$ |
|  |  | Type $B_n$ ($n \geq 2$): $G/P_{\text{adjoint}} \simeq \mathrm{OGr}(2,2n+1)$ $G/P_{\text{coadjoint}} \simeq Q^{2n-1}$ |
|  |  | Type $C_n$ ($n \geq 3$): $G/P_{\text{adjoint}} \simeq \mathbb{P}^{2n-1}$ $G/P_{\text{coadjoint}} \simeq \mathrm{IG}(2,2n)$ |
|  | | Type $D_n$ ($n \geq 4$): $G/P_{\text{(co)adjoint}} \simeq \mathrm{OGr}(2,2n)$ |
|  | | Type $E_6$ |
|  | | Type $E_7$ |
|  | | Type $E_8$ |
|  |  | Type $F_4$ |
|  |  | Type $G_2$: $G/P_{\text{coadjoint}} \simeq Q^5$ |

**Table 1:** (Co)adjoint flag varieties.

Our work is analogous to the work of Richmond, Tarigradschi and Xu in [11], where they proved that two cominuscule Schubert varieties are isomorphic if and only if their corresonding labeled posets of roots are isomorphic. Other related works include the classification of a class of smooth Schubert varieties in type A partial flag varieties by Develin, Martin, Reiner in [4] and Richmond and Slofstra's work on the isomorphism problem of Schubert varieties in complete flag varieties in [13]. The cases of Grassmannian Schubert varieties are handled using Young diagrams, by Tarigradschi and Xu in [15].

We illustrate examples in Section 2 and discuss preliminaries in Section 3. Then in Section 4, we will construct the Chevalley–Hasse diagram associated with a (co)adjoint Schubert variety. Our proof of Theorem 1.1 will be sketched very briefly in Section 6. The direction from isomorphic (co)adjoint Schubert varieties to isomorphic Chevalley–Hasse diagram is easy by definition, see Proposition 4.4. However, the converse part of Theorem 1.1 is more complicated: two main techniques, namely minimal embeddings and foldings, are applied to embed (co)adjoint Schubert varieties into other classes of Schubert varieties.

## 2 Examples of Theorem 1.1

Let us first apply Theorem 1.1 in various examples.

*Convention* 2.1. **Throughout the remaining sections**: for convenience, for a fixed root system $\Phi$ of some Lie type, we index the simple roots in $\Phi$ following Table 1. A general root $\alpha$ can be expressed uniquely as a linear combination of simple roots, and we will denote $\alpha$ simply by the coefficients of this expression. For example, the root $\alpha_2 + \alpha_4 + \alpha_5$ in $E_7$ will be denoted 0101100, and the root $-\alpha_3 - 2\alpha_4$ in $B_4$ will be denoted $-(0012)$.

We will omit the directions of edges when we present the Chevalley–Hasse diagram of a (co)adjoint Schubert variety $X_\alpha$; instead, we will give the set of vertices a partial order such that the "heights" of vertices (defined as the minimal length of paths to the minimal vertex) represent the dimension $d$ of the corresponding class in $\mathrm{CH}_*(X_\alpha)$. And every edge will automatically be directed from the lower-dimensional class to the higher-dimensional one. See Section 4 for the full definition.

We will denote in this article the (co)adjoint Schubert variety and its Chevalley–Hasse diagram respectively by $X(T, \alpha)$ and $P(T, \alpha)$, where $T$ is the Cartan type of the root system $\Phi$ and $\alpha$ is a short/long root in $\Phi$ labeling the Schubert variety. We will sometimes use $X_\alpha$ and $P_\alpha$ if the partial flag variety is clear in the context.

**Example 2.2.** Consider the root system $\Phi$ of Cartan type $C_2$; we know that the adjoint partial flag variety corresponding to this type is $\mathbb{P}^3$. Letting $\alpha = (01) = \alpha_2$, we get an adjoint Schubert variety $X(C_2, \alpha)$.

On the other hand, consider the root system $\Phi'$ of Cartan type $B_3$; the corresponding adjoint flag variety is $\mathrm{OGr}(2, 2n+1)$. Letting $\beta = -(012)$, we get another Schubert variety $X(B_3, \beta)$.

