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Colored hooks and poset structure of cylindric diagrams

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Abstract. Cylindric diagrams admit structures of infinite d-complete posets with natural ordering. The purpose of this paper is to provide a realization of a cylindric diagram as a subset of an affine root system of type A via colored hook lengths, and to present several characterizations of its poset structure. Furthermore, the set of finite order ideals of a cylindric diagram is described as a weak right Bruhat interval of the affine Weyl group.

Résumé. Les diagrammes cylindriques admettent des structures de posets infinis d-complets avec un ordre naturel. Le but de cet article est de fournir une réalisation d'un diagramme cylindrique en tant que sous-ensemble d'un système affine de racines de type A via des longueurs de crochets colorés, et de présenter plusieurs caractérisations de sa structure de poset. De plus, l'ensemble des idéaux d'ordre fini d'un diagramme cylindrique est décrit comme un intervalle de Bruhat droit faible du groupe de Weyl affine.

Keywords: d-complete poset, cylindric diagrams, hooks.

1 Introduction

An order filter of a cylinder $\mathbb{Z}^2/\mathbb{Z}\omega$ ($\omega \in \mathbb{Z}_{\geqq 1} \times \mathbb{Z}_{\leqq -1}$) is called a *cylindric (Young) diagram*. A diagram given as set-difference of two cylindric diagrams is called a *cylindric skew diagram* (see Fig. 2).

We note that cylindric skew diagrams have been known to parameterize a certain class of irreducible modules over the Cherednik algebras (double affine Hecke algebras) ([10, 12]) and the (degenerate) affine Hecke algebras ([1, 5]) of type A, where standard tableaux on those diagrams also appear.

Let $\omega = (m, -\ell)$ and let θ be a cylindric diagram in $\mathbb{Z}^2/\mathbb{Z}\omega$. The lattice $\mathbb{Z}^2$ admits a partial order $\leq$ defined by

$$(a, b) \leq (c, d) \iff a \geqq c \text{ and } b \geqq d,$$

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which induces a poset structure on $\mathbb{Z}^2/\mathbb{Z}\omega$ and also on θ. Together with the content map $\mathbf{c} : \theta \to \mathbb{Z}/\kappa\mathbb{Z}$, where $\mathbf{c}(a,b) = b - a \mod \kappa$ and $\kappa = \ell + m$, the cylindric digram θ is a locally finite $\mathbb{Z}/\kappa\mathbb{Z}$-colored d-complete poset in the sense of [8, 9].

Our first goal of the present paper is to describe the poset structure $(\theta, \leq)$. We briefly review a description in the classical case. Let λ be a finite Young diagram. The associated Grassmannian permutation w_λ is an element of the Weyl group of a finite root system R of type A. In [7], J. R. Stembridge has shown that the poset $(\lambda, \leq)$ is dually isomorphic to the poset $(R(w_\lambda^{-1}), \leq^{\mathrm{or}})$, where $R(w_\lambda^{-1}) := R_+ \cap w_\lambda^{-1} R_-$ and $\leq^{\mathrm{or}}$ is the ordinary order (or the standard order) defined by

$$\alpha \leq^{\mathrm{or}} \beta \iff \beta - \alpha \text{ is a sum of simple roots.}$$

Let θ be a cylindric diagram in $\mathbb{Z}^2/\mathbb{Z}\omega$. We would like to describe the poset $(\theta, \leq)$ in terms of the root system of type $A^{(1)}_{\kappa-1}$ with $\kappa = \ell + m$. A key ingredient in our approach is the *colored hook length* ([2, 4]), which is given by

$$\mathbf{h}(x) = \sum_{y \in H(x)} \alpha_{\mathbf{c}(y)} \quad (x \in \theta),$$

where $H(x)$ denotes the hook at x and α_i are simple roots. (See Section 2.4 for precise definitions.) We will show that the map $\mathbf{h}$ embeds the cylindric diagram θ into the set R_+ of positive (real) roots, and that the image $\mathbf{h}(\theta)$ is also characterized as the subset $D(\zeta_\theta)$ of R_+ consisting of those elements satisfying

$$\langle \zeta_\theta, \alpha^\vee \rangle = -1,$$

where ζ_θ is a predominant integral weight determined by θ (see Section 2.5 for details). The set $D(\zeta_\theta)$ can be thought as the inversion set $R(w_\theta)$ associated with a "semi-infinite Grassmannian permutation" w_θ.

