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Permutation representations of classical Weyl groups on mod q lattices

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Abstract.

For a given linear action of a finite group on a lattice and a positive integer q, the mod q permutation representation is a quasi-polynomial in q. In this paper, we compute the multiplicity of each irreducible representation in the mod q permutation representation of a classical Weyl group on the two types of lattices, generated by the standard basis and by coroots.

Keywords: classical Weyl groups, lattices, mod q permutation representations, quasi-polynomials, integer partitions

1 Introduction

1.1 Quasi-polynomials

Let R be a commutative ring. A function $f : \mathbb{Z}_{>0} \to R$ is called a **quasi-polynomial** if there exists a positive integer $\widetilde{n} \in \mathbb{Z}_{>0}$ and polynomials $g_1(t), \ldots, g_{\widetilde{n}}(t) \in R[t]$ such that

$$f(q) = g_r(q), \quad \text{if } q \equiv r \mod \widetilde{n} \quad (1 \le r \le \widetilde{n}).$$

The positive integer $\widetilde{n}$ is called a **period** and each polynomial g_r is called the **constituent** of f. The quasi-polynomial f has degree d if all the constituents have degree d. Moreover, the quasi-polynomial f has the **gcd-property** if the polynomial g_r depends on r only through $\gcd\{\widetilde{n}, r\}$. In other words, $g_{r_1} = g_{r_2}$ if $\gcd\{\widetilde{n}, r_1\} = \gcd\{\widetilde{n}, r_2\}$. Quasi-polynomials play an important role in many areas of mathematics and appear frequently, especially as counting functions.

Example 1.1 (The Ehrhart quasi-polynomial)**.** Let $\mathcal{P}$ be a rational polytope in $\mathbb{R}^\ell$. For $q \in \mathbb{Z}_{\ge 0}$, define

$$\mathrm{L}_{\mathcal{P}}(q) := \#(q\mathcal{P} \cap \mathbb{Z}^\ell).$$

Then $\mathrm{L}_{\mathcal{P}}(q)$ is a quasi-polynomial ([1, Theorem 3.23]), known as the **Ehrhart quasi-polynomial**.

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Example 1.2 (The characteristic quasi-polynomial)**.** Let $L \simeq \mathbb{Z}^\ell$ be a lattice and $L^\vee := \mathrm{Hom}_\mathbb{Z}(L, \mathbb{Z})$ be the dual lattice. Given $\alpha_1, \dots, \alpha_n \in L^\vee$, we can associate a hyperplane arrangement $\mathscr{A} := \{H_1, \dots, H_n\}$ in $\mathbb{R}^\ell \simeq L \otimes \mathbb{R}$, where

$$H_i := \{\, x \in L \otimes \mathbb{R} \mid \alpha_i(x) = 0 \,\}.$$

For a positive integer $q \in \mathbb{Z}_{>0}$, define the mod q complement of the arrangement by

$$\begin{aligned} M(\mathscr{A}, q) &:= (L/qL)^\ell \setminus \bigcup_{i=1}^{\ell} \overline{H}_i \\ &= \{\, \bar{x} \in L/qL \mid \alpha_i(x) \not\equiv 0 \mod q \quad \text{for all } i \in \{1, \dots, n\} \,\}. \end{aligned}$$

It is known ([4, Theorem 2.4]) that

$$\chi_{\mathrm{quasi}}(\mathscr{A}, q) := \#M(\mathscr{A}, q)$$

is a quasi-polynomial with gcd-property. It is called a **characteristic quasi-polynomial**. Furthermore, the first constituent of $\chi_{\mathrm{quasi}}(\mathscr{A}, t)$ is equal to the **characteristic polynomial** $\chi(\mathscr{A}, t)$ of $\mathscr{A}$, the most important invariant of $\mathscr{A}$.

Example 1.3 (Equivariant Ehrhart theory)**.** Let $L \simeq \mathbb{Z}^\ell$ be a lattice and let Γ be a finite group acting linearly on L. Suppose that $\mathcal{P}$ is a Γ-invariant lattice polytope. For a positive integer $q \in \mathbb{Z}_{>0}$, the group acts on the lattice points $q\mathcal{P} \cap L$. Let $\chi_{q\mathcal{P}}$ denote the character of this permutation representation. Then the map

$$F : q \longmapsto \chi_{q\mathcal{P}}$$

is a quasi-polynomial ([10, Theorem 5.7, Corollary 5.9]). It is an equivariant version of Ehrhart quasi-polynomials. In fact, for the identity element 1 of Γ, then $\chi_{q\mathcal{P}}(1) = \#(q\mathcal{P} \cap L)$, hence it is a generalization of Ehrhart theory. Furthermore, the multiplicity of a fixed irreducible character χ in $\chi_{q\mathcal{P}}$ is also a quasi-polynomial in q.

1.2 mod q permutation representations

In [11], we consider mod q permutation representations for linear finite group actions on lattices toward an equivariant version of characteristic quasi-polynomials.