The combinatorial view of Chevalley–Hasse diagrams will be introduced later. Let us admit the following results: the Chevalley–Hasse diagrams $P(C_2, \alpha)$ and $P(B_3, \beta)$ are both isomorphic to the diagram in Figure 1. Therefore, Theorem 1.1 says $X(C_2, \alpha)$ and $X(B_3, \beta)$ are isomorphic as algebraic varieties.

Note that there is exactly one vertex of each height in Figure 1, which means the Chow group of this algebraic variety is free of rank 1 in dimensions $0, 1, 2$. This is verified by the fact that $X(C_2, \alpha)$ and $X(B_3, \beta)$ are both isomorphic to $\mathbb{P}^2$.

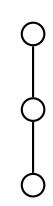


**Figure 1:** $P(C_2, \alpha) \simeq P(B_3, \beta)$

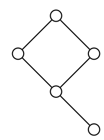


**Figure 2:** $P(B_4, -(0112)) \simeq P(C_4, -(0111)) \simeq P(D_4, -(0101))$

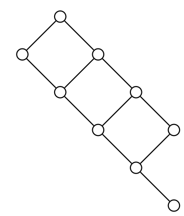


**Figure 3:** $P(C_4, -(0100))$

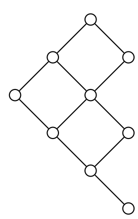


**Figure 4:** $P(C_4, -(0010))$

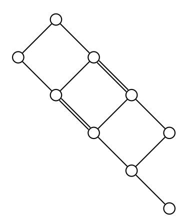


**Figure 5:** $P(B_4, -(0100))$

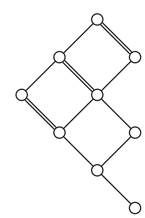


**Figure 6:** $P(B_4, -(0010))$

**Example 2.3.** Now, consider the root system of type $D_4$ with the root $-(0101)$, the root system of type $C_4$ with the *short* root $-(0111)$, and the root system $B_4$ with the *long* root $-(0112)$. So we get a coadjoint Schubert variety $X(C_4, -(0111))$, an adjoint Schubert variety $X(B_4, -(0112))$, and a Schubert variety $X(D_4, -(0101))$ which is both coadjoint and adjoint (as $D_4$ is simply laced).

The Chevalley–Hasse diagrams of all the three varieties are isomorphic to the one in Figure 2. Since Theorem 1.1 is applicable to one adjoint Schubert variety and one coadjoint Schubert variety, we deduce that the three Schubert varieties are all isomorphic. They are all isomorphic to the following Schubert variety in the Grassmannian $\mathrm{Gr}(2, 8)$:

$$\mathrm{Gr}_{2,4}(2, 8) := \{\Sigma \in \mathrm{Gr}(2, 8) \,|\, \dim(\Sigma \cap E_2) \geq 1,\ \Sigma \subseteq E_4\},$$

where $0 = E_0 \subset E_1 \subset \ldots \subset E_8 = \mathbb{C}^8$ is a given flag in $\mathbb{C}^8$.

**Example 2.4.** Consider the coadjoint Schubert varieties $X(C_4, -(0100))$, $X(C_4, -(0010))$ and the adjoint Schubert varieties $X(B_4, -(0100))$, $X(B_4, -(0010))$. As is illustrated in Figure 3, Figure 4, Figure 5, and Figure 6, their Chevalley–Hasse diagrams are pairwise non-isomorphic. Therefore, although the Chow groups these varieties are free of the same rank in each dimension, Theorem 1.1 says that these varieties are pairwise non-isomorphic. The point is that an isomorphism between the Chow groups of two distinct varieties in this example cannot preserve the action of Picard groups.

