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A charge monomial basis of the Garsia–Procesi ring

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Abstract. We construct a basis of the Garsia–Procesi ring using the catabolizability type of standard Young tableaux and the charge statistic, providing the first direct connection between the structure of the Garsia–Procesi ring and the catabolizability formula for the modified Hall–Littlewood polynomial that gives its graded Frobenius character.

Keywords: Garsia–Procesi ring, modified Hall–Littlewood polynomials, charge, descent monomials, catabolizability type

1 Introduction

The polynomial ring $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \dots, x_n]$ has a natural S_n action permuting the variables. Taking the quotient of $\mathbb{C}[\mathbf{x}]$ by the ideal generated by S_n-invariant polynomials with no constant term defines the classical **coinvariant ring** R_{1^n}. As graded algebras with S_n actions, the coinvariant ring is isomorphic to the cohomology ring of the complex flag variety $\mathcal{F}_n$. The Hilbert series $\mathrm{Hilb}_q(R_{1^n})$ is equal to $[n]_q!$.

The **Garsia–Stanton descent basis** of R_{1^n} is given by the following set of monomials indexed by permutations:

$$\{g_w(\mathbf{x}) = \prod_{i \in \mathrm{Des}(w)} x_{w_1} \cdots x_{w_i} \mid w \in S_n\}. \tag{1.1}$$

where $\mathrm{Des}(w) := \{i \mid w_i > w_{i+1}\}$ denotes the descent set of w. We refer to these monomials $g_w(\mathbf{x})$ as **descent monomials**. The descent monomials correspond to the permutation statistic maj, defined to be $\mathrm{maj}(w) = \sum_{i \in \mathrm{Des}(w)} i$; in particular, we can see that $\deg(g_w(\mathbf{x})) = \mathrm{maj}(w)$. Using the fact that maj is Mahonian, we have the following equality:

$$\sum_{w \in S_n} q^{\deg(g_w(\mathbf{x}))} = \sum_{w \in S_n} q^{\mathrm{maj}(w)} = [n]_q!.$$

From this, it is evident that the number of descent monomials of degree d matches the coefficient of q^d in $[n]_q!$. This gives a combinatorial explanation of $\mathrm{Hilb}_q(R_{1^n}) = [n]_q!$.

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For $\mu \vdash n$, the **Garsia–Procesi ring** R_μ is a quotient of R_{1^n} that corresponds to the cohomology ring of the Springer fiber $\mathcal{F}_\mu \subset \mathcal{F}_n$. The concrete presentation of R_μ as a quotient of $\mathbb{C}[\mathbf{x}]$ is due to De Concini–Procesi [4] and Tanisaki [14]. This quotient is also a graded S_n representation under the same action, where the graded Frobenius character of R_μ is given by the **modified Hall–Littlewood polynomial** $\widetilde{H}_\mu[\mathbf{x}; q]$ [9]. One combinatorial formula for $\widetilde{H}_\mu[X; q]$, due to Lascoux [10] and Butler [2], is:

$$\widetilde{H}_\mu[X;q] = \sum_{\substack{T \in \mathrm{SYT}_n \\ \mathrm{ctype}(T) \trianglerighteq \mu}} q^{\mathrm{cocharge}(T)} s_{\mathrm{shape}(T)}(X). \tag{1.2}$$

The sum is over all standard Young tableaux T satisfying $\mathrm{ctype}(T) \trianglerighteq \mu$, where $\trianglerighteq$ denotes the dominance order on partitions and $\mathrm{ctype}(T)$ is a partition associated to T called its catabolizability type [10].

The question of finding a subset of the Garsia–Stanton descent monomials that is a monomial basis of R_μ was unresolved until recently, when Carlsson–Chou gave a construction using shuffles of descent compositions [3]. However, this construction does not give a direct combinatorial explanation for the Hilbert series of R_μ analogous to how the descent monomials do for the coinvariant ring.