Let L be a lattice, and $\{\beta_1, \dots, \beta_\ell\}$ be a $\mathbb{Z}$-basis of L, that is, $L = \mathbb{Z}\beta_1 \oplus \cdots \oplus \mathbb{Z}\beta_\ell \simeq \mathbb{Z}^\ell$. We identify an element $x = x_1\beta_1 + \cdots + x_\ell\beta_\ell$ of L with the row vector $x = (x_1, \dots, x_\ell)$ of $\mathbb{Z}^\ell$.

Let Γ be a finite group and let $\rho : \Gamma \to \mathrm{GL}(L)$ be a group homomorphism. Let us denote the representation matrix of $\rho(\gamma)$ by R_γ, and we consider the right multiplication, namely,

$$\rho(\gamma) : L \longrightarrow L, \quad x \longmapsto xR_\gamma.$$

For a positive integer $q \in \mathbb{Z}_{>0}$, define $\mathbb{Z}_q := \mathbb{Z}/q\mathbb{Z}$. We will consider the following ***q*-reduction** of a vector $x = (x_1, \dots, x_\ell) \in \mathbb{Z}^\ell$ and an integer matrix $A = (a_{ij})_{ij}$:

$$[x]_q := ([x_1]_q, \dots, [x_\ell]_q) \in L_q^\ell, \qquad [A]_q := \Big([a_{ij}]_q\Big)_{ij},$$

where $[z]_q = z + q\mathbb{Z} \in \mathbb{Z}/q\mathbb{Z}$ for $z \in \mathbb{Z}$. Then, for a $\mathbb{Z}$-homomorphism $\varphi : \mathbb{Z}^\ell \to \mathbb{Z}^\ell$ represented by an integer matrix A, we can define the induced morphism $\varphi_q : \mathbb{Z}_q^\ell \to \mathbb{Z}_q^\ell$ by $x \longmapsto x[A]_q$.

Let $L_q := L/qL \simeq \mathbb{Z}_q^\ell$. The action of Γ on L_q is induced by $\rho(\gamma)_q : L_q \longrightarrow L_q$. Let χ_{L_q} denote the character of the permutation representation of L_q, and consider its irreducible decomposition:

$$\chi_{L_q} = m(\chi_1; q) \cdot \chi_1 + \cdots + m(\chi_k; q) \cdot \chi_k,$$

where $\{\chi_1, \dots, \chi_k\}$ is the set of all irreducible characters of Γ and $m(\chi_i; q)$ denotes the multiplicity of χ_i in χ_{L_q}.

In [11], it is shown that the multiplicity $m(\chi; q)$ of the any irreducible character χ in χ_{L_q} is a quasi-polynomial in q with gcd-property ([11, Corollary 2.2]). More explicity,

$$m(\chi; q) = \frac{1}{\#\Gamma} \sum_{\gamma \in \Gamma} \chi(\gamma) \left(\prod_{j=1}^{r(\gamma)} \gcd\{e_{\gamma,j}, q\} \right) q^{\ell - r(\gamma)}, \tag{1.1}$$

where $r(\gamma) := \operatorname{rank}(R_\gamma - I)$ with the identity matrix I and $e_{\gamma,1}, \dots, e_{\gamma,r(\gamma)}$, with $e_{\gamma,1} \mid e_{\gamma,2} \mid \cdots \mid e_{\gamma,r(\gamma)}$, are the elementary divisors of the matrix $R_\gamma - I$. Obviously, the map $q \longmapsto \chi_{L_q}$ is also a quasi-polynomial with gcd-property. Moreover, their quasi-polynomial have a period $\operatorname{lcm}\{\, e_{\gamma,r(\gamma)} \mid \gamma \in \Gamma \,\}$ (**lcm-period**).

1.3 Classical Weyl groups

In this paper, we compute $m(\chi; q)$ for each irreducible character χ when a group Γ is a classical Weyl group. In this section, we recall classical Weyl groups and their conjugacy classes and irreducible characters. See also [2, 12, 13] for details.

For $\ell \in \mathbb{Z}_{>0}$, an **integer partition** of ℓ is an integer sequence $\lambda = (\lambda_1, \dots, \lambda_t)$ satisfying $\lambda_1 \geq \cdots \geq \lambda_t > 0$ and $\lambda_1 + \cdots + \lambda_t = \ell$. We define $\ell(\lambda)$ as the length t of λ and $|\lambda| := \lambda_1 + \cdots + \lambda_t$. We consider that the empty sequence $\varnothing = ()$ to be an integer partition of 0, with $\ell(\varnothing) = 0$ and $|\varnothing| = 0$.

