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α-chromatic symmetric functions

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Abstract. In this extended abstract, we introduce the *α-chromatic symmetric functions* $\chi_\pi^{(\alpha)}[X;q]$, extending Shareshian and Wachs' chromatic symmetric functions with an additional real parameter α. We present positive combinatorial formulas with explicit interpretations. Notably, we show an explicit monomial expansion in terms of the α-binomial basis and an expansion into certain chromatic symmetric functions in terms of the α-falling factorial basis. We also exhibit various Schur positivity results.

Keywords: α-chromatic symmetric functions, Schur positivity, α-binomial coefficients, rook polynomials, hit polynomials

1 Introduction

In his seminal paper [13], Stanley introduced the *chromatic symmetric function* $\chi_G[X]$ as a symmetric function generalization of the chromatic polynomial for graphs. Subsequently, Shareshian and Wachs [12] refined this by defining the *chromatic (quasi)symmetric function* $\chi_G[X;q]$ with an additional parameter q. Notably, Shareshian and Wachs showed that $\chi_{G(\pi)}[X;q]$ is symmetric if the graphs $G(\pi)$ are associated with Dyck paths π.

In this extended abstract, we consider even further generalization of Shareshian and Wachs' chromatic symmetric functions. We introduce a new parameter α to define the *α-chromatic symmetric function* $\chi_\pi^{(\alpha)}[X;q]$ for a Dyck path π, for $\alpha \in \mathbb{R}$, as

$$\chi_\pi^{(\alpha)}[X;q] := \chi_\pi\left[\frac{q^\alpha - 1}{q-1}X;q\right].$$

Notably, for a Dyck path π of size n (referred to as an n-Dyck path from now on), the coefficients of the α-chromatic symmetric function $\chi_\pi^{(\alpha)}[X;q]$ lie within the $\mathbb{C}(q)$-span of

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$\{[\alpha]_q, [\alpha]_q^2, \dots, [\alpha]_q^n\}$, where $[\alpha]_q := \dfrac{q^\alpha - 1}{q-1}$. We can express each coefficient in terms of two alternative bases:

$$\left\{ \begin{bmatrix} \alpha + k \\ n \end{bmatrix}_q \right\}_{k=0,1,\dots,n-1} \qquad \text{and} \qquad \left\{ [\alpha]_q^{\underline{k}} \right\}_{k=1,2,\dots,n}. \tag{1.1}$$

Here, $[\alpha]_q^{\underline{k}} := [\alpha]_q[\alpha-1]_q \cdots [\alpha-k+1]_q$ is the q-falling factorial and $\begin{bmatrix} \alpha + k \\ n \end{bmatrix}_q = \dfrac{[\alpha+k]_q^{\underline{n}}}{[n]_q!}$ denotes the q-binomial coefficient.

We want to remark that the expansions of $\chi_\pi^{(\alpha)}[X;q]$ in terms of α-binomial bases have close connections to the original chromatic symmetric functions and the Schur expansion of the unicellular LLT polynomials, namely, the coefficients of the α-chromatic symmetric functions expressed in terms of the q-falling factorial basis serve as an interpolation between the chromatic symmetric functions and the unicellular LLT polynomials. For more details, refer to [9].

This extended abstract presents various positive combinatorial formulas for the α-chromatic symmetric functions in these bases or their specialization at $q = 1$.

2 Backgrounds

2.1 Combinatorics

Throughout this paper, we set $[n] = \{1, 2, \dots, n\}$. We shall mainly follow the notations and definitions of [11, 14] for symmetric functions. For a partition λ of n, we let $m_\lambda[X]$, $e_\lambda[X]$, $h_\lambda[X]$, $p_\lambda[X]$, $s_\lambda[X]$ denote the *monomial*, *elementary*, *complete homogeneous*, *power sum* and *Schur symmetric functions*, respectively. Let Λ_F denote the F-algebra of symmetric functions with coefficients in $F = \mathbb{Q}(q,t)$. The elements in Λ_F may be viewed as formal power series over $\mathbb{Q}(q,t)$ in infinitely many variables $X = (x_1, x_2, \dots)$ with finite degrees that are invariant under permutations of variables. In this extended abstract, we often replace t by q^α, for some α.