Unlike the classical case, the ordinary order in $R(w_\theta)$ does not lead a poset isomorphism, and we need to introduce a modified order $\trianglelefteq$ in $R(w_\theta)$ by

$$\alpha \trianglelefteq \beta \iff \beta - \alpha \text{ is a sum of elements of } \Pi_\theta$$

to obtain a poset isomorphism $(\theta, \leq) \cong (R(w_\theta), \trianglelefteq)$, where Π_θ is a certain subset of the affine root system (see Section 3).

Another description of the poset θ is given by a linear extension or (reverse) standard tableau $\mathfrak{t}$ on θ, namely a bijective order preserving map $\theta \to \mathbb{Z}_{\geqq 1}$. A linear extension $\mathfrak{t} : \theta \to \mathbb{Z}_{\geqq 1}$ brings a partial order $\leq^{\mathrm{hp}}_{\mathfrak{t}}$ to $\mathbb{Z}_{\geqq 1}$ and the resulting poset is an infinite analogue of a heap [7]. In summary, we have the following:

Theorem 1.1 (Theorem 3.6, Proposition 3.10)**.** *The followings are poset isomorphisms:*

$$(\mathbb{Z}_{\geqq 1}, \leq^{\mathrm{hp}}_{\mathfrak{t}}) \xleftarrow{\mathfrak{t}} (\theta, \leq) \xrightarrow{\mathbf{h}} (R(w_\theta), \trianglelefteq).$$

We will also give several other description of the order in $R(w_\theta)$ (Propositions 3.8 and 3.12).

Another goal of this paper is to describe the poset structure $\mathcal{J}(\theta)$. For a finite Young diagram λ, it is known that the set $\mathcal{J}(\lambda)$ of order ideals of λ is isomorphic to the interval $[e, w_\lambda] = \{u \in W \mid e \preceq u \preceq w_\lambda\}$ with weak right Bruhat order $\preceq$ ([4, Proposition I]). For a cylindric diagram θ, we define a "semi-infinite weak right Bruhat interval" $[e, w_\theta)$, and we have the following:

Theorem 1.2 (Theorem 3.14)**.** *The map* $\Psi : ([e, w_\theta), \preceq) \to (\mathcal{J}(R(w_\theta)), \subset) \simeq (\mathcal{J}(\theta), \subset)$ *given by* $\Psi(w) = R(w)$ *is a poset isomorphism.*

The detailed proofs of the statements in this exteded abstract can be read in [3].

2 Cylindric diagrams

2.1 Cylindric diagrams as posets

Let $(P, \leq)$ be a poset. For $x, y \in P$, define an *interval* $[x, y]$ by

$$[x, y] = \{z \in P \mid x \leq z \leq y\}.$$

We say that y *covers* x if $[x, y] = \{x, y\}$ and $x < y$.

Definition 2.1. Let $(P, \leq)$ be a (possibly infinite) poset. A subset J of P is called an *order filter* (resp. *order ideal*) if the following condition holds:

$$x \in J,\ x \leq y \implies y \in J \qquad (\text{resp. } x \in J,\ x \geq y \implies y \in J).$$

The set of finite order ideals of P is denoted by $\mathcal{J}(P)$ and we regard it as a poset with the inclusion relation.

For $\omega \in \mathbb{Z}_{\geqq 1} \times \mathbb{Z}_{\leqq -1}$, we let $\mathbb{Z}\omega$ denote the subgroup of (the additive group) $\mathbb{Z}^2$ generated by ω, and define the cylinder $\mathcal{C}_\omega$ by

$$\mathcal{C}_\omega = \mathbb{Z}^2 / \mathbb{Z}\omega.$$

Let $\pi : \mathbb{Z}^2 \to \mathcal{C}_\omega$ be the natural projection. The cylinder $\mathcal{C}_\omega$ inherits a $\mathbb{Z}^2$-module structure via π.