1.3.1 Type $A_{\ell-1}$

The Weyl group of type $A_{\ell-1}$ is the symmetric group $\mathfrak{S}_\ell$. The order of $\mathfrak{S}_\ell$ is $\#\mathfrak{S}_\ell = \ell!$. Each element $\sigma \in \mathfrak{S}_\ell$ may be decomposed uniquely into a product of cyclic permutations up to the order of factors:

$$\sigma = (i_{11} \cdots i_{1\lambda_1}) \cdots\cdots (i_{t1} \cdots i_{t\lambda_t}), \quad \lambda_1 \geq \cdots \geq \lambda_t.$$

Then the integer partition $(\lambda_1, \dots, \lambda_t)$ of ℓ is called the **cycle type** of $\sigma \in \mathfrak{S}_\ell$. Two elements of $\mathfrak{S}_\ell$ are conjugate if and only if they have the same cycle type. In other words, conjugacy classes of $\mathfrak{S}_\ell$ are parameterized by integer partitions of ℓ.

Irreducible characters of $\mathfrak{S}_\ell$ are also parameterized by integer partition of ℓ. We denote the irreducible character defined by a integer partition λ by χ^λ.

1.3.2 Type B_ℓ and C_ℓ

The Weyl group of type B_ℓ and C_ℓ are the hyperoctahedral group $\mathfrak{H}_\ell$, isomorphic to the semidirect product $\mathfrak{S}_2^\ell \rtimes \mathfrak{S}_\ell$ (the wreath product $\mathfrak{S}_2 \wr \mathfrak{S}_\ell$). The order of $\mathfrak{H}_\ell$ is $\#\mathfrak{H}_\ell = 2^\ell \ell!$. The hyperoctahedral group $\mathfrak{H}_\ell$ can be regarded as a subgroup of the symmetric group $\mathfrak{S}_{I_\ell}$ of $I_\ell := \{-\ell, \dots, -1, 1, \dots, \ell\}$ by

$$\mathfrak{H}_\ell = \{\, \eta \in \mathfrak{S}_{I_\ell} \mid \eta(-i) = -\eta(i) \text{ for all } i \in I_\ell \,\}.$$

Especially, $\mathfrak{S}_2^\ell$ is to be regarded as a subgroup generalized by the sign transpositions $\tau_1, \dots, \tau_\ell$:

$$\tau_i : \begin{cases} j \longmapsto -j & \text{if } j = i,\ -i; \\ j \longmapsto j & \text{if } j \neq i,\ -i \end{cases} \quad (1 \leq i \leq \ell).$$

The hyperoctahedral group $\mathfrak{H}_\ell$ has two types of cyclic permutations, **even cyclic permutations** $(i_1 \cdots i_t)_+$ and **odd cyclic permutations** $(i_1 \cdots i_t)_-$:

$$(i_1 \cdots i_t)_+ = \begin{pmatrix} i_1 & i_2 & \cdots & i_{t-1} & i_t \\ i_2 & i_3 & \cdots & i_t & i_1 \end{pmatrix}, \qquad (i_1 \cdots i_t)_- = \begin{pmatrix} i_1 & i_2 & \cdots & i_{t-1} & i_t \\ i_2 & i_3 & \cdots & i_t & -i_1 \end{pmatrix},$$

where $i_1, \dots, i_t \in I_\ell$ and they have pairwise different absolute values. Note that an odd cyclic permutation $(i)_-$ of length 1 is equal to the sign transposition τ_i. Each element $\eta \in \mathfrak{H}_\ell$ may be decomposed uniquely into a product of two types of cyclic permutations up to the order of factors:

$$\eta = (i_{11} \cdots i_{1\lambda_1})_+ \cdots\cdots (i_{t1} \cdots i_{t\lambda_t})_+ (j_{11} \cdots j_{1\mu_1})_- \cdots\cdots (j_{s1} \cdots j_{s\mu_s})_-, \\ \lambda_1 \geq \cdots \geq \lambda_t, \quad \mu_1 \geq \cdots \geq \mu_s.$$

Then the pair of two integer partitions $\big((\lambda_1, \dots, \lambda_t),\ (\mu_1, \dots, \mu_s)\big)$ is called the **cycle type** of $\eta \in \mathfrak{H}_\ell$. Two elements of $\mathfrak{H}_\ell$ are conjugate if and only if they have the same cycle type. Hence conjugacy classes of $\mathfrak{H}_\ell$ are parameterized by the pair of integer partitions (λ, μ)

with $|\lambda| + |\mu| = \ell$. Note that an element $\eta \in \mathfrak{H}_\ell$ with cycle type $(\lambda, \emptyset)$ can be regarded as an element of $\mathfrak{S}_\ell$ with cycle type λ. Hence we consider that $\mathfrak{S}_\ell$ is a subgroup of $\mathfrak{H}_\ell$.