# 3 Preliminaries

## 3.1 Adjoint Schubert varieties and Coadjoint Schubert varieties

First, let us recall some definitions. Let $G$ be a complex reductive algebraic group with a fixed Borel subgroup $B$ and a maximal torus $T$. The triple $(G, B, T)$ determines a root system $\Phi$ in the weight lattice $M = \mathrm{Hom}(T, \mathbb{C}^*)$, with a root basis $\Delta$ and Weyl group $W = N_G(T)/T = \langle s_\alpha \,|\, \alpha \in \Delta \rangle$, where $s_\alpha$ is the reflection with respect to the simple root $\alpha$. We consider the cases when $W$ is finite. For a subset $I \subseteq \Delta$, $W_I$ denotes $\langle s_\alpha \,|\, \alpha \in I \rangle \subset W$. There is a minimal parabolic subgroup $P \leqslant G$ containing $B$ and $\langle s_\alpha \,|\, -\alpha \in I \rangle$. The quotient space $G/P$ is called a partial flag variety, and the closure of orbits $X_w = \overline{BwP/P}$ are called Schubert varieties. All Schubert varieties are indexed by $W/W_I$, or equivalently by $W^I$, the set of all minimal length coset representatives of $W/W_I$. If $w \in W^I$, $\dim X_w = \ell(w)$.

Let $\Theta$ be the highest (long) root in the root system. The weight $\varpi = \Theta$ is called *adjoint weight*. The partial flag variety $G/P$ is called *adjoint* if $P$ is the parabolic subgroup associated to $\varpi$[1]. In Table 1, set $I$ consists of the unfilled nodes in the Dynkin diagram. In this case, we have $W_I = \mathrm{Stab}_W(\Theta)$, and $W$ acts transitively on $\Phi_{\mathrm{long}}$; thus there is a bijection

$$\begin{array}{rcl} W^I & \longrightarrow & \Phi_{\mathrm{long}} \\ w & \longmapsto & w(-\Theta), \end{array}$$

so we may just define $X_{w(-\Theta)} := X_w$.[2] Therefore, the Schubert varieties in an adjoint partial flag variety can be indexed by long roots in the root system.

The coadjoint case is pretty much the same, except that 'long' shall be replaced with 'short': We define $\varpi = \theta$, the highest short root, as the *coadjoint* weight[3]; there is a

[1] That is, $P$ corresponds to the simple roots $\alpha_i$ such that $\langle \varpi, \alpha_i \rangle = 0$.

[2] In fact, our indexing of roots differs from the definition in literature, such as in [10], by an action of $-w_0$ on the root space. Here $w_0 \in W$ is the element with maximal Coxeter length. But note that $-w_0$ permutates the simple roots which induces an automorphism of the Dynkin diagram, so this difference does not affect anything in our consideration.

[3] Note that our definition of coadjoint weight agrees with [3] but differs from [1] by a multiplicity of

bijection between coadjoint Schubert varieties and the set of short roots. Also, we define $X_{w(-\theta)} := X_w$.

## 3.2 Chow groups, Picard groups and the Chevalley formula

Here, we list several well-known results for the Chow groups of the Schubert varieties, which we will refer to in the rest of the paper.

**Lemma 3.1** ([5, Corollary of Theorem 1]; also see [7, Example 1.9.1])**.** *The Schubert classes $[X_u]$ such that $X_u \subseteq X_w$ are exactly the minimal elements in the extremal rays of the effective cone in $\mathrm{CH}_*(X_w)$. Moreover, these classes form an integral basis of $\mathrm{CH}_*(X_w)$.*

**Lemma 3.2** ([3, Proposition 2.8, part (ii)])**.**

$$X_\alpha \subseteq X_\beta \Leftrightarrow \begin{cases} \beta - \alpha = \sum\limits_{\alpha_i \in \Delta} c_i \alpha_i \text{ with } c_i \geq 0, & \text{if } \alpha, \beta \text{ are of the same sign;} \\ \mathrm{Supp}(\alpha) \cup \mathrm{Supp}(\beta) \text{ is connected}, & \text{if } \alpha \in \Phi^- \text{ and } \beta \in \Phi^+. \end{cases}$$

*Here, the connectedness is counted as in the Dynkin diagram.*

We mainly focus on the multiplication of $\mathrm{Pic}(X_w)$ on $\mathrm{CH}_*(X_w)$, however, the following lemma allows us to reduce to the multiplication of $\mathrm{Pic}(X)$ on $\mathrm{CH}_*(X)$ by projection formula.