Define a poset structure on $\mathbb{Z}^2$ by

$$(a, b) \leq (a', b') \iff a \geqq a' \text{ and } b \geqq b' \text{ as integers.}$$

For $x, y \in \mathcal{C}_\omega$, write $x \leq y$ if there exists $\tilde{x}, \tilde{y} \in \mathbb{Z}^2$ such that $\pi(\tilde{x}) = x$, $\pi(\tilde{y}) = y$ and $\tilde{x} \leq \tilde{y}$. It is not difficult to see the following:

Lemma 2.2. *Let $\omega \in \mathbb{Z}_{\geqq 1} \times \mathbb{Z}_{\leqq -1}$. Then the relation $\leq$ on $\mathcal{C}_\omega$ is a partial order, and the projection π is order preserving.*

In the rest of this section, we fix $\omega \in \mathbb{Z}_{\geqq 1} \times \mathbb{Z}_{\leqq -1}$.

Definition 2.3. 1. A non-trivial order filter of $\mathcal{C}_\omega$ is called a *cylindric diagram*.

2. A finite order ideal of a cylindric diagram is called a *cylindric skew diagram*.

3. A non-trivial order filter Θ of $\mathbb{Z}^2$ is called a *periodic diagram of period ω* if $\Theta + \omega = \Theta$.

Lemma 2.4. *1. For a cylindric diagram θ in $\mathcal{C}_\omega$, the inverse image $\pi^{-1}(\theta)$ is a periodic diagram of period ω.*

2. For a periodic diagram Θ of period ω, the image $\pi(\Theta)$ is a cylindric diagram in $\mathcal{C}_\omega$.

Fig. 1 indicates a periodic diagram of period $\omega = (4, -5)$. The set consisting of colored cells is a fundamental domain with respect to the action of $\mathbb{Z}\omega$, and it is in one to one correspondence with the associated cylindric diagram.

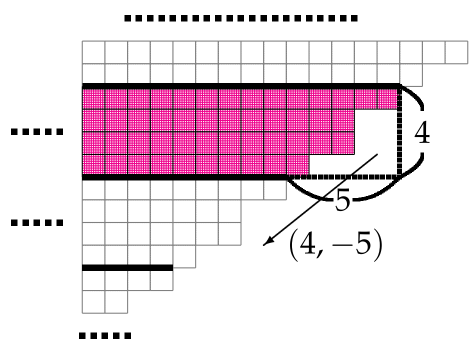

Figure 1: A periodic diagram of period $\omega = (4, -5)$.

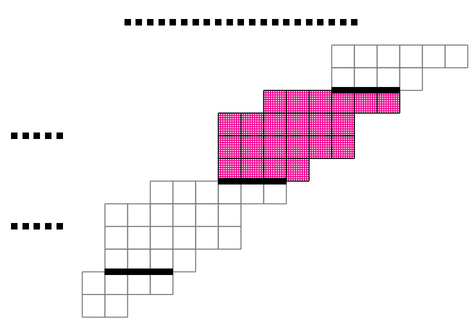

Figure 2: A cylindric skew diagram.

2.2 Content map and bottom set

Let Θ be a periodic diagram of period ω. Define the *content map*

$$\mathbf{c} : \Theta \to \mathbb{Z}$$

by $\mathbf{c}(a, b) = b - a$. Put $\kappa = |\mathbf{c}(\omega)|$. Let $\theta = \pi(\Theta)$. Since $\mathbf{c}(x + \omega) = \mathbf{c}(x) - \kappa$, the content map $\mathbf{c}$ induces the map

$$\theta \to \mathbb{Z}/\kappa\mathbb{Z},$$

which we denote by the same symbol $\mathbf{c}$. It is easy to show the following:

Proposition 2.5. *For $x, y \in \theta$, the followings hold:*

1. If $\mathbf{c}(x) - \mathbf{c}(y) \equiv 0, \pm 1 \mod \kappa$, then x and y are comparable.

2. If x is covered by y, then $\mathbf{c}(x) - \mathbf{c}(y) \equiv \pm 1 \mod \kappa$.

Remark 2.6. By Proposition 2.5, cylindric diagrams are infinite (locally finite) "$\mathbb{Z}/\kappa\mathbb{Z}$-colored d-complete posets" in the sense of [8, 9].

Let $i \in \mathbb{Z}/\kappa\mathbb{Z}$. By Proposition 2.5 (1), the inverse image $\mathbf{c}^{-1}(i)$ is non-empty totally ordered subset of θ. Let b_i denote the minimum element in $\mathbf{c}^{-1}(i)$.