Our main contribution is a construction of a monomial basis of R_μ, consisting of descent monomials, using a catabolizability type condition on standard Young tableaux (Theorem 4.1.) Our basis coincides with that of Carlsson–Chou, though the two different constructions are independent. We make use of this equality to show that our set is indeed a basis. Our construction provides a natural way of describing this set of monomials in light of (1.2). The description of the basis is compatible with the Hilbert series $\mathrm{Hilb}_q(R_\mu)$ in the same way that the descent basis is compatible with $\mathrm{Hilb}_q(R_{1^n})$. We further highlight the connection between our basis and (1.2) by using our construction to give an elementary proof of the fact that $\mathrm{Frob}_q(R_\mu) = \widetilde{H}_\mu[X; q]$ (Proposition 1, Corollary 3) directly from (1.2) using just the ungraded Frobenius character of R_μ, without relying on other identities regarding $\widetilde{H}_\mu[X; q]$. Full details of this work are in [8].

2 Background

For any word $w = w_1 w_2 \dots w_n$, we write $\mathrm{rev}(w) := w_n w_{n-1} \dots w_1$ for the reverse word. For two words $z^{(1)}, z^{(2)}$ of length l_1, l_2 we define $\mathrm{Sh}(z^{(1)}, z^{(2)})$ to be the collection of all words u of length $(l_1 + l_2)$ such that $z^{(1)}$ and $z^{(2)}$ are two disjoint subwords of u. An element u of $\mathrm{Sh}(z^{(1)}, z^{(2)})$ is a **shuffle** of $z^{(1)}$ and $z^{(2)}$. We can extend this definition and let $\mathrm{Sh}(z^{(1)}, \dots, z^{(l)})$ denote the set of all shuffles of $z^{(1)}, \dots, z^{(l)}$.

Throughout, we use French notation for Young diagrams, meaning that the first part corresponds to the bottom row. We denote the transpose of a partition λ by λ^t.

The **dominance** ordering on partitions, denoted $\trianglerighteq$, is defined by $\mu \trianglerighteq \lambda \Leftrightarrow \mu_1 + \cdots \mu_k \geq \lambda_1 + \cdots \lambda_k$ for all k. It is well known that $\mu \trianglerighteq \lambda \quad \Leftrightarrow \quad \mu^t \trianglelefteq \lambda^t$. If we move a box of λ to a lower row so that the resulting shape μ is a partition, we have $\mu \trianglerighteq \lambda$.

For any partition λ, we denote the set of **standard Young tableaux** of shape λ by $\mathrm{SYT}(\lambda)$. We denote the set of standard Young tableaux of all shapes of size n by SYT_n. For $T \in \mathrm{SYT}(\lambda)$, we let $T^t \in \mathrm{SYT}(\lambda^t)$ denote the filling of λ^t we get by swapping the rows and columns of T. We define the descent set of T to be $\mathrm{Des}(T) = \{i \mid i \text{ appears below } i + 1 \text{ in } T\}$.

For a **semistandard Young tableau** T of shape λ, we say the **weight** of T is the tuple $(m_1, m_2, \dots)$ where m_i is the number of times i appears in T. For any tableau T, we denote the shape of T by $\mathrm{shape}(T)$ and the row reading word of a tableau T by $\mathrm{rw}(T)$.

The **RSK (Robinson–Schensted–Knuth) correspondence** gives a bijection from permutations $w \in S_n$ to pairs $(P(w), Q(w))$ of standard Young tableau of size n of the same shape. We say $P(w)$ is the **insertion tableau** and $Q(w)$ is the **recording tableau** of w. We note that for any given standard tableau T of shape λ, the number of permutations w with $P(w) = T$ is $|\mathrm{SYT}(\mathrm{shape}(T))|$, since we have one such permutation for each possible choice of $Q(w)$. We recall the following proprety of RSK that we use throughout this paper: for a more detailed exposition of RSK, see [13, Chapter 7].

Proposition 1 ([13, Corollary A1.2.11]). *For any $w \in S_n$, we have $P(\mathrm{rev}(w)) = (P(w))^t$.*

2.1 Symmetric functions and representation theory of S_n

We refer to [13, Chapter 7] or [12] for background on symmetric functions or the representation theory of the symmetric group. We follow Macdonald's notation [12] for the monomial symmetric functions m_λ, the elementary symmetric functions e_λ, the complete (homogeneous) symmetric functions h_λ, and the Schur functions s_λ.