**Lemma 3.3** ([9, Proposition 6]; also see [2, Proposition 2.2.8, part (ii)] )**.** *The pullback map $\mathrm{Pic}(X) \to \mathrm{Pic}(X_w)$ is surjective.*

By Poincaré duality we may identify the cup product with the cap product $H^2(X) \times H_{2N-2q}(X) \to H_{2N-2q-2}(X)$. By [7, Example 19.1.11] and [7, Proposition 19.2], we may moreover identify homology groups and Chow groups of $X$, while the cup product is identified with the intersection product $\mathrm{Pic}(X) \times \mathrm{CH}_{N-q}(X) \to \mathrm{CH}_{N-q-1}(X)$. Using these identifications, we can give an explicit Chevalley formula for coadjoint and adjoint partial flag variety. We first define a suitable divisor such that the formula for its action on the Chow group has a unified form.

**Definition 3.4.** Given an adjoint partial flag variety $X$, we define $D'_X$ to be the divisor corresponding to the adjoint weight $\varpi = \Theta$. Let $D_X := \frac{1}{2} D'_X$ in adjoint type $C$, and let $D_X := D'_X$ in other cases. The definition of $D'_X$ in coadjoint cases is similar; moreover, in coadjoint cases, we always let $D_X := D'_X$.

Finally, if $X_w \subseteq X$ is a (co)adjoint Schubert variety, we define $D_{X_w} := i^* D_X$, where $i : X_w \hookrightarrow X$ is the embedding.

2 in type $B$. The resulting parabolic subgroups and varieties are the same, but we will need $\varpi = \theta$ for convenience in Section 4.

*Remark* 3.5. In adjoint cases, it turns out that $D'_X := D_{\alpha_1} + D_{\alpha_n}$ in type $A$, $D'_X := 2D_{\alpha_1}$ in type $C$, and $D'_X := D_{\alpha_i}$ ($\alpha_i \in \Delta \setminus I$ is unique) in other cases. In coadjoint cases, it turns out that $D'_X := D_{\alpha_1} + D_{\alpha_n}$ in type $A$, and $D'_X := D_{\alpha_i}$ ($\alpha_i \in \Delta \setminus I$ is unique) in other cases.

The reason for the case-by-case definition of $D_X$ can be seen in Table (1) in [3]: in adjoint type C, the weight $\Theta$ and the simple root $\varpi_1$ differ by a scalar 2, so in order to make $D_{X_w}$ an invariant on the algebraic structure of $X_w$, we need to offset this scalar.

Now, we restate the Chevalley formula by Fulton and Woodward for $D'_X$:

**Lemma 3.6** ([6, Lemma 8.1])**.** *Let $X$ be an adjoint (resp. coadjoint) partial flag variety. Then for $\alpha \in \Phi_{long\ (resp.\ short)}$ we have*

$$D'_X \cdot [X_\alpha] = \begin{cases} \displaystyle\sum_{\gamma\in\Delta,\ (\gamma^\vee,\alpha)>0} (\gamma^\vee, \alpha)[X_{s_\gamma \alpha}] & \text{if } \alpha \notin \Delta_{long\ (resp.\ short)}; \\ \displaystyle\sum_{\gamma\in\Delta_{long\ (resp.\ short)}} (\gamma^\vee, \alpha)[X_{-\gamma}] & \text{if } \alpha \in \Delta_{long\ (resp.\ short)}. \end{cases}$$

# 4 Chevalley–Hasse Diagrams

## 4.1 Intersection Product and the Definition of Chevalley–Hasse Diagram

**Definition 4.1** (Chevalley–Hasse diagram for (co)adjoint Schubert varieties)**.** Let $X_\beta$ be a Schubert variety in the adjoint partial flag variety $X = X_\Theta$. We then construct the Chevalley–Hasse diagram $P_\beta$ as follows:

**Vertices:** all Schubert classes $[X_\alpha]$ such that $X_\alpha \subseteq X_\beta$. By Lemma 3.2, these classes coincide with those long roots $\alpha$ such that $\alpha \leq \beta$ when they are of the same sign, or such that $\mathrm{Supp}(\alpha) \cup \mathrm{Supp}(\beta)$ is connected when they are of different signs. The vertices have a natural partial order of inclusion relationship .