Definition 2.7. Define the *bottom set* Γ of θ by

$$\Gamma = \{b_i \mid i \in \mathbb{Z}/\kappa\mathbb{Z}\}.$$

The yellowed cells indicate the elements of bottom sets in each diagram of Fig. 3

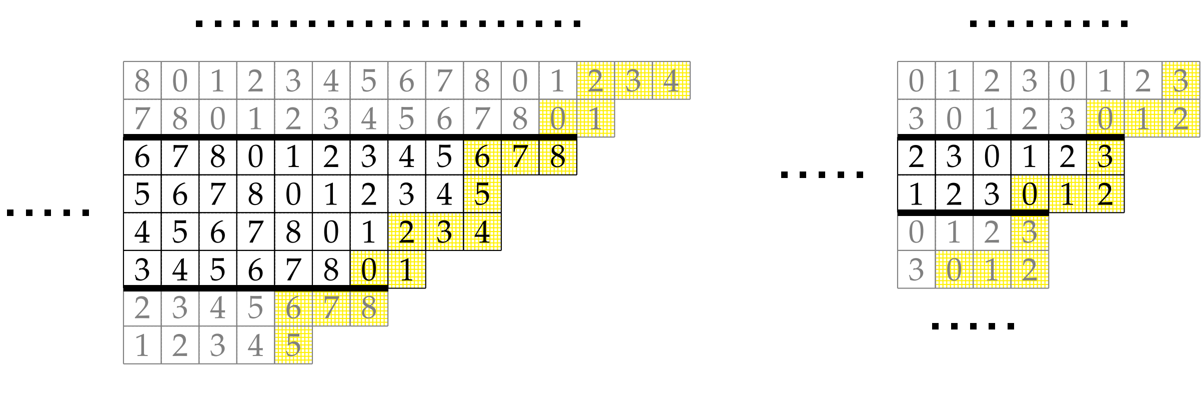

Figure 3: Contents and bottom sets

2.3 Root systems and affine Weyl groups of type $A^{(1)}_{\kappa-1}$

Let $\kappa \in \mathbb{Z}_{\geqq 2}$. In the rest, we often identify $\mathbb{Z}/\kappa\mathbb{Z}$ with $\{0, 1, \ldots, \kappa-1\}$. We use the following notations:

$$\begin{aligned}
\mathfrak{h} &: \text{the cartan subalgebra of } \widehat{\mathfrak{sl}_\kappa},\\
\mathfrak{h}^* &: \text{the dual of } \mathfrak{h},\\
\langle\ ,\ \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C} &: \text{the natrual pairing},\\
\Pi = \{\alpha_0, \alpha_1, \cdots, \alpha_{\kappa-1}\}(\subset \mathfrak{h}^*) &: \text{the set of simple roots},\\
\Pi^\vee = \{\alpha_0^\vee, \alpha_1^\vee, \cdots, \alpha_{\kappa-1}^\vee\}(\subset \mathfrak{h}) &: \text{the set of simple coroots},
\end{aligned}$$

The integers a_{ij} $(i, j \in \mathbb{Z}/\kappa\mathbb{Z})$ are defined by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ -1 & \text{if } i - j = \pm 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } \kappa \geqq 3, \qquad a_{ij} = \begin{cases} 2 & \text{if } i = j \\ -2 & \text{if } i \neq j \end{cases} \quad \text{for } \kappa = 2.$$

The pairing satisfies $\langle \alpha_j, \alpha_i^\vee \rangle = a_{ij}$. Put $\delta = \alpha_0 + \alpha_1 + \cdots + \alpha_{\kappa-1}$ which is called the null root. For $i \in \mathbb{Z}/\kappa\mathbb{Z}$, define the simple reflection $s_i \in GL(\mathfrak{h}^*)$ by

$$s_i(\zeta) = \zeta - \langle \zeta, \alpha_i^\vee \rangle \alpha_i \quad (\zeta \in \mathfrak{h}^*).$$

Define the *affine Weyl group* W of type $A^{(1)}_{\kappa-1}$ as the subgroup of $GL(\mathfrak{h}^*)$ generated by simple reflections:

$$W = \langle s_i \mid i \in \mathbb{Z}/\kappa\mathbb{Z} \rangle.$$

Define the action of W on $\mathfrak{h}$ by $s_i(h) = h - \langle \alpha_i, h \rangle \alpha_i^\vee \quad (h \in \mathfrak{h})$. The following is well-known:

Proposition 2.8. *The group W has the following fundamental relations*:

$$s_i^2 = 1, \tag{2.1}$$
$$s_i s_j = s_j s_i \quad (i - j \neq 0, \pm 1), \tag{2.2}$$
$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}. \tag{2.3}$$

We put

$$Q = \left\{ \sum_{i \in \mathbb{Z}/\kappa\mathbb{Z}} c_i \alpha_i \;\middle|\; c_i \in \mathbb{Z} \right\}, \qquad Q_+ = \left\{ \sum_{i \in \mathbb{Z}/\kappa\mathbb{Z}} c_i \alpha_i \;\middle|\; c_i \in \mathbb{Z}_{\geqq 0} \right\}.$$

The set Q is called the root lattice. Put

$$R = W\Pi \subset \mathfrak{h}^*, \qquad R^\vee = W\Pi^\vee \subset \mathfrak{h}.$$

Then R (resp. $R^\vee$) is the set of real roots (resp. coroots) and $R \sqcup \mathbb{Z}\delta$ is the affine root system. Define the set R_+ of positive real roots and the set R_- of negative real roots by

$$R_+ = R \cap Q_+ = \left\{ \sum_{i=0}^{\kappa-1} c_i\alpha_i \in R \mid c_i \in \mathbb{Z}_{\geqq 0} \right\}, \quad R_- = \left\{ \sum_{i=0}^{\kappa-1} c_i\alpha_i \in R \mid c_i \in \mathbb{Z}_{\leqq 0} \right\}.$$

For $\beta = \sum_{i=0}^{\kappa-1} k_i\alpha_i \in R$, define $\beta^\vee = \sum_{i=0}^{\kappa-1} k_i\alpha_i^\vee \in R^\vee$. Then the correspondence $\beta \mapsto \beta^\vee$ gives a bijection $R \to R^\vee$. Define the set of positive (resp. negative) coroots $R_+^\vee$ (resp. $R_-^\vee$) as the image of R_+ (resp. R_-) by this bijection.

For $w \in W$, we define the *length* $\ell(w)$ of w as the smallest r for which an expression $w = s_{i_1}s_{i_2}\cdots s_{i_r} \in W \ \ (i_j \in \mathbb{Z}/\kappa\mathbb{Z})$ exists. An expression $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ is said to be *reduced* if $\ell(w) = r$. The set $R(w) = R_+ \cap wR_-$ is called the *inversion set* of w. It is known for any reduced expression $w = s_{i_1}s_{i_2}\cdots s_{i_\ell}$ that $\ell(w) = |R(w)|$ and

$$R(w) = \{\alpha_{i_1}, s_{i_1}\alpha_{i_2}, s_{i_1}s_{i_2}\alpha_{i_3}, \ldots, s_{i_1}s_{i_2}\cdots s_{i_{\ell-1}}\alpha_{i_\ell}\}.$$

2.4 Colored hook length

In this section, we will introduce colored hook length, which is a key ingredient in this paper. In the rest of this paper, we use the following notations:

$$\alpha(x) = \alpha_{\mathbf{c}(x)}, \ \ s(x) = s_{\mathbf{c}(x)} \ \text{ for } x \in \theta.$$

Definition 2.9. For $x \in \theta$, put

$$\mathrm{Arm}(x) = \{x + (0,k) \in \theta \mid k \in \mathbb{Z}_{\geqq 1}\}, \qquad \mathrm{Leg}(x) = \{x + (k,0) \in \theta \mid k \in \mathbb{Z}_{\geqq 1}\},$$

and define

$$\mathbf{h}(x) = \alpha(x) + \sum_{y \in \mathrm{Arm}(x)} \alpha(y) + \sum_{y \in \mathrm{Leg}(x)} \alpha(y).$$

We call $\mathbf{h}(x)$ the *colored hook length* at x. It is straightforward to see that the colored hook length is a positive real root $\mathbf{h}(x) \in R_+$.

Remark 2.10. 1. For $x \in \theta$, the "multiset" $\mathrm{H}(x) := \{x\} \sqcup \mathrm{Arm}(x) \sqcup \mathrm{Leg}(x)$ is a cylindric analogue of the hook at x.