Irreducible characters of $\mathfrak{H}_\ell$ are also parameterized by the pair of integer partitions (λ, μ) with $|\lambda| + |\mu| = \ell$. Let $\chi^{\lambda,\mu}$ denote the irreducible character defined by (λ, μ). Then the dimensions of $\chi^{\lambda,\mu}$, χ^λ and χ^μ are connected by the following relation:

$$\frac{\chi^{\lambda,\mu}(1)}{\ell!} = \frac{\chi^\lambda(1)\chi^\mu(1)}{|\lambda|! \cdot |\mu|!}. \tag{1.2}$$

1.3.3 Type D_ℓ

The Weyl group of type D_ℓ is a subgroup of index 2 of the hyperoctahedral group $\mathfrak{H}_\ell$ denoted by $\widetilde{\mathfrak{H}}_\ell$. An element $\eta \in \mathfrak{H}_\ell$ is in $\widetilde{\mathfrak{H}}_\ell$ if and only if η has cycle type (λ, μ) such that $\ell(\mu)$ is even. In other words, we can consider that

$$\widetilde{\mathfrak{H}}_\ell = \{ \eta \in \mathfrak{H}_\ell \mid \#\{ i \in [\ell] \mid \eta(i) < 0 \} \text{ is even} \}.$$

Let $C_{\lambda,\mu}$ denote the conjugacy class of $\mathfrak{H}_\ell$ parameterized by the pair of partitions (λ, μ). Let $\widetilde{C}_{\lambda,\mu} := C_{\lambda,\mu} \cap \widetilde{\mathfrak{H}}_\ell$. If (λ, μ) satisfies

$$\lambda_1 \equiv \cdots \equiv \lambda_{\ell(\lambda)} \equiv 0 \pmod 2 \quad \text{and} \quad \mu = \emptyset, \tag{1.3}$$

then $\widetilde{C}_{\lambda,\mu}$ is the union of two conjugacy classes of $\widetilde{\mathfrak{H}}_\ell$, which denote $\widetilde{C}^1_{\lambda,\mu}$ and $\widetilde{C}^2_{\lambda,\mu}$. These classes have the following relation:

$$\widetilde{C}^1_{\lambda,\mu} = \tau_1^{-1}\widetilde{C}^2_{\lambda,\mu}\tau_1 = \{ \tau_1^{-1}\gamma\tau_1 \mid \gamma \in \widetilde{C}^2_{\lambda,\mu} \}. \tag{1.4}$$

If (λ, μ) does not satisfy (1.3), then $\widetilde{C}_{\lambda,\mu}$ is also a conjugacy class of $\widetilde{\mathfrak{H}}_\ell$. All conjugacy classes of $\widetilde{\mathfrak{H}}_\ell$ can be obtained in this way.

Let (λ, μ) be the pair of integer partitions satisfying $|\lambda| + |\mu| = \ell$. Let $\widetilde{\chi}^{\lambda,\mu}$ denote the restriction to $\widetilde{\mathfrak{H}}_\ell$ of the irreducible character $\chi^{\lambda,\mu}$ of $\mathfrak{H}_\ell$. If $\lambda \neq \mu$, then $\widetilde{\chi}^{\lambda,\mu}$ is also an irreducible character of $\widetilde{\mathfrak{H}}_\ell$, and furthermore $\widetilde{\chi}^{\lambda,\mu} = \widetilde{\chi}^{\mu,\lambda}$. But if $\lambda = \mu$, then $\widetilde{\chi}^{\lambda,\lambda}$ is the sum of two irreducible characters of $\widetilde{\mathfrak{H}}_\ell$, which we denote as $\widetilde{\chi}^{\lambda,\lambda}_1$ and $\widetilde{\chi}^{\lambda,\lambda}_2$. All irreducible characters of $\widetilde{\mathfrak{H}}_\ell$ can be obtained in this way. Moreover, the characters $\chi^{\lambda,\mu}$, $\chi^{\mu,\lambda}$ and $\widetilde{\chi}^{\lambda,\mu}$ are connected by the following formula:

$$\frac{\chi^{\lambda,\mu}(\eta) + \chi^{\mu,\lambda}(\eta)}{2} = \begin{cases} \widetilde{\chi}^{\lambda,\mu}(\eta) & \text{if } \eta \in \widetilde{\mathfrak{H}}_\ell; \\ 0 & \text{if } \eta \notin \widetilde{\mathfrak{H}}_\ell. \end{cases} \tag{1.5}$$

The irreducible characters $\widetilde{\chi}^{\lambda,\lambda}_1$ and $\widetilde{\chi}^{\lambda,\lambda}_2$ are connected by the following relation:

$$\widetilde{\chi}^{\lambda,\lambda}_1(\eta) = \widetilde{\chi}^{\lambda,\lambda}_2(\tau_1^{-1}\eta\tau_1) \quad \text{for all } \eta \in \widetilde{\mathfrak{H}}_\ell. \tag{1.6}$$

Note that η and $\tau_1^{-1}\eta\tau_1$ are not necessarily conjugate as seen in (1.4).