We briefly review plethystic notation. Let $E = E(t_1, t_2, \dots)$ be a formal Laurent series with rational coefficients in $t_1, t_2, \dots$. The *plethystic substitution* $p_k[E]$ is defined by replacing each t_i in E by t_i^k, i.e., $p_k[E] = E\left(t_1^k, t_2^k, \dots\right)$ and $p_\lambda[E] = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}[E]$. The plethystic substitution has the following properties: (1). If $X = x_1 + x_2 + \cdots$, then $p_k[X] = p_k(x_1, x_2, \dots)$; (2). For a real parameter z, $p_k[zX] = z^k p_k[X]$; (3). $p_k[(1-t)X] = \sum_i x_i^k (1-t^k) = (1-t^k)p_k[X]$; (4). $p_k\left[\dfrac{X}{1-q}\right] = \sum_i \dfrac{x_i^k}{1-xq^k} = \dfrac{1}{1-q^k} p_k[X]$.

Then for an arbitrary symmetric function f, we define $f[E]$ by $\sum_\lambda c_\lambda p_\lambda[E]$ if $f = \sum_\lambda c_\lambda p_\lambda$. Due to the above property (1), it easily follows that for any $f \in \Lambda_F$, $f[X] = f(x_1, x_2, \dots)$. For this reason, it is a common convention in plethystic expression that X stands for $x_1 + x_2 + \cdots$. In a similar vein, for $X = x_1 + x_2 + \cdots, Y = y_1 + y_2 + \cdots$, we have $f[X+Y] = f(x_1, x_2, \dots, y_1, y_2, \dots)$ and $f[XY] = f(x_1y_1, x_1y_2, \dots, x_2y_1, x_2y_2, \dots,)$. Refer to [8, 11] for a more comprehensive account.

Given a Dyck path π, there is a natural way to associate a Hessenberg function $h(\pi)$, a natural unit interval order $P(\pi)$, and a graph $G(\pi)$ for a Dyck path π. We outline those correspondences.

A *Hessenberg function* is a nondecreasing function $h : [n] \to [n]$ such that $h(i) \geq i$ for all $1 \leq i \leq n$. For a Dyck path π, if we let $h(\pi)$ be a function whose value at i is the i-th column height of π, then $h(\pi)$ is a Hessenberg function. Therefore, a Dyck path determines a Hessenberg function and vice versa.

The *natural unit interval order* is a poset arising from an arrangement of unit intervals. There is a well-known correspondence between Dyck paths and natural unit interval orders and we will take it as a definition of natural unit interval order. For a Dyck path π, $P = P(\pi)$ is a poset on $[n]$ whose order relation is defined by $i <_{P(\pi)} j$ if $i < n$ and $h(\pi)_i + 1 \leq j \leq n$. In this case, we also denote the order relation by $i <_\pi j$ for $i <_{P(\mathbf{m})} j$. In addition, we say (i, j) is an *attacking pair* of π if $i \not<_\pi j$.

Given a poset P, the *incomparability graph* $\mathrm{inc}(P)$ is a graph whose vertex set is the elements of P and edges are connecting pairs of incomparable elements in P. Notice that for a natural unit interval order $P(\pi)$ associated with a Dyck path π, the corresponding incomparability graph is encoded in the cells between the Dyck path and the main diagonal. We denote this graph by $G(\pi)$.

When referring to a Dyck path π, we implicitly encompass the associated objects discussed above. That is, π will represent not only the Dyck path itself but also its corresponding Hessenberg function $h(\pi)$, the natural unit interval order $P(\pi)$, and the associated incomparability graph $G(\pi)$.

2.2 Chromatic symmetric functions and unicellular LLT polynomials

We define the *inversion* statistic as $\mathrm{inv}_\pi(w) = |\{(i,j) \ : \ i < j, i \not<_\pi j, \text{ and } w(i) > w(j)\}|$, for a word $w \in \mathbb{Z}^n_{>0}$ of length n.

Given a Dyck path π, the *chromatic symmetric function* of π is defined by

$$\chi_\pi[X; q] = \sum_{\substack{w \in \mathbb{Z}^n_{>0} \\ \text{proper}}} q^{\mathrm{inv}_\pi(w)} x^w, \tag{2.1}$$

where the sum is over 'proper' colorings of $G(\pi)$, that is, the sum is over the words of length n such that $w(i) \neq w(j)$ for $i \not<_\pi j$. In definition (2.1) for χ_π, if we remove the proper condition for w, then it becomes the *unicellular LLT polynomial* indexed by π, namely

$$\mathrm{LLT}_\pi[X;q] = \sum_{w\in\mathbb{Z}^n_{>0}} q^{\mathrm{inv}_\pi(w)} x^w. \tag{2.2}$$

Carlsson and Mellit [4] found an explicit relationship between unicellular LLT polynomials and chromatic symmetric functions via a plethystic substitution.