2. A conjectural hook formula concerning the number of standard tableaux on cylindric skew diagrams is proposed in [11], where the hook length at $x \in \theta$ is given by $|H(x)| = |\mathrm{Arm}(x)| + |\mathrm{Leg}(x)| + 1$.

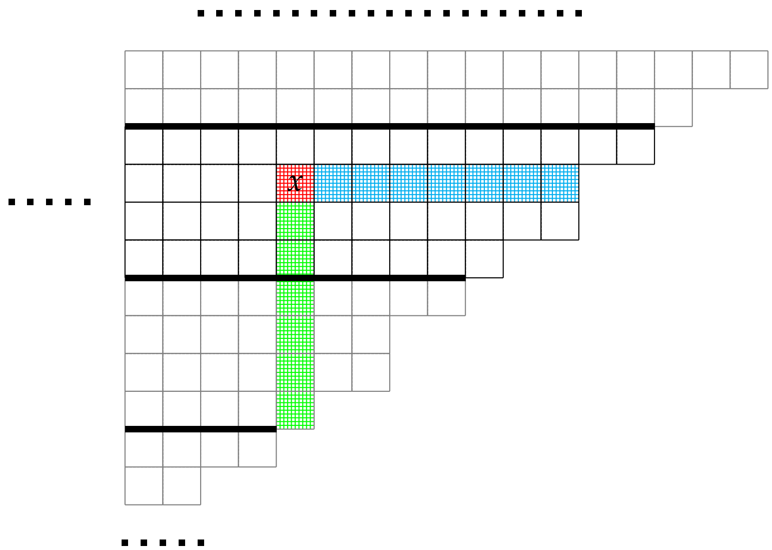

Figure 4: The sets $\mathrm{Arm}(x)$ and $\mathrm{Leg}(x)$ for x in the cylindric diagram.

2.5 Predominant weights and hooks

We take an integral weight $\zeta_\theta \in \mathfrak{h}^*$ such that

$$\langle \zeta_\theta, \alpha_i^\vee \rangle = \begin{cases} 1 & \text{if } b_i \in \Gamma_{\max} \\ -1 & \text{if } b_i \in \Gamma_{\min} \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Note that maximal and minimal elements are lined up alternatively in Γ. This implies that the weight ζ_θ is predominant, namely, $\langle \zeta_\theta, \alpha^\vee \rangle \geqq -1$ for all $\alpha^\vee \in R_+^\vee$. Define

$$D(\zeta_\theta) = \{\alpha \in R_+ \mid \langle \zeta_\theta, \alpha^\vee \rangle = -1\}.$$

Theorem 2.11 ([3, Proposition 28])**.** *The correspondence $x \mapsto \mathbf{h}(x)$ gives a bijection*

$$\mathbf{h} : \theta \to D(\zeta_\theta).$$

Definition 2.12. Let $\zeta \in \mathfrak{h}^*$ be an integral weight.

1. An element w of W is said to be ζ-*pluscule* if $\langle \zeta, \alpha^\vee \rangle = -1$ for all $\alpha \in R(w)$.

2. An element w of W is said to be ζ-*minuscule* if $\langle \zeta, \alpha^\vee \rangle = 1$ for all $\alpha \in R(w^{-1})$.

Remark 2.13. 1. An element $w \in W$ is ζ-pluscule if and only if w is $(w^{-1}\zeta)$-minuscule.

2. An element $w \in W$ is said to be *fully commutative* if any reduced expression of w can be obtained from any other by using only the relations (2.2). It is known that if w is ζ-minuscule for some integral weight ζ then w is fully commutative ([6]).

Proposition 2.14. *We symbolically put* $R(w_\theta) := \bigcup_{w:\zeta_\theta\text{-pluscule}} R(w)$*. Then we have*

$$D(\zeta_\theta) = R(w_\theta)$$

The set $R(w_\theta)$ can be thought as the "inversion set" associated with the semi-infinite word w_θ.