2 Lattices generated by the standard basis

2.1 Calculation results of the multiplicities

In this section, suppose that $L = \mathbb{Z}^\ell$. Let W be a Weyl group of type $A_{\ell-1}$, B_ℓ, C_ℓ or D_ℓ. We compute the multiplicity $m(\chi; q)$ of each irreducible character χ in $\chi_{\mathbb{Z}_q^\ell}$. Take a $\mathbb{Z}$-basis $\{e_1, \ldots, e_\ell\}$ of $\mathbb{Z}^\ell$, that is, $\mathbb{Z}^\ell = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_\ell$. Then W acts on $\mathbb{Z}^\ell$ by

$$w(e_i) = \begin{cases} e_{w(i)} & \text{if } w(i) > 0; \\ -e_{-w(i)} & \text{if } w(i) < 0 \end{cases} \qquad (1 \leq i \leq \ell,\ w \in W).$$

Obviously, W also acts on $\mathbb{R}^\ell = \mathbb{R}e_1 \oplus \cdots \oplus \mathbb{R}e_\ell$ in the same way. This is the natural action as a finite reflection group.

For $w \in W$, let R_w be the representation matrix of w and $r(w)$ denote the rank of matrix $R_w - I$, where I is the identity matrix.

Lemma 2.1. *For $w \in W$, the following values are all equal to $r(w)$:*

(1) The number of eigenvalues of w which are not equal to 1 (counted with multiplicity);

(2) $\ell - \ell(\lambda)$;

(3) $\ell - \dim(\mathbb{R}^\ell)^w$;

(4) The minimum number of reflections required to express w as a product of reflections,

where (λ, μ) is the cycle type of w and $(\mathbb{R}^\ell)^w = \{ x \in \mathbb{R}^\ell \mid w(x) = x \}$.

We obtain the elementary divisors $e_{w,1}, \ldots, e_{w,r(w)}$ of $R_w - I$ by computed directly as follows:

Lemma 2.2. *Let $w \in W$ with cycle type (λ, μ). Then*

$$e_{w,j} = \begin{cases} 1 & \text{if } j \leq r(w) - \ell(\mu); \\ 2 & \text{if } j > r(w) - \ell(\mu) \end{cases} \qquad (1 \leq j \leq r(w)).$$

Hence, using (1.1), $m(\chi; q)$ is obtained as follows:

Theorem 2.3. *For an irreducible character χ of W, we have*

$$m(\chi; q) = \frac{1}{\#W} \sum_{w \in W} \chi(w) g(q)^{\ell(\mu_w)} q^{\ell(\lambda_w)},$$

where (λ_w, μ_w) is the cycle type of w and $g(q) := \gcd\{2, q\}$.

Let **1** denote the trivial character of W. By the Shephard–Todd formula ([9], see also [8, Theorem 4.6]), the first constituent $m(\mathbf{1};\, t)_1$ of the quasi-polynomial $m(\mathbf{1};\, t)$ can be expressed as

$$m(\mathbf{1};\, t)_1 = \frac{1}{\#W}(t+m_1)\cdots(t+m_\ell), \tag{2.1}$$

where $m_1, \ldots, m_\ell$ are the exponents of W.

Let δ be the determinant character of W. By (2.1) and the reciprocity theorem ([11, Theorem 2.10]), the first constituent $m(\delta;\, t)_1$ of the quasi-polynomial $m(\delta;\, t)$ can be expressed as

$$m(\delta;\, t)_1 = \frac{1}{\#W}(t-m_1)\cdots(t-m_\ell).$$

It is well known that the above polynomial on the right hand side is equal to the characteristic polynomial of the Coxeter arrangement of W up to scalar (see [8, Corollary 3.3]).

2.2 Factorization of the multiplicity

Let W be a classical Weyl group. For a character χ of W, define $M_W(\chi;\, s,t) \in \mathbb{Z}[s,t]$ by

$$M_W(\chi;\, s,t) := \sum_{w\in W} \chi(w) s^{\ell(\mu_w)} t^{\ell(\lambda_w)},$$

where (λ_w, μ_w) is the cycle type of w. For simplicity, we denote $M_W(\chi;\, 1,t)$ (for $s=1$) by $M_W(\chi;\, t)$. Define a function $g(q) := \gcd\{2, q\}$. From Theorem 2.3, we have

$$m(\chi;\, q) = \frac{M_W(\chi;\, g(q), q)}{\#W}. \tag{2.2}$$

Let λ be an integer partition. Define a multiset A_λ as

$$A_\lambda := \bigsqcup_{i=1}^{\ell(\lambda)} \{\, i-j \mid 1 \le j \le \lambda_i \,\}.$$

2.2.1 Type $A_{\ell-1}$

Suppose that $W = \mathfrak{S}_\ell$ (type $A_{\ell-1}$). Let χ^λ denote the irreducible character parameterized by an integer partition λ. Since $\ell(\mu_\sigma) = 0$ for all $\sigma \in \mathfrak{S}_\ell$, then $M_{\mathfrak{S}_\ell}(\chi^\lambda;\, t) = M_{\mathfrak{S}_\ell}(\chi^\lambda;\, s,t)$. It is known that the factorization formula for $M_{\mathfrak{S}_\ell}(\chi^\lambda;\, t)$:

Theorem 2.4 (Littlewood [6, p. 56], Molchanov [7, Theorem 1]). *The roots of the polynomial $M_{\mathfrak{S}_\ell}(\chi^\lambda;\, t)$ are the numbers in A_λ:*

$$M_{\mathfrak{S}_\ell}(\chi^\lambda;\, t) = \chi^\lambda(1) \prod_{a \in A_\lambda} (t-a) = \chi^\lambda(1) \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (t-i+j).$$

Corollary 2.5. *Let χ^λ be the irreducible character of $\mathfrak{S}_\ell$ parameterized by an integer partition λ. Then we have*

$$m(\chi^\lambda;\, q) = \frac{\chi^\lambda(1)}{\ell!} \prod_{a \in A_\lambda} (q-a) = \frac{\chi^\lambda(1)}{\ell!} \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (q-i+j).$$

Hence $m(\chi^\lambda;\, q)$ is a quasi-polynomial with the minimal period 1 (just a polynomial).

2.2.2 Type B_ℓ and C_ℓ

Suppose that $W = \mathfrak{H}_\ell$ (type B_ℓ or C_ℓ). For the irreducible character $\chi^{\lambda,\mu}$, it is known by Young [13] in the following formula:

$$M_{\mathfrak{H}_\ell}(\chi^{\lambda,\mu};\, s,t) = \frac{2^\ell \ell!}{|\lambda|!|\mu|!} M_{\mathfrak{S}_{|\lambda|}}\left(\chi^\lambda;\, \frac{t+s}{2}\right) M_{\mathfrak{S}_{|\mu|}}\left(\chi^\mu;\, \frac{t-s}{2}\right), \tag{2.3}$$

where we consider that $M_{\mathfrak{S}_{|\emptyset|}}(\chi^\emptyset;\, \cdot) = 1$.

Theorem 2.6. *Let $\chi^{\lambda,\mu}$ be the irreducible character of $\mathfrak{H}_\ell$ parameterized by (λ,μ). Then we have*

$$\begin{aligned} m(\chi^{\lambda,\mu};\, q) &= m\left(\chi^\lambda;\, \frac{q+g(q)}{2}\right) \cdot m\left(\chi^\mu;\, \frac{q-g(q)}{2}\right) \\ &= \frac{\chi^{\lambda,\mu}(1)}{2^\ell \ell!} \prod_{a \in A_\lambda} (q+g(q)-2a) \prod_{b \in A_\mu} (q-g(q)-2b). \end{aligned}$$

Hence $m(\chi^{\lambda,\mu};\, q)$ is a quasi-polynomial with the minimal period 2.

Proof. It follows from equations (2.2), (2.3) and the dimensional relation (1.2). □

2.2.3 Type D_ℓ

Suppose that $W = \widetilde{\mathfrak{H}}_\ell$ (type D_ℓ). Let (λ,μ) be the pair of integer partitions satisfying $|\lambda| + |\mu| = \ell$. From the equation (1.5), we have

$$M_{\widetilde{\mathfrak{H}}_\ell}(\widetilde{\chi}^{\lambda,\mu};\, s,t) = \sum_{\eta \in \mathfrak{H}_\ell} \frac{\chi^{\lambda,\mu}(\eta) + \chi^{\mu,\lambda}(\eta)}{2} s^{\ell(\mu_\eta)} t^{\ell(\lambda_\eta)} = \frac{M_{\mathfrak{H}_\ell}(\chi^{\lambda,\mu};\, s,t) + M_{\mathfrak{H}_\ell}(\chi^{\mu,\lambda};\, s,t)}{2}. \tag{2.4}$$

Theorem 2.7. *Let $\widetilde{\chi}^{\lambda,\mu}$ be the irreducible character of $\widetilde{\mathfrak{H}}_\ell$ parameterized by (λ, μ) satisfying $\lambda \neq \mu$. Then we have*

$$\begin{aligned} m(\widetilde{\chi}^{\lambda,\mu}; q) &= m(\chi^{\lambda,\mu}; q) + m(\chi^{\mu,\lambda}; q) \\ &= m\left(\chi^{\lambda}; \frac{q+g(q)}{2}\right) \cdot m\left(\chi^{\mu}; \frac{q-g(q)}{2}\right) \\ &\qquad + m\left(\chi^{\mu}; \frac{q+g(q)}{2}\right) \cdot m\left(\chi^{\lambda}; \frac{q-g(q)}{2}\right). \end{aligned}$$

Hence $m(\widetilde{\chi}^{\lambda,\mu}; q)$ is a quasi-polynomial with the minimal period 2.