Proposition 2.1. *[4] For a Dyck path π, we have*

$$(q-1)^n \chi_\pi[X;q] = \mathrm{LLT}_\pi[(q-1)X;q]. \tag{2.3}$$

3 α-chromatic symmetric functions

Throughout this paper, we let $Q_\alpha = \frac{q^\alpha-1}{q-1}$.

Definition 3.1. Given a Dyck path π and $\alpha \in \mathbb{R}$, we define the *α-chromatic symmetric function*

$$\chi_\pi^{(\alpha)}[X;q] := \chi_\pi\left[Q_\alpha X;q\right] = \frac{\mathrm{LLT}_\pi[(q^\alpha-1)X;q]}{(q-1)^n}. \tag{3.1}$$

Given a Dyck path π, for $1 \le i \le n$, let $b_i(\pi)$ and $a_i(\pi)$ denote the number of cells below π and strictly above the diagonal $y = x$, in the ith column (row, respectively), from the right (top, respectively). Then, by applying the *superization technique* (cf. [8]) on the right most side of (3.1), we can prove the following proposition.

Proposition 3.2. *For an n-Dyck path π and a real parameter α,*

$$\chi_\pi^{(\alpha)}[1;q] = \chi_\pi[Q_\alpha;q] = q^{\mathrm{area}(\pi)} \prod_{i=1}^n [\alpha - a_i(\pi)]_q = q^{\mathrm{area}(\pi)} \prod_{i=1}^n [\alpha - b_i(\pi)]_q. \tag{3.2}$$

Remark 3.3. The products occurring in the middle and the right hand side terms of (3.2) are of the form so called *rook product*. Note that the close connection between the chromatic quasisymmetric functions and rook theory has already been revealed in various works (cf. [1, 5]). We also observe many interesting combinatorial properties of rook theory related to the α-chromatic symmetric functions. In particular, we obtain a new solution to Garsia and Remmel's problem of finding a combinatorial description for their q-hit numbers for any Ferrers board. See [9] for the details.

Remark 3.4. Recently, Hikita [10] announced a proof of the Stanley–Stembridge conjecture, and his approach uses probabilistic argument observing that

$$\sum_{\lambda\vdash n} q^{|\mathsf{e}|-|\mathsf{e}_\lambda|} \frac{c_{\lambda,\pi}(q)}{\prod_i [\lambda_i]_q!} = 1 \tag{3.3}$$

where $\chi_\pi(X;q) = \sum_{\lambda \vdash n} c_{\lambda,\pi}(q) e_\lambda[X]$ is the e-expansion of the chromatic quasisymmetric function indexed by a Dyck path π, $|\mathsf{e}|$ is the number of cells outside of π and $|\mathsf{e}_\lambda| = \sum_{i<j} \lambda_i \lambda_j$. We can easily derive (3.3) from (3.2), by applying the principal specialization of the elementary symmetric functions and by letting $\alpha \to \infty$.

3.1 Monomial expansion into α-binomial bases

Given an n-Dyck path π, we define an ordering $\prec_\pi$ for *bi-letters* $(i,j) \in \mathbb{Z}_{>0} \times [n]$ as follows:

$$(i,j) \prec_\pi (i',j') \text{ if and only if } \begin{cases} i < i' & \text{or} \\ i = i' & \text{and} \;\; j <_\pi j'. \end{cases}$$

Here, we consider the second coordinate $j \in [n]$ as elements in the poset $P(\pi)$ corresponding to the Dyck path π. For $\lambda \vdash n$, let $m_i(\lambda)$ denote the number of times i occurs as a part in λ, and let $M(\lambda)$ be the set of all words of *weight* λ, which means all words where i occurs λ_i times for $i \geq 1$. For a bi-word $(w,\sigma) \in M(\lambda) \times \mathfrak{S}_n$, we say that $i \in [n-1]$ is an *ascent* of (w,σ) with respect to π if

$$(w(i), \sigma(i)) \prec_\pi (w(i+1), \sigma(i+1)).$$

We define $\mathrm{asc}_\pi(w,\sigma)$ as the number of ascents of (w,σ) with respect to π. We let $\mathrm{comaj}_\pi(w,\sigma)$ be the sum of positions where the ascents of (w,σ) occur. Finally, for a bi-word $(w,\sigma) \in M(\lambda) \times \mathfrak{S}_n$, we define a statistic stat_π by