We denote by $\preceq$ the weak right Bruhat order, namely,

$$v \preceq w \iff \ell(v) + \ell(v^{-1}w) = \ell(w), \qquad (v, w \in W).$$

Proposition 2.15. *Let* $v, w \in W$*. If* w *is* ζ_θ*-pluscule and* $v \preceq w$*, then* v *is also* ζ_θ*-pluscule.*

We denote by $[e, w_\theta)$ the set of ζ_θ-pluscule elements. Proposition 2.15 justifies using the notation $[e, w_\theta)$.

3 Poset structure of cylindric diagrams

Recall that Q denote the root lattice: $Q = \bigoplus_{i \in \mathbb{Z}/\kappa\mathbb{Z}} \mathbb{Z}\alpha_i$.

Definition 3.1. Define the partial order $\leq^{\text{or}}$ on Q by

$$\alpha \leq^{\text{or}} \beta \iff \beta - \alpha \in Q_+ = \bigoplus_{i \in \mathbb{Z}/\kappa\mathbb{Z}} \mathbb{Z}_{\geqq 0}\alpha_i$$

The order $\leq^{\text{or}}$ is called the *ordinary order*.

The restriction of the ordinary order defines a poset structure on $R(w_\theta) = D(\zeta_\theta)$.

Let θ be a cylindric diagram in $\mathcal{C}_\omega$ with $|\omega| = \kappa$. We have introduced a poset structure on θ and also have seen that the map $\mathbf{h}$ gives a bijection between θ and $R(w_\theta)$. Remark that this is not a poset isomorphism as seen in the following example:

Example 3.2. In the cylindric diagram θ in Fig. 3 (right), the cells $x = \pi(3,3)$ and $y = \pi(2,4)$ are incomparable. On the other hand, $\mathbf{h}(x) = \delta + \alpha_2$ and $\mathbf{h}(y) = \alpha_1 + \alpha_2 + \alpha_3$, and hence $\mathbf{h}(x) - \mathbf{h}(y) = \alpha_0 + \alpha_2$. This implies $\mathbf{h}(y) \leq^{\text{or}} \mathbf{h}(x)$.

We will introduce a modified ordinary order $\trianglelefteq$, for which we will have $(\theta, \leq) \cong (R(w_\theta), \trianglelefteq)$.

Let $\Gamma = \{b_i \mid i \in \mathbb{Z}/\kappa\mathbb{Z}\}$ be the bottom set of θ, where b_i is the element such that $\mathbf{c}(b_i) = i$. Let $\Gamma_{\max}$ (resp. $\Gamma_{\min}$) denote the set of maximal (resp. minimal) elements in Γ.

Definition 3.3. Define

$$\Pi_\theta = \Pi_\theta^0 \sqcup \Pi_\theta^{\mathrm{arm}} \sqcup \Pi_\theta^{\mathrm{leg}}.$$

Here,

$$\Pi_\theta^0 = \{\alpha(x) \mid x \in \Gamma \setminus (\Gamma_{\max} \sqcup \Gamma_{\min})\},$$
$$\Pi_\theta^{\mathrm{arm}} = \left\{ \alpha(x) + \sum_{y \in \mathrm{Arm}(x)} \alpha(y) \;\middle|\; x \in \Gamma_{\max} \right\},$$
$$\Pi_\theta^{\mathrm{leg}} = \left\{ \alpha(x) + \sum_{y \in \mathrm{Leg}(x)} \alpha(y) \;\middle|\; x \in \Gamma_{\max} \right\}.$$

Note that $\Pi_\theta \subset R_+ \sqcup \mathbb{Z}_{\geqq 0}\delta$.

Example 3.4. For the cylindric diagram described in Fig. 3 (left), we have

$$\Pi_\theta^0 = \{\alpha_3, \alpha_5, \alpha_7\},$$
$$\Pi_\theta^{\mathrm{arm}} = \{\alpha_6 + \alpha_7 + \alpha_8, \alpha_2 + \alpha_3 + \alpha_4, \alpha_0 + \alpha_1\},$$
$$\Pi_\theta^{\mathrm{leg}} = \{\alpha_4 + \alpha_5 + \alpha_6, \alpha_1 + \alpha_2, \alpha_0 + \alpha_8\}.$$

Definition 3.5. Define the partial order $\trianglelefteq$ on $R(w_\theta)$ by

$$\alpha \trianglelefteq \beta \iff \beta - \alpha \in \sum_{\gamma \in \Pi_\theta} \mathbb{Z}_{\geqq 0}\gamma = \left\{ \sum_{\gamma \in \Pi_\theta} k_\gamma \gamma \;\middle|\; k_\gamma \in \mathbb{Z}_{\geqq 0} \ (\forall \gamma \in \Pi_\theta) \right\}. \tag{3.1}$$

Theorem 3.6 ([3, Theorem 47]). *The map*

$$\mathbf{h} : (\theta, \leq) \to (R(w_\theta), \trianglelefteq)$$

is a poset isomorphism.