Proof. It follows from the equation (2.4) and Theorem 2.6. □

Suppose that $\lambda = \mu$. By the relation (1.6), we have

$$M_{\widetilde{\mathfrak{H}}_\ell}(\widetilde{\chi}_1^{\lambda,\lambda}; s, t) = M_{\widetilde{\mathfrak{H}}_\ell}(\widetilde{\chi}_2^{\lambda,\lambda}; s, t).$$

Since $\widetilde{\chi}^{\lambda,\lambda} = \widetilde{\chi}_1^{\lambda,\lambda} + \widetilde{\chi}_2^{\lambda,\lambda}$, then

$$M_{\widetilde{\mathfrak{H}}_\ell}(\widetilde{\chi}_i^{\lambda,\lambda}; s, t) = \frac{M_{\widetilde{\mathfrak{H}}_\ell}(\widetilde{\chi}_1^{\lambda,\lambda}; s, t) + M_{\widetilde{\mathfrak{H}}_\ell}(\widetilde{\chi}_2^{\lambda,\lambda}; s, t)}{2} = \frac{M_{\widetilde{\mathfrak{H}}_\ell}(\widetilde{\chi}^{\lambda,\lambda}; s, t)}{2}. \tag{2.5}$$

for $i \in \{1, 2\}$.

Theorem 2.8. *Let $\widetilde{\chi}_1^{\lambda,\lambda}$ and $\widetilde{\chi}_2^{\lambda,\lambda}$ be the irreducible characters of $\widetilde{\mathfrak{H}}_\ell$ parameterized by (λ, λ). Then we have*

$$m(\widetilde{\chi}_1^{\lambda,\lambda}; q) = m(\widetilde{\chi}_2^{\lambda,\lambda}; q) = m(\chi^{\lambda,\lambda}; q) = m\left(\chi^{\lambda}; \frac{q+g(q)}{2}\right) \cdot m\left(\chi^{\lambda}; \frac{q-g(q)}{2}\right).$$

Proof. It follows from equations (2.4), (2.5) and Theorem 2.6. □

In the case of type $A_{\ell-1}$, B_ℓ and C_ℓ, all the roots of each constituent of the quasi-polynomial $m(\chi; q)$ are integers. However, it is not always the case in type D_ℓ. This phenomenon has also been observed in characteristic quasi-polynomials of the arrangements of root systems (see [5, Example 3.5]).

Example 2.9. Let $\ell = 7$, $\lambda = (2, 1, 1)$ and $\mu = (3)$. Then

$$m(\chi^{\lambda,\mu}; q) = \begin{cases} \dfrac{105(q+1)(q-1)(q+3)(q-3) \cdot (q+1)(q-1)(q+3)}{2^7 7!} & \text{if } \gcd\{2, q\} = 1; \\ \dfrac{105q(q+2)(q-2)(q+4) \cdot q(q+2)(q-2)}{2^7 7!} & \text{if } \gcd\{2, q\} = 2, \end{cases}$$

$$m(\chi^{\mu,\lambda}; q) = \begin{cases} \dfrac{105(q+1)(q+3)(q+5) \cdot (q+1)(q-1)(q-3)(q-5)}{2^7 7!} & \text{if } \gcd\{2, q\} = 1; \\ \dfrac{105(q+2)(q+4)(q+6) \cdot q(q-2)(q-4)(q-6)}{2^7 7!} & \text{if } \gcd\{2, q\} = 2 \end{cases}$$

and

$$m(\widetilde{\chi}^{\lambda,\mu};q) = m(\chi^{\lambda,\mu};q) + m(\chi^{\mu,\lambda};q)$$
$$= \begin{cases} \dfrac{105(q+1)^2(q-1)(q+3)(q-3)(q^2+q-14)}{2^6 7!} & \text{if } \gcd\{2,q\}=1; \\ \dfrac{105q(q+2)(q-2)(q+4)(q^3-2q^2-20q+72)}{2^6 7!} & \text{if } \gcd\{2,q\}=2. \end{cases}$$

Note that t^2+t-14 and $t^3-2t^2-20t+72$ are irreducible polynomials over $\mathbb{Z}[t]$.

3 Coroot lattices

Let Φ be a root system of type $A_{\ell-1}$, B_ℓ, C_ℓ or D_ℓ and $W = W(\Phi)$ be the Weyl group of Φ. In this section, suppose that L is the coroot lattice $\check{Q} = \check{Q}(\Phi)$ of Φ:

$$\check{Q} = \check{Q}(\Phi) = \sum_{\alpha \in \Phi} \mathbb{Z}\alpha^\vee, \quad \alpha^\vee := \frac{2\alpha}{(\alpha,\alpha)}.$$

We compute the multiplicity $m(\chi;q)$ of each irreducible character χ in $\chi_{\check{Q}_q^\ell}$. Especially in this case, $m(\mathbf{1};q)$ is the Ehrhart quasi-polynomial of the fundamental alcove of Φ with respect to $\check{Q}$. Haiman [3, Section 7.4] details the case where q is relatively prime to the Coxeter number.