$$\mathrm{stat}_\pi(w,\sigma) = \mathrm{inv}_\pi(w) + \sum_{i=1}^{\ell(\lambda)} \mathrm{inv}_{\pi[w^{-1}(i)]}(\sigma[w^{-1}(i)]) + \binom{n}{2} - nk + \mathrm{comaj}_\pi(w,\sigma),$$

where $\pi[A]$ is the induced subgraph π on A. Using these notions, we state a monomial expansion of $\chi_\pi^{(\alpha)}[X;q]$ into the basis $\left\{ \begin{bmatrix} \alpha+k \\ n \end{bmatrix}_q \right\}_{0 \leq k \leq n-1}$.

Theorem 3.5. *Given a Dyck path π, we have*

$$\chi_\pi^{(\alpha)}[X;q] = \sum_{\lambda \vdash n} \sum_{k=0}^{n-1} \sum_{\substack{(w,\sigma) \in M(\lambda) \times \mathfrak{S}_n \\ \mathrm{asc}_\pi(w,\sigma) = k}} q^{\mathrm{stat}_\pi(w,\sigma)} \begin{bmatrix} \alpha+k \\ n \end{bmatrix}_q m_\lambda[X].$$

Proof. We briefly sketch the proof.

We consider an ordering $1 < \overline{1} < 2 < \overline{2} < \cdots < n < \overline{n}$ on the set of signed alphabets $\mathcal{A}_+ \cup \mathcal{A}_-$. By applying the superization technique to the LLT polynomials, we get

$$\mathrm{LLT}_\pi[tX - Y; q] = \sum_{w \in \mathbb{Z}_{>0}^n} q^{\mathrm{inv}_\pi(w)} \prod_{i=1}^n (t - q^{N_\pi(i)}) x^w,$$

where $N_\pi(i)$ is the number of j's such that $i < j$, $i \not<_\pi j$ and $w(i) = w(j)$. On the left-hand side, by replacing t by q^α and dividing by $(q-1)^n$, we get the α-chromatic symmetric function:

$$\begin{aligned}\chi_\pi^{(\alpha)}[X;q] &= \frac{\mathrm{LLT}_\pi[(q^\alpha-1)X;q]}{(q-1)^n} \\ &= \sum_{\lambda\vdash n}\sum_{w\in M(\lambda)} q^{\mathrm{inv}_\pi(w)} \prod_{i=1}^{\ell(\lambda)} \left(q^{\mathrm{area}(\pi[w^{-1}(i)])} \prod_{j=1}^{\lambda_i} [\alpha - a_j(\pi[w^{-1}(i)])]_q \right) m_\lambda[X]. \quad (3.4)\end{aligned}$$

For a given positive integer α, a word (coloring) $c \in \mathbb{Z}_{>0}^n$, and another word (decoration) $d \in \{0,1,\dots,\alpha-1\}^n$, we say that the bi-word (c,d) is an *α-decorated proper coloring* of π of weight λ if the weight of c is λ, and for any cell (i,j) with $i<j$ and $i \not<_\pi j$, we have $(c(i),d(i)) \neq (c(j),d(j))$. Let $\mathcal{C}_{\pi,\lambda}^{(\alpha)}$ denote the set of α-decorated proper colorings of weight λ. By Proposition 3.2 we have

$$\prod_{j=1}^{n} [\alpha - a_j(\pi)]_q = q^{-|\,\mathrm{area}(\pi)|} \sum_{\substack{\sigma\in\{0,1,\dots,\alpha-1\}^n \\ \text{proper coloring of } \pi}} q^{\mathrm{inv}_\pi(\sigma)+|\sigma|}.$$

By utilizing (3.4) and the identity above, we get the following monomial expansion

$$\chi_\pi^{(\alpha)}[X;q] = \sum_{\lambda\vdash n} \sum_{(c,d)\in\mathcal{C}_{\pi,\lambda}^{(\alpha)}} q^{\mathrm{inv}_\pi(c)+|d|} \prod_{i=1}^{\ell(\lambda)} q^{\mathrm{inv}_{\pi[c^{-1}(i)]}(d^{-1}(i))} m_\lambda[X]. \quad (3.5)$$

To prove Theorem 3.5, it suffices to verify it for n distinct values of α. To that end, we fix an integer α and prove the identity using the bijection below.