We denote the **Frobenius character** of an S_n representation V by $\mathrm{Frob}(V)$. Recall that $\mathrm{Frob}(V_\lambda) = s_\lambda$, where V_λ is the irreducible S_n representation indexed by λ. For a graded vector space $V = \oplus_{d \geq 0} V_d$, the **graded Frobenius character** is $\mathrm{Frob}_q(V) = \sum_{d \geq 0} q^d \,\mathrm{Frob}(V_d)$.

For any S_n representation V, we can recover $\mathrm{Hilb}_q(V)$ from $\mathrm{Frob}_q(V)$ by the relation $\mathrm{Hilb}_q(V) = \langle h_{1^n}, \mathrm{Frob}_q(V)\rangle$ where $\langle -, - \rangle$ denotes the Hall inner product on symmetric functions. We can also determine $\mathrm{Frob}_q(V)$ using the Hilbert series of certain subspaces of V. For the Young subgroup $S_\gamma = S_{\gamma_1} \times \cdots \times S_{\gamma_l} \subset S_n$ indexed by $\gamma \vdash n$, we define $N_\gamma = \sum_{\sigma \in S_\gamma} \mathrm{sgn}(\sigma)\sigma$ to be the antisymmetrizer with respect to S_γ. For any $\mathbb{C}S_n$-module V, the vector space $N_\gamma V$ is the subspace of elements of V that are antisymmetric with respect to S_γ. Let $\mathcal{E} \uparrow_{S_\gamma}^{S_n}$ denote the induction of the sign representation $\mathcal{E}$ of S_γ to S_n. Using Frobenius reciprocity (see [5, Chapter 3.3]) and the fact that $\mathrm{Frob}(\mathcal{E} \uparrow_{S_\gamma}^{S_n}) = e_\gamma$, we get the following result:

Proposition 2. *For any graded* $\mathbb{C}S_n$*-module* $V = \oplus_{d\geq 0} V_d$ *and Young subgroup* $S_\gamma \subset S_n$*, we have* $\mathrm{Hilb}_q(N_\gamma V) = \langle e_\gamma, \mathrm{Frob}_q(V)\rangle$.

From this, we can see that $\mathrm{Frob}_q(V)$ is uniquely determined by $\mathrm{Hilb}_q(N_\gamma V)$ for all partitions $\gamma \vdash n$. We use this result in Section 5 in our proof of $\mathrm{Frob}_q(R_\mu) = \widetilde{H}_\mu[X;q]$.

2.2 Charge and cocharge

Charge and cocharge are statistics on tableaux that arise in the study of Hall–Littlewood polynomials. The **modified Hall–Littlewood polynomial** $\widetilde{H}_\mu[X;q]$ is given by the formula $\widetilde{H}_\mu[X;q] = \sum_\lambda \widetilde{K}_{\lambda,\mu}(q)s_\lambda(X)$ where the coefficients $\widetilde{K}_{\lambda,\mu}(q)$ can be expressed using the statistic **cocharge** on SYT:

$$\widetilde{K}_{\lambda,\mu}(q) = \sum_{\substack{T\in\mathrm{SYT}(\lambda)\\ \mathrm{ctype}(T)\trianglerighteq\mu}} q^{\mathrm{cocharge}(T)}. \tag{2.1}$$

We define $\mathrm{ctype}(T)$ in Section 2.3. Note that (2.1) is equivalent to the more familiar expression for $\widetilde{K}_{\lambda,\mu}(q)$ with respect to semistandard Young tableaux under a cocharge preserving bijection from $\{T \in \mathrm{SYT}(\lambda) \mid \mathrm{ctype}(T) \trianglerighteq \mu\}$ to $\{S \in \mathrm{SSYT}(\lambda) \mid \mathrm{weight}(S) = \mu\}$, due to Lascoux [10]. From the description using semistandard Young tableaux, we can see that $\widetilde{K}_{\lambda,\mu}(1) = K_{\lambda,\mu}$, the Kostka number counting semistandard tableaux of shape λ, weight μ. From this, we can see that $\widetilde{H}_\mu[X;1] = h_\mu(X)$.

We first define these statistics on the level of permutations and then extend it to SYT_n. For $w \in S_n$, we define $\mathrm{charge}(w) = \mathrm{maj}(\mathrm{rev}(w^{-1}))$ and $\mathrm{cocharge}(w) = \binom{n}{2} - \mathrm{charge}(w)$. Lascoux–Schützenberger [11] proved that charge is the unique statistic on words that satisfies a set of properties, one of them being the following:

Theorem 2.3 (Lascoux–Schützenberger [11])**.** *For* $w, w' \in S_n$ *such that* $P(w) = P(w')$*, we have* $\mathit{charge}(w) = \mathit{charge}(w')$.