3.1 Type $A_{\ell-1}$

Suppose that Φ is a root system of type $A_{\ell-1}$. Then $W = \mathfrak{S}_\ell$ and the coroot lattice $\check{Q}$ can be expressed by

$$\check{Q} = \mathbb{Z}(e_1 - e_2) \oplus \cdots \oplus \mathbb{Z}(e_{\ell-1} - e_\ell).$$

Similar to Lemma 2.1, for $\sigma \in \mathfrak{S}_\ell$, $r(\sigma)$ is obtained as $r(\sigma) = \ell - \ell(\lambda)$, where λ is the cycle type of σ. We obtain the elementary divisors $e_{\sigma,1}, \ldots, e_{\sigma,r(\sigma)}$ of $R_\sigma - I$ by computed directly as follows:

Lemma 3.1. *Let $\sigma \in \mathfrak{S}_\ell$ with cycle type λ. Then*

$$e_{\sigma,1} = \cdots = e_{\sigma,r(\sigma)-1} = 1, \quad e_{\sigma,r(\sigma)} = \gcd\{\lambda_1, \ldots, \lambda_{\ell(\lambda)}\}.$$

Theorem 3.2. *Let χ^λ be the irreducible character of $\mathfrak{S}_\ell$. Then $m(\chi^\lambda;q)$ is a quasi-polynomial with minimal period ℓ. In particular, if $\gcd\{\ell,q\}=1$, then*

$$m(\chi^\lambda;q) = \frac{q^{-1}M_{\mathfrak{S}_\ell}(\chi^\lambda;q)}{\ell!} = \frac{\chi^\lambda(1)}{\ell!} q^{-1} \prod_{a \in A_\lambda} (q-a).$$

3.2 Type B_ℓ

Suppose that Φ is a root system of type B_ℓ. The coroot lattice $\check{Q}$ can be expressed by

$$\check{Q} = \mathbb{Z}(e_1 - e_2) \oplus \cdots \oplus \mathbb{Z}(e_{\ell-1} - e_\ell) \oplus \mathbb{Z}(2e_\ell).$$

Lemma 3.3. *Let $\eta \in \mathfrak{H}_\ell$ with cycle type (λ, μ). Then $r(\eta) = \ell - \ell(\lambda)$ and the following holds:*

(1) If λ has at least an odd part, then $e_{\eta,j} = \begin{cases} 1 & \text{if } j \leq r(\eta) - \ell(\mu) + 1; \\ 2 & \text{if } j > r(\eta) - \ell(\mu) + 1 \end{cases} \quad (1 \leq j \leq r(\eta)).$

(2) If λ has all even parts and $\mu = \emptyset$, then $e_{\eta,j} = \begin{cases} 1 & \text{if } j < r(\eta); \\ 2 & \text{if } j = r(\eta). \end{cases}$

(3) If λ has all even parts and μ has all even or all odd parts, $e_{\eta,j} = \begin{cases} 1 & \text{if } j \leq r(\eta) - \ell(\mu); \\ 2 & \text{if } j > r(\eta) - \ell(\mu). \end{cases}$

(4) If λ has all even parts, μ has both even and odd parts, then

$$e_{\eta,j} = \begin{cases} 1 & \text{if } j \leq r(\eta) - \ell(\mu) + 1; \\ 2 & \text{if } r(\eta) - \ell(\mu) + 1 < j < r(\eta); \\ 4 & \text{if } j = r(\eta). \end{cases}$$

Theorem 3.4. *Let χ be the irreducible character of $\mathfrak{H}_\ell$. Then $m(\chi; q)$ is a quasi-polynomial with minimal period 2 for $\ell = 2$ and 4 for $\ell \geq 3$.*

3.3 Type C_ℓ

Suppose that Φ is a root system of type C_ℓ. The coroot lattice $\check{Q}$ cen be expressed by

$$\check{Q} = \mathbb{Z}(e_1 - e_2) \oplus \cdots \oplus \mathbb{Z}(e_{\ell-1} - e_\ell) \oplus \mathbb{Z}e_\ell.$$

Since $\check{Q}$ is isomorphic to $\mathbb{Z}^\ell$, $r(\eta)$ and $e_{\eta,1}, \ldots, e_{\eta,r(\eta)}$ are as in Lemma 2.1 and Lemma 2.2.

Theorem 3.5. *Let χ be the irreducible character of $\mathfrak{H}_\ell$. Then $m(\chi; q)$ is a quasi-polynomial with minimal period 2.*

3.4 Type D_ℓ

Suppose that Φ is a root system of type D_ℓ. The coroot lattice $\check{Q}$ cen be expressed by

$$\check{Q} = \mathbb{Z}(e_1 - e_2) \oplus \cdots \oplus \mathbb{Z}(e_{\ell-1} - e_\ell) \oplus \mathbb{Z}(e_{\ell-1} + e_\ell).$$

Since $\check{Q}$ is isomorphic to the coroot lattice of type B_ℓ, $r(\eta)$ and $e_{\eta,1}, \ldots, e_{\eta,r(\eta)}$ are as in Lemma 3.3.

Theorem 3.6. *Let $\widetilde{\chi}$ be the irreducible character of $\widetilde{\mathfrak{H}}_\ell$. Then $m(\widetilde{\chi}; q)$ is a quasi-polynomial with minimal period 2 for $\ell = 2$ and 4 for $\ell \geq 3$.*

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