**Edges:** for two long roots $\alpha, \alpha'$ in the diagram with[4] $\mathrm{depth}(\alpha) = \mathrm{depth}(\alpha') - 1$, we draw $n$ oriented edge(s) from $\alpha$ to $\alpha'$ if the coefficient of $[X_\alpha]$ in $D_{X_\beta} \cdot [X_{\alpha'}]$ under the selected basis is $n$. Now Lemma 3.6, together with the relations between $D_{X_\beta}$, $D_X$ , and $D'_X$, allows us to write the weight as an inner product, which relies only on combinatorial data of the root system.

We do a similar construction for coadjoint Schubert varieties, with 'long' replaced by 'short'.

[4]The depth of a root is a standard notation in Coxeter systems. For $\alpha \in \Phi^+$, $\mathrm{depth}(\alpha)$ is the minimal number of consecutive simple reflections needed to send $\alpha$ to $\Phi^-$, and for $\alpha \in \Phi^-$, $\mathrm{depth}(\alpha)$ is just $1 - \mathrm{depth}(-\alpha)$.

*Remark* 4.2. (1) Let $p$ be a vertex in the Chevalley–Hasse diagram. Then the dimension of the associated Schubert class is the minimal length of the path from the (co)adjoint root to $p$.

(2) If $X_\beta \subseteq X$ is a Schubert variety in the (co)adjoint partial flag variety $X$. By definition, $D_{X_\beta} = i^* D_X$. Let $P_\beta$ and $P_X$ be the corresponding Chevalley–Hasse diagrams. Then it turns out that $P_\beta$ is a lower ideal of $P_X$ with the unique maximal vertex $\beta$.

(3) The Chevalley–Hasse diagrams of partial flag varieties are all vertically symmetrical, with the upper half corresponding to the positive roots and the lower half corresponding to the negative roots. For simplicity, most of the diagrams shown in this article are only the lower half and the negative roots of the whole diagram.

(4) One may now use Definition 4.1, together with the formula given in Lemma 3.6, to revisit Section 2 and work out the diagrams explicitly!

**Example 4.3.** Consider the root system of type $F_4$; the corresponding coadjoint partial flag variety, which is itself the maximal coadjoint variety, is denoted $X(F_4, \theta)$. Its Chevalley–Hasse diagram is illustrated in Figure 7. Similarly, the Chevalley–Hasse diagram of $X(E_7, \theta)$ is illustrated in Figure 8.

Note that the colored parts in Figure 7 and Figure 8 are isomorphic. In fact, as an example of the techniques we applied to prove Theorem 1.1, we will show the isomorphism between those Schubert varieties indexed by the roots marked in blue in $F_4$ and $E_7$ later in Example 5.3. We will deal with the red parts similarly in Example 5.5.

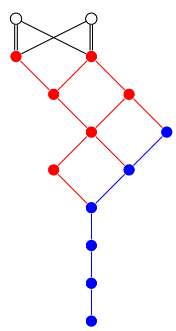


**Figure 7:** Lower half of $P(F_4, \theta)$

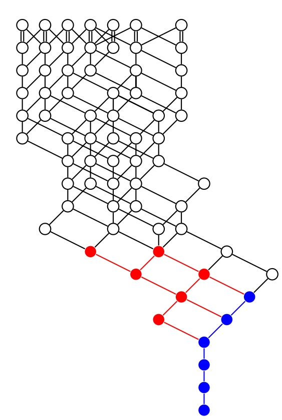


**Figure 8:** Lower half of $P(E_7, \theta)$

### 4.2 Chevalley–Hasse diagrams serve as Isomorphic Invariants

**Proposition 4.4.** *If two (co)adjoint Schubert varieties are isomorphic, then they have the same Chevalley–Hasse diagrams. In other words, we have derived one direction of Theorem 1.1.*

*Sketch of proof.* The Chow groups of $X$, together with the chosen basis $\{[X_\alpha]\}$, can be recovered from purely algebraic structures of $X_\beta$ by Lemma 3.1. It suffices to give a characterization of $D_{X_\beta}$ independent of the root system. It can be easily verified that

(†) $D = D_{X_\beta}$ is the only divisor in $\mathrm{Pic}(X_\beta)$ such that $D \cdot [X_\alpha] = [\mathrm{pt}]$ for every Schubert class $[X_\alpha] \in \mathrm{CH}_1(X_\beta)$.