Definition 3.6. Consider an integer $0 \le \alpha \le n-1$. Let $\binom{[\alpha+k]}{n}$ denote the multiset of size n whose elements are in $\{0,1,\dots,\alpha+k-n\}$, namely

$$\binom{[\alpha+k]}{n} = \{\tau = (\tau(1),\tau(2),\dots,\tau(n)) \mid 0 \le \tau(i) \le \alpha+k-n,\ \tau(1)\le\cdots\le\tau(n)\}.$$

Note that the size of this set is equal to $\binom{\alpha+k}{n}$. We define

$$\Phi : \bigcup_{k=0}^{n-1} \{(w,\sigma)\in M(\lambda)\times\mathfrak{S}_n : \mathrm{asc}(w,\sigma)=k\} \times \binom{[\alpha+k]}{n} \to \mathcal{C}_{\pi,\lambda}^{(\alpha)}$$

by $\Phi(w,\sigma,\tau)_{\sigma(i)} = (w(i), \tau(i)+(i-1)-\mathrm{asc}_\pi^{<i}(w,\sigma))$. Here, $\mathrm{asc}_\pi^{<i}(w,\sigma)$ counts the ascents occurring in the first $i-1$ positions.

Then the fact that the map Φ is a bijection gives a way to collect $\binom{\alpha+k}{n}$ terms in the monomial expansion of Theorem 3.5. See [9] for a fuller account. □

3.2 The XY technique and $(\alpha)^{\underline{k}}$ expansion

Now we consider the $(\alpha)^{\underline{k}}$ basis expansion.

Given two sets of variables X, Y, let $XY = \{x_i y_j : i, j \geq 1\}$ and π be a Dyck path.

Theorem 3.7.

$$\chi_\pi[XY;q] = \sum_{\lambda \vdash n} m_\lambda[X] \sum_{w \in M(\lambda)} q^{\mathrm{inv}_\pi(w)} \chi_{\beta(\pi,w)}(Y;q), \tag{3.6}$$

where the n-Dyck path $\beta(\pi, w)$ corresponds to the graph G_β obtained by starting with G_π colored by w, and then removing all edges which connect two vertices a, b with different colors (i.e. vertices satisfying $w(a) \neq w(b)$).

Proof. Define a total order on ordered pairs of positive integers (a, b) as follows:

$$\text{if } a < c \text{ then } (a,b) < (c,d) \text{ for all } b,d \text{ and also } (b,a) < (b,c) \text{ for all } b. \tag{3.7}$$

By definition,

$$\chi_\pi[XY;q] = \sum_{\substack{C \in (\mathbb{Z}^2)^n \\ C \text{ proper}}} q^{\mathrm{inv}_\pi(C)} \prod_{i \geq 1} x_{w_i} y_{z_i}, \tag{3.8}$$

where $C_i = (w_i, z_i)$ with both w and z in $\mathbb{Z}_+^n$, and we use (3.7) to determine inequalities among the C_i's. Now for any $i < j$ where $(j, i) \in G_\pi$, if $w_i \neq w_j$, then whether or not C_i and C_j contribute to $\mathrm{inv}_\pi(C)$ is completely determined by the values of w_i and w_j. On the other hand, if $w_i = w_j$, then whether C_i and C_j contribute to $\mathrm{inv}_\pi(C)$ is completely determined by the values of z_i and z_j, and in this case we also must have $z_i \neq z_j$, otherwise the coloring C of G_π would not be proper. It follows that

$$\mathrm{inv}_\pi(C) = \mathrm{inv}_\pi(w_1, w_2, \ldots, w_n) + \mathrm{inv}_{\beta(\pi,w)}(z_1, z_2, \ldots, z_n), \tag{3.9}$$

where the first term on the right counts the inversions amongst C_i, C_j which have unequal first coordinates, and $\mathrm{inv}_{\beta(\pi,w)}$ counts the inversions amongst C_i, C_j which have equal first coordinates. Thus the contribution to $\chi_\pi[XY;q]$ from all colorings C whose first coordinate w has weight λ on the right-hand-side of (3.8) factors into a term of weight λ in the X variables, times a sum of terms in the Y variables, which are over proper colorings of $G_{\beta(\pi,w)}$. We leave it to the reader to show that $G_{\beta(\pi,w)}$ is a unit interval order when G_π is. □

Corollary 3.8. *Given an n-Dyck path π, let*

$$\chi_\pi^{(\alpha)}[X;q] = \sum_{\lambda \vdash n} C_{\pi,\lambda}(\alpha) s_\lambda[X]. \tag{3.10}$$

Then if $\alpha \in \mathbb{N}$, $C_{\pi,\lambda}(\alpha) \in \mathbb{N}[q]$ and furthermore has a combinatorial interpretation.