From this, we can define charge (resp. cocharge) on standard Young tableau T by setting $\mathrm{charge}(T) = \mathrm{charge}(\mathrm{rw}(T))$ (resp. $\mathrm{cocharge}(T) = \mathrm{cocharge}(\mathrm{rw}(T))$) and always have $\mathrm{charge}(P(w)) = \mathrm{charge}(w)$ for any $w \in S_n$.

Explicitly, we can compute $\mathrm{charge}(w)$ by assigning labels to each letter in w. The **charge word** of w (denoted $\mathrm{c}(w)$) is a word of length n consisting of the charge labelling of w, which we define in the following way. We label 1 of w with 0. We proceed by labelling the numbers in increasing order: if we label i with a k, then we label $(i+1)$ with a k if it is to the left of i. We label $(i+1)$ with a $(k+1)$ if it is to the right of i. We compute **charge**(w) by taking the sum of the letters in $\mathrm{c}(w)$. Using the algorithm for computing charge, we obtain the following lemma about charge words, which we make use of in Section 4 when proving that the set of monomials we construct is indeed a basis of R_μ.

Lemma 4. *For $w \in S_n$ and nonnegative integer x where $x = \mathrm{c}(w)_i$ for some i, any word in* $\mathrm{Sh}(\mathrm{c}(w), x)$ *is a charge word as well.*

Example 5. Consider $w = 45132$. The corresponding charge word is $\mathrm{c}(w) = 12011$, thus $\mathrm{charge}(w) = 5$. If $x = 2$, we can see that any word in $\mathrm{Sh}(12011, 2)$ is a charge word. For example, $120\underline{2}11 = \mathrm{c}(461532)$.

We use these charge words to construct the monomials that appear in our basis. To each $w \in S_n$, we associate a monomial $\mathbf{x}^{\mathrm{c}(w)}$, where the exponents are given by the charge word of w. We refer to this monomial as the **charge monomial** of w.

Example 6. For $w = 45132$, the corresponding charge monomial is $\mathbf{x}^{\mathrm{c}(w)} = x_1 x_2^2 x_4 x_5$.

We can see that charge monomials of S_n are exactly the descent monomials given in (1.1). Let D_n denote the set of exponents of descent monomials. We refer to these words as **descent words.**

Example 7. Using (1.1), we can compute $g_{321}(\mathbf{x}) = (x_3)(x_3 x_2) = x_2 x_3^2$. Thus $012 \in D_3$. In this way, we can compute $D_3 = \{012, 011, 101, 001, 010, 000\}$.

Since the definition of charge follows from maj, it is clear from definition that $\mathbf{x}^{\mathrm{c}(w)} = g_\sigma(\mathbf{x})$ for any $w \in S_n$, where $\sigma = \mathrm{rev}(w^{-1})$. From this, we get the following fact:

Lemma 8. *We have $D_n = \{c(w) \mid w \in S_n\}$.*

We define the **cocharge** word $\mathrm{cc}(w)$ of permutation w by $\mathrm{cc}(w) = \mathrm{rev}(\mathrm{c}(\mathrm{rev}(w)))$ We can see that $\mathrm{cc}(w)$ can be computed by the same algorithm we use to compute the charge word, except with left and right exchanged. We compute **cocharge**(w) by taking the sum of the letters in $\mathrm{cc}(w)$. We make the following observation from Proposition 1 and the relation between cocharge, which we use to switch between the two statistics when describing $\widetilde{H}_\mu[X; q]$.

Proposition 9. *If T is a standard Young tableau, then charge(T) = cocharge(T^t).*

2.3 Catabolism and catabolizability

To each standard Young tableau T, we associate a partition called its **catabolizability type** $\mathrm{ctype}(T)$. Originally introduced by Lascoux [10], this partition keeps track of a sequence of catabolisms to apply to T in order to produce a one row tableau.