Now (†) determines $D_{X_\beta}$ by the algebraic structure of $X_\beta$ only. □

## 5 Two Techniques for Embedding

Now, we introduce some techniques that allow us to embed some of the (co)adjoint Schubert varieties into a smaller partial flag variety $X' = G'/P'$ (minimal embedding) or larger partial flag variety $\widehat{G}/\widehat{P}$ (folding). Note that these embeddings are isomorphisms when we restrict them to Schubert varieties.

### 5.1 Minimal embedding

To define the minimal embedding of Schubert varieties, let us first recall the support of a Weyl group element.

**Definition 5.1.** Let $W$ be a Weyl group generated by the simple reflections in $S$. For $w \in W$, the support $S(w)$ of $w$ is defined as the set of all simple reflections that have appeared in reduced expressions of $w$.

**Lemma 5.2** ([12, Lemma 4.8])**.** *Let $G'$ be the reductive subgroup of $P_{S(w)}$ with Weyl group $W' = W_{S(w)}$ and $P'_{I'} = G' \cap P_I$ as the parabolic subgroup of $G'$, corresponding to $I' := I \cap S(w)$. Set $X' := G'/P'$. Note that $w \in W'^{I'}$.*

*Now $X'$ is again a partial flag variety, whose Dynkin diagram is the full subdiagram $D'$ of $D$ with underlying set $D \cap S(w)$. Let $X'_w$ be the Schubert variety in $X'$ induced by $w \in W'^{I'}$. Then the natural embedding $X' \hookrightarrow X$ induces an isomorphism $X'_w \cong X_w$.*

**Example 5.3.** In the coadjoint partial flag variety of type $F_4$, take $\beta = -(1110)$. Then $P_\beta$ is a path of length 5 (see Figure 10; it is exactly the subdiagram marked in blue in Figure 7). We have

$$\beta = s_4 s_3 s_2 s_3 s_4(-\theta),$$

with the simple reflections marked in Figure 10. By Lemma 5.2 we can embed $X_\beta$ as $X_w$ in $X'$, a partial flag variety with $D' = \{\alpha_2, \alpha_3, \alpha_4\}$ of type $C_3$ and $I' = D' \setminus \{\alpha_4\}$.

This procedure is illustrated in Figure 9. As $X_\beta \subseteq X' = C_3/P_1 \simeq \mathbb{P}^5$, it follows that $X_\beta \simeq \mathbb{P}^5$ as $\dim X_\beta = 5$. Moreover, $[X_{-(1111)}]$ is an effective Cartier divisor in $\mathbb{P}^5$ which is a generator of $\mathrm{CH}_4(\mathbb{P}^5) \cong \mathrm{CaDiv}(\mathbb{P}^5)$, so we must have $X_{-(1111)} \simeq \mathbb{P}^4$. Similarly, $X_{-(1121)} \simeq \mathbb{P}^3$ and so on.

Similarly, we can show that the blue part in Figure 8 also corresponds to some Schubert varieties in $E_7$ isomorphic to projective spaces. This fact presents isomorphisms between Schubert varieties from coadjoint types $F_4$ and $E_7$, as promised in Example 4.3.

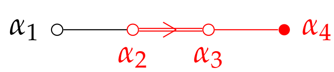



**Figure 9:** taking support of $w$ in $F_4$

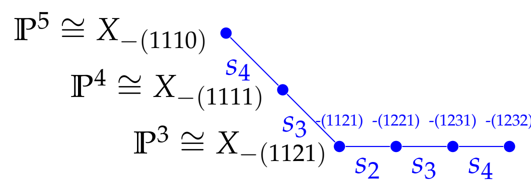



**Figure 10:** diagram $P(F_4, -(1110))$

## 5.2 Folding

| $G/P$ | $\widehat{G}/\widehat{P}$ |
|---|---|
| $B_n/P_1 = Q_{2n-1}$ | $D_{n+1}/P_1 = Q_{2n}$ |
| $C_n/P_2 = IG(2,2n)$ | $A_{2n-1}/P_2 = G(2,2n)$ |
| $F_4/P_4$ | $E_6/P_1$ |
| $G_2/P_1 = Q_5$ | $D_4/P_1 = Q_6$ |