Proof. Let $X = Q_\alpha$ in (3.6). Clearly $m_\lambda[Q_\alpha] \in \mathbb{N}[q]$ since $\alpha \in \mathbb{N}$. Since all the Schur coefficients of the $\chi_{\beta(\pi,w)}(Y;q)$ are in $\mathbb{N}[q]$ and count weighted P-tableaux by the result of Shareshian and Wachs, everything in (3.6) is positive and has a combinatorial interpretation. $\square$

A *set partition* of $[n]$ is a collection of nonempty subsets of $[n]$ such that each element in $[n]$ is included in exactly one subset. Each subset in a set partition is called a *part*. The set of set partitions of $[n]$ into k parts is denoted by $\mathrm{S}(n,k)$.

For an n-Dyck path π and a set partition S of $[n]$, we assign a Dyck path $\beta(\pi,S)$ as follows. First, order the parts of S as $S = \{S^{(1)}, \dots, S^{(k)}\}$ and let w_S be the word obtained by replacing elements in $S^{(i)}$ with i. Now define

$$\beta(\pi,S) := \beta(\pi,w_S).$$

Using this, we present the expansion of α-chromatic symmetric functions into $(\alpha)^{\underline{k}}$ basis.

Theorem 3.9. *For an n-Dyck path π, the α-chromatic symmetric function $\chi_\pi^{(\alpha)}[X;1]$ at $q=1$ in terms of the falling factorial basis $\{(\alpha)^{\underline{k}}\}_{1\le k\le n}$ is*

$$\chi_\pi^{(\alpha)}[X;1] = \sum_{k=1}^{n} (\alpha)^{\underline{k}} \sum_{S\in \mathrm{S}(n,k)} \chi_{\beta(\pi,S)}[X;1]. \tag{3.11}$$

In particular, the α-chromatic symmetric function $\chi_\pi^{(\alpha)}[X;1]$ is positively expanded in terms of the basis $\{(\alpha)^{\underline{k}} s_\lambda\}_{\substack{1\le k\le n,\\ \lambda\vdash n}}$.

Proof of Theorem 3.9. Let w be a word of length n. Define $S(w)$ to be the set partition such that i and j belong to the same part of $S(w)$ if and only if $w(i) = w(j)$. For example, $S(31321) = \{\{1,3\},\{2,5\},\{4\}\}$. By letting $X = Q_\alpha$ and $Y = X$ in (3.6), we have

$$\chi_\pi^{(\alpha)}[X;q] = \sum_{\lambda\vdash n} m_\lambda[Q_\alpha] \sum_{w\in M(\lambda)} q^{\mathrm{inv}_\pi(w)} \chi_{\beta(\pi,w)}(Y;q),$$

We let $q=1$ to obtain

$$\chi_\pi^{(\alpha)}[X;1] = \sum_{\lambda\vdash n}\sum_{w\in M(\lambda)} \binom{\alpha}{\ell(\lambda)} \binom{\ell(\lambda)}{m_1(\lambda), m_2(\lambda), \cdots, m_{\ell(\lambda)}(\lambda)} \chi_{\beta(\pi,w)}[X;1], \tag{3.12}$$

where $m_i(\lambda)$ denotes the multiplicity of i in the partition λ.

For a set partition S of n, one can associate a partition $\lambda(S)$ by reordering the sizes of each part of S to a non-increasing sequence. Let $\mathrm{S}(n,\lambda)$ be the set of set partitions S with $\lambda(S) = \lambda$.

Fix a partition $\lambda \vdash n$ and a set partition $S \in \mathrm{S}(n,\lambda)$. Note that the number of words w satisfying $S(w) = S$ is given by $m_1(\lambda)!m_2(\lambda)!\cdots m_{\ell(\lambda)}(\lambda)!$. Therefore, we can rewrite (3.12) as

$$\begin{aligned}\chi_\pi^{(\alpha)}[X;1] &= \sum_{\lambda\vdash n}\sum_{S\in\mathrm{S}(\lambda)}\binom{\alpha}{\ell(\lambda)}\binom{\ell(\lambda)}{m_1(\lambda),m_2(\lambda),\dots}m_1(\lambda)!\cdots m_{\ell(\lambda)}(\lambda)!\chi_{\beta(\pi,S)}[X;1]\\ &= \sum_{\lambda\vdash n}\sum_{S\in\mathrm{S}(\lambda)}(\alpha)^{\underline{\ell(\lambda)}}\chi_{\beta(\pi,S)}[X;1],\end{aligned}$$

which finishes the proof. $\square$

Remark 3.10. Due to Hikita's proof of Stanley–Stembridge conjecture [10], Theorem 3.9 gives e-positivity of the α-chromatic symmetric functions.