In general, many things are unknown about catabolism and catabolizability, though the operation of catabolism itself is very easy to compute. However, there is a concrete algorithm, due to Blasiak [1], called the **catabolism insertion algorithm**, which computes the catabolizability type of a permutation w, where we define $\mathrm{ctype}(w) := \mathrm{ctype}(P(w))$.

We define a modification of the catabolism insertion to use as our main tool in showing that the set we define in Section 4 is indeed a basis of the Garsia–Procesi rings. For this extended abstract, we reverse our conventions from that used in the full version of this work [8] for simplicity. We first define the following function:

Definition 10. Consider a tuple (i, x, T) consisting of an integer $i \in [n]$, a word x (with potentially empty spots), and a filling T of partition shape. We construct a new word x' and filling T' from (x, i, T) in the following way:

- If x_i is empty, set $T' = T, x' = x$.
- If not, set T' to be the filling we obtain by adding a box containing i to row $(x_i + 1)$ of T if the resulting shape is a partition. We create x' by replacing x_i with an empty spot.
- If we cannot add a box to row $(x_i + 1)$ of T to make a partition shape, we set $T' = T$. We obtain x' from x by replacing x_i with $(x_i + 1)$.

We define the function f on such tuples by $f(i, x, T) = (i + 1, x', T')$ if $i < n$ and $f(n, x, T) = (1, x', T')$.

Algorithm 11. Let $(1, \mathrm{c}(w), \emptyset)$ be the initial input, where $w \in S_n$. Apply f repeatedly to this tuple until it results in the tuple $(j, \emptyset, T_w)$ for some index j, where T_w is a standard filling (but not necessarily a SYT) of a partition shape.

We illustrate this algorithm in Example 15. Note that this algorithm may not terminate if the word in the initial tuple is not a charge word. For example: if $z = 20$, the second letter will never be deleted, since we cannot add a box to the partition (1) in rows 2 or higher. The shape of the resulting filling T_w gives us the catabolizability type of a certain permutation.

Proposition 12 (Blasiak[1])**.** *We have* $\mathrm{ctype}(\mathrm{rev}(w)) = \mathrm{shape}(T_w)$*, where* T_w *is the filling produced by Algorithm 11.*

The original algorithm of Blasiak produced a partition shape with no filling. We choose to fill the shape in order to construct disjoint subwords of $\mathrm{c}(w)$ using the entries in the columns of T_w. We make use of these subwords when proving the main result in Section 4.

We now describe these subwords. Consider $w \in S_n$ where $\mathrm{shape}(T_w) = \lambda$. For a column j of λ, define $\mathrm{c}(w)^{(j)}$ to be the subword of $\mathrm{c}(w)$ consisting of the entries $\mathrm{c}(w)_k$ where k appears in column j of T_w. We give examples of this construction in Example 15. Using Lemma 4, we can prove the following proposition.

Proposition 13. *For* $w \in S_n$ *where* $\mathrm{shape}(T_w) = \lambda$*, consider* j *such that* $1 \le j \le \lambda_1$*. Then* $\mathrm{c}(w)^{(j)} = \mathrm{c}(\sigma)$ *for some* $\sigma \in S_r$*.*

From this, we can see that each of the subwords of $\mathrm{c}(w)$ corresponding to columns of T_w are themselves charge words. This observation will be essential when proving the main result in Section 4.

Remark 14. The fact that $\mathrm{c}(w)^{(j)}$ is a charge word is not obvious a priori, since it is not true that any subword of a charge word is a charge word. For example, consider $w = 12$, which has $\mathrm{c}(w) = 01$. Though 1 is a subword of $\mathrm{c}(w)$, it is not a charge word.

Example 15. Consider $w = 52143$, with $\mathrm{c}(w) = 10011$. The position of the word that we are considering at each step is underlined. Our initial input is $(1, \underline{\mathbf{1}}0011, \varnothing)$. We can see that at the first step, the resulting tuple is $(2, 20011, \varnothing)$, since we cannot add a box to row 2 of the empty partition. We proceed through the rest of the word, where we underline x_i when considering the tuple (i, x, T):