**Table 2:** Hyperplane sections

The main reference of this type of embedding is [10]. Firstly, we introduce the folding operation between linear algebraic groups. Let $\widehat{G}$ be a simply connected simple algebraic group whose Dynkin diagram is as in the second column of Table 2. Let $\sigma \in \mathrm{Aut}(\widehat{G})$ be the automorphism induced by the symmetry of the Dynkin diagram of $\widehat{G}$. Define a closed subgroup $G := \widehat{G}^\sigma$ as the fixed locus of $\sigma$; other standard subgroups of $G$ are defined as $T := G \cap \widehat{T}$, $B := G \cap \widehat{B}$, $P := G \cap \widehat{P}$. The embedding of maximal tori gives rise to a surjective map of root systems $\pi : \Phi_{\widehat{G}} \twoheadrightarrow \Phi_G$. There is a closed embedding $j$ from $G/P$ to $\widehat{G}/\widehat{P}$. Moreover, $G/P$ can be identified as hyperplane sections of $\widehat{G}/\widehat{P}$. For more details, see [10, Lemma 4.1].

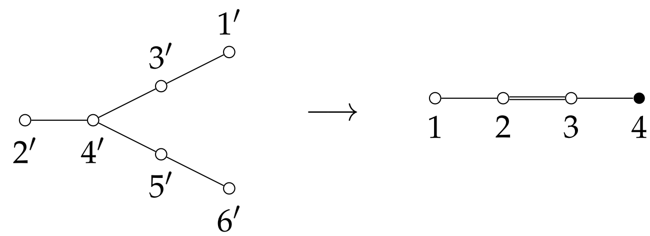
The map of root systems $\pi$ induces an embedding $\iota : W_G \to W_{\widehat{G}}$ of Weyl groups, by sending each $s_\alpha$ to $\prod_{\widehat{\alpha} \in \pi^{-1}(\alpha)} s_{\widehat{\alpha}}$. The product is well-defined since for every $\alpha \in \Phi$ and every $\widehat{\alpha}, \widehat{\alpha}' \in \pi^{-1}(\alpha)$, the reflections $s_{\widehat{\alpha}}$ and $s_{\widehat{\alpha}'}$ commute.

The lemma below serves as a useful tool for verifying isomorphisms between Schubert varieties when the minimal embedding method fails.

**Lemma 5.4** ([10, Lemma 4.5])**.** *For $w \in W_G^P$ with $2\ell(w) \leq \dim G/P$, the closed embedding gives rise to a isomorphism between Schubert varieties $j((G/P)_w) = (\widehat{G}/\widehat{P})_{w_*}$.*

Now let us witness the power of Lemma 5.4 by revisiting examples in Figures 7 and 8.

**Example 5.5.** The folding of Dynkin diagrams from $E_6$ to $F_4$ is illustrated in Figure 11. By applying Lemma 5.4, the Schubert varieties of coadjoint type $F_4$ indexed by the red




$$\iota : W_{F_4} \hookrightarrow W_{E_6}$$
$$s_1 \mapsto s_2'$$
$$s_2 \mapsto s_4'$$
$$s_3 \mapsto s_3's_5'$$
$$s_4 \mapsto s_1's_6'$$

**Figure 11:** Folding from $E_6$ to $F_4$

vertices in Figure 7 are isomorphic to some Schubert varieties in the *cominuscule* partial flag variety $E_6/P_1$. On the other hand, by applying Lemma 5.2, the Schubert varieties of type $E_7$ indexed by the red vertices in Figure 8 are isomorphic to the same set of Schubert varieties in $E_6/P_1$. Therefore we have shown the result promised in Example 4.3.

# 6 Proof Sketch of Theorem 1.1

We give a brief sketch of our strategy proving the other direction of Theorem 1.1:

(1) We classify the Schubert varieties $X(T, \beta)$ whose Chevalley–Hasse diagrams are unique, up to at most an automorphism of the Dynkin diagram. For example, the (co)adjoint Schubert varieties corresponding to positive roots in most Cartan types are unique in their isomorphic types.

(2) We use the two techniques for embedding to give the isomorphism between two Schubert varieties whose Chevalley–Hasse diagrams fall into the same isomorphic class. Example 5.3 and Example 5.5 provide a good demonstration.

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