3.3 Further results on Schur expansions

This section proves the Schur positivity of α-chromatic symmetric functions into two binomial bases $\left\{\begin{bmatrix}\alpha+k\\ n\end{bmatrix}_q\right\}_{0\le k\le n-1}$ and $\left\{\begin{bmatrix}\alpha\\ k\end{bmatrix}_q\right\}_{1\le k\le n}$. Notably, we establish a general result for Schur positivity of $s_\lambda[Q_\alpha X]$.

Lemma 3.11. *For a partition λ, $s_\lambda[Q_\alpha X]$ is Schur positive in terms of the bases*

$$\left\{\begin{bmatrix}\alpha+k\\ n\end{bmatrix}_q\right\}_{0\le k\le n-1} \text{ and } \left\{\begin{bmatrix}\alpha\\ k\end{bmatrix}_q\right\}_{1\le k\le n}.$$

As a Corollary, the Schur positivity of α-chromatic symmetric function $\chi_\pi^{(\alpha)}[X;q]$ in terms of $\left\{\begin{bmatrix}\alpha+k\\ n\end{bmatrix}_q\right\}_{0\le k\le n-1}$ and $\left\{\begin{bmatrix}\alpha\\ k\end{bmatrix}_q\right\}_{1\le k\le n}$ follows from the Schur positivity of the usual chromatic symmetric functions [6, 12].

Corollary 3.12. *Consequently, if a symmetric function f is Schur positive, then $f[Q_\alpha X]$ also exhibits Schur positivity in the bases $\left\{\begin{bmatrix}\alpha+k\\ n\end{bmatrix}_q\right\}_{0\le k\le n-1}$ and $\left\{\begin{bmatrix}\alpha\\ k\end{bmatrix}_q\right\}_{1\le k\le n}$. In particular, the Schur coefficients of α-chromatic symmetric function $\chi_\pi^{(\alpha)}[X;q]$ of a Dyck path π are in $\mathbb{N}[q]$ when $\chi_\pi^{(\alpha)}[X;q]$ is expanded in terms of aforementioned bases.*

It is noteworthy that the positivity in terms of the second basis is *weaker* than the positivity in the q-falling factorial basis $[\alpha]_q^{\underline{k}}$. In fact, $s_\lambda[Q_\alpha X]$ might not be in $\mathbb{N}[q]$ in terms of the q-falling factorial basis $[\alpha]_q^{\underline{k}}$. For example, $s_{(2)}[Q_\alpha X]$ at $q=1$ is

$$\left(\frac{1}{2}\alpha^2-\frac{1}{2}\alpha\right)s_{(2)}[X]+\left(\frac{1}{2}\alpha^2+\frac{1}{2}\alpha\right)s_{(1,1)}[X]=\frac{1}{2}(\alpha)^{\underline{2}}s_{(2)}[X]+\left(\frac{1}{2}(\alpha)^{\underline{2}}+(\alpha)^{\underline{1}}\right)s_{(1,1)}[X],$$

in which the coefficients are rational in terms of the falling factorial basis (even for $q=1$). Nevertheless, we conjecture that the Schur coefficients of the α-chromatic symmetric functions lie in $\mathbb{N}[q]$ in terms of the q-falling factorial basis.

Conjecture 3.13. For an n-Dyck path π, the coefficients of the α-chromatic symmetric function $\chi_\pi^{(\alpha)}[X;q]$ in $\left\{[\alpha]_q^{\underline{k}}s_\lambda[X]\right\}_{1\le k\le n,\,\lambda\vdash n}$ expansion are in $\mathbb{N}[q]$.

Lastly, we present a symmetry relation for the Schur expansion of the α-chromatic symmetric functions.