Definition 3.7. Let $\trianglelefteq^{\mathrm{tc}}$ denote the transitive closure of the relations

$$\alpha \trianglelefteq^{\mathrm{tc}} \beta \text{ whenever } \beta - \alpha \in \Pi_\theta. \tag{3.2}$$

Proposition 3.8. *The partial order $\trianglelefteq$ on $R(w_\theta)$ coincides with $\trianglelefteq^{\mathrm{tc}}$.*

Definition 3.9. Let $\mathrm{t} : \theta \to \mathbb{Z}_{\geqq 1}$ be a standard tabuleau (or a linear extension), namely t be an order preserving bijection. Define a partial order $\preceq_{\mathrm{t}}$ on $\mathbb{Z}_{\geqq 1}$ as the transitive closure of the relations

$$a \preceq_{\mathrm{t}} b \text{ whenever } a \leqq b \text{ and either } s_{i_a} s_{i_b} = s_{i_b} s_{i_a} \text{ or } i_a = i_b.$$

where $i_k = \mathbf{c}(\mathrm{t}^{-1}(k))$ for $k \in \mathbb{Z}_{\geqq 1}$. The poset $(\mathbb{Z}_{\geqq 1}, \preceq_{\mathrm{t}})$ is called the *heap* of w_θ.

Proposition 3.10. *Let θ be a cylindric diagram and $\mathfrak{t}$ a standard tableau on θ. Then, the map $\mathfrak{t} : \theta \to \mathbb{Z}_{\geqq 1}$ gives a poset isomorphism*

$$(\theta, \leq) \cong (\mathbb{Z}_{\geqq 1}, \preceq_{\mathfrak{t}}).$$

The posets $(\mathbb{Z}_{\geqq 1}, \preceq_{\mathfrak{t}})$ are thought as semi-infinite analogue of heaps introduced by Stembridge [6]. Stembridge also introduced the heap order on the inversion sets.

We treat a slightly modified version of heap order by Nakada [2].

Definition 3.11. Define a partial order $\leq^{\mathrm{hp}}$ on $D(\zeta_\theta) = R(w_\theta)$ as the transitive closure of the relations

$$\alpha \leq^{\mathrm{hp}} \beta \text{ whenever } \alpha \leq^{\mathrm{or}} \beta \text{ and } \langle \alpha, \beta^\vee \rangle \neq 0.$$

Proposition 3.12. *The partial order $\leq^{\mathrm{hp}}$ on $D(\zeta_\theta) = R(w_\theta)$ coincides with $\trianglelefteq$.*

In summary, we get the following poset isomorphisms:

- $(\theta, \leq) \xrightarrow{\mathfrak{t}} (\mathbb{Z}_{\geqq 1}, \leq^{\mathrm{hp}}_{\mathfrak{t}})$.
- $(\theta, \leq) \xrightarrow{\mathbf{h}} (R(w_\theta), \trianglelefteq) = (R(w_\theta), \trianglelefteq^{\mathrm{tc}}) = (R(w_\theta), \leq^{\mathrm{hp}})$.

A poset isomorphism $\theta \simeq R(w_\theta)$ (Theorem 3.6) induces a poset isomorphism $\mathcal{J}(\theta) \simeq \mathcal{J}(R(w_\theta))$.

Proposition 3.13. *Let w be a ζ_θ-pluscule element. Then the inversion set $R(w)$ is a finite order ideal of $(R(w_\theta), \trianglelefteq)$.*

Theorem 3.14 ([3, Theorem 58])**.** *The map $\Psi : ([e, w_\theta), \preceq) \to (\mathcal{J}(R(w_\theta)), \subset) \simeq (\mathcal{J}(\theta), \subset)$ given by $\Psi(w) = R(w)$ is a poset isomorphism.*

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