$$\begin{gathered}(2,\ 2\,\underline{\mathbf{0}}\,0\,1\,1,\ \varnothing) \xrightarrow{f} (3,\ 2\,\textvisiblespace\,\underline{\mathbf{0}}\,1\,1,\ \begin{ytableau} 2 \end{ytableau}) \xrightarrow{f} (4,\ 2\,\textvisiblespace\,\textvisiblespace\,\underline{\mathbf{1}}\,1,\ \begin{ytableau} 2 & 3 \end{ytableau}) \\ \xrightarrow{f} (5,\ 2\,\textvisiblespace\,\textvisiblespace\,\textvisiblespace\,\underline{\mathbf{1}},\ \begin{ytableau} 4 \\ 2 & 3 \end{ytableau}) \xrightarrow{f} (1,\ \underline{\mathbf{2}}\,\textvisiblespace\,\textvisiblespace\,\textvisiblespace\,\textvisiblespace,\ \begin{ytableau} 4 & 5 \\ 2 & 3 \end{ytableau}) \xrightarrow{f} (2,\ \textvisiblespace\,\textvisiblespace\,\textvisiblespace\,\textvisiblespace\,\textvisiblespace,\ \begin{ytableau} 1 \\ 4 & 5 \\ 2 & 3 \end{ytableau})\end{gathered}$$

Hence we have $T_{52143} = \begin{ytableau} 1 \\ 4 & 5 \\ 2 & 3 \end{ytableau}$, $\mathrm{ctype}(34125) = \mathrm{ctype}(\mathrm{rev}(52143)) = (2, 2, 1)$. In this case, we see that the subwords corresponding to the columns of T_w are $\mathrm{c}(w)^{(1)} = \mathrm{c}(w)_1\,\mathrm{c}(w)_2\,\mathrm{c}(w)_4 = 101 = \mathrm{c}(312)$ and $\mathrm{c}(w)^{(2)} = 01 = \mathrm{c}(12)$.

3 Descent basis of Carlsson–Chou

We now review the construction of the Carlsson–Chou descent basis.

For $\mu = (\mu_1, \ldots, \mu_l) \vdash n$, define $\mathcal{D}_\mu$ to be $\mathcal{D}_\mu = \bigcup_{(z^{(1)},\ldots,z^{(l)})} \mathrm{Sh}(z^{(1)}, \ldots, z^{(l)})$ where $(z^{(1)}, \ldots, z^{(l)})$ ranges over all l-tuples in $\mathcal{D}_{\mu_1} \times \cdots \times \mathcal{D}_{\mu_l}$.

Theorem 3.1 (Carlsson–Chou [3])**.** *The set $\mathbf{x}^{\mathcal{D}_\mu} = \{\mathbf{x}^\alpha \mid \alpha \in \mathcal{D}_\mu\}$ is a monomial basis of R_{μ^t}.*

Example 2. Let $\mu = (3, 1)$. Since $D_3 = \{012, 011, 101, 001, 010, 000\}$ and $D_1 = \{0\}$, we have $\mathcal{D}_{3,1} = \{0012,\ 0102,\ 0120,\ 0011,\ 0101,\ 1001,\ 1010,\ 0110,\ 0001,\ 0010,\ 0100,\ 0000\}$. The corresponding set of monomials, given below, is a basis of $R_{2,1,1}$ since $(3, 1)^t = (2, 1, 1)$:

$$\{x_3x_4^2,\ x_2x_4^2,\ x_2x_3^2,\ x_3x_4,\ x_2x_4,\ x_1x_4,\ x_1x_3,\ x_2x_3,\ x_4,\ x_3,\ x_2,\ 1\}.$$

Carlsson–Chou show that their basis is a subset of the descent basis. Though the description of the basis is combinatorial, it lacks nice combinatorial properties that the descent basis had. For one, it is not obvious that $|\mathcal{D}_\mu| = \dim(R_{\mu^t})$, since multiple shuffles can correspond to the same descent word. For example, $\mathbf{0}101 \in \mathrm{Sh}(101, \mathbf{0})$ and $01\mathbf{0}1 \in \mathrm{Sh}(011, \mathbf{0})$ both correspond to the same element in $\mathcal{D}_{3,1}$. It is also not obvious how to see directly for which $w \in S_n$ we have $g_w(\mathbf{x}) \in \mathbf{x}^{\mathcal{D}_\mu}$ without computing $\mathcal{D}_\mu$.

This construction also fails to provide direct connections to the $q = 0$ specialization of the modified Macdonald polynomials $\