Proposition 3.14. *For an n-Dyck path π, if we let*

$$\chi_\pi^{(\alpha)}[X;q]=\sum_{k=0}^{n-1}\sum_{\lambda\vdash n}c_{\pi,\lambda,k}^{(\alpha)}(q)\begin{bmatrix}\alpha+k\\ n\end{bmatrix}_q s_\lambda[X],$$

then

$$c_{\pi,\lambda,k}^{(\alpha)}(q)=q^{\mathrm{area}(\pi)+\binom{n}{2}}c_{\pi,\lambda',n-1-k}^{(\alpha)}(q^{-1}).$$

Proof. This symmetry relation easily follows the following relation of the unicellular LLT polynomials [2, Proposition 4.1.4]

$$\omega(\mathrm{LLT}_\pi[X;q])=q^{\mathrm{area}(\pi)}\,\mathrm{LLT}_\pi[X;q^{-1}].$$

For the details, see [9]. □

4 Final Remarks

4.1 Schur expansion of unicellular LLT polynomials

Proposition 4.1. *For an n-Dyck path π, let $c_{\pi,\lambda}^k(q)$ and $d_{\pi,\lambda}^k(q)$ be the Schur coefficients of α-chromatic symmetric function into the bases $\left\{\begin{bmatrix}\alpha+k\\ n\end{bmatrix}_q\right\}_{0\le k\le n-1}$ and $\left\{[\alpha]_q^{\underline{k}}\right\}_{1\le k\le n}$, respectively:*

$$\chi_\pi^{(\alpha)}[X;q]=\sum_{\lambda\vdash n}\sum_{k=0}^{n-1}c_{\pi,\lambda}^k(q)\begin{bmatrix}\alpha+k\\ n\end{bmatrix}_q s_\lambda[X]=\sum_{\lambda\vdash n}\sum_{k=1}^{n}d_{\pi,\lambda}^k(q)[\alpha]_q^{\underline{k}}s_\lambda[X].$$

Then we have

$$\mathrm{LLT}_\pi[X;q]=\sum_{\lambda\vdash n}\frac{\sum_{k=0}^{n-1}c_{\pi,\lambda}^k(q)}{[n]_q!}s_\lambda[X]=\sum_{\lambda\vdash n}q^{-\binom{n}{2}}d_{\pi,\lambda}^n(q)s_\lambda[X].$$

At $\alpha = 1$, we have $[\alpha]_q^{\underline{k}} = 0$ for all k except for $k = 1$. On the other hand, the α-chromatic symmetric function evaluated at $\alpha = 1$ coincides with the original chromatic symmetric function. Consequently, in the expansion of α-chromatic symmetric functions $\chi_\pi^{(\alpha)}[X;q]$ using the $\left\{[\alpha]_q^{\underline{k}}\right\}$ basis, the coefficient corresponding to $[\alpha]_q^{\underline{1}}$ is the chromatic symmetric function $\chi_\pi[X;q]$. Based on this observation, Proposition 4.1 implies that the coefficients of the α-chromatic symmetric functions expressed in terms of the q-falling factorial basis serve as an interpolation between the chromatic symmetric functions and the unicellular LLT polynomials.

4.2 Geometric interpretations

For an n-Dyck path π, let $\mathrm{Hess}(\pi)$ be the (regular semisimple) Hessenberg variety associated to π (or the Hessenberg function $h(\pi)$). Tymoczko's dot action on torus equivariant cohomology induces an $\mathfrak{S}_n$ action on the cohomology $H^*(\mathrm{Hess}(\pi))$. Brosnan–Chow [3], and Guay-Paquet [7] independently proved the conjecture of Shareshian–Wachs [12] that this representation coincides with the chromatic symmetric function:

$$\omega\chi_\pi[X;q] = \mathrm{Frob}(H^*(\mathrm{Hess}(\pi));q),$$

where Frob is the (graded) Frobenius characteristic map.

Since α-chromatic symmetric functions enjoy various Schur positivity phenomena and symmetry relations, it is tempting to seek a geometric model behind it. For fixed $\alpha \in \mathbb{Z}_{>0}$, it is quite direct to obtain the following geometric interpretation for α-chromatic symmetric function.

Proposition 4.2. *Let π be an n-Dyck path and $\alpha \in \mathbb{Z}_{>0}$ be a fixed positive integer. Then we have*

$$\omega\chi_\pi^{(\alpha)}[X;q] = \mathrm{Frob}\left(H^*\left(\mathrm{Hess}(\pi) \times (\mathbb{P}^{\alpha-1})^n\right);q\right).$$

Here $\mathbb{P}^{\alpha-1}$ is the complex projective space of dimension $\alpha - 1$ where $\mathfrak{S}_n$ acts by permuting coordinates.

Unfortunately, if we let α be general (as a parameter), it is unclear how to interpret the α-chromatic symmetric functions in terms of geometry. Such an approach could guide us toward achieving the desired Schur positivity in Conjecture 3.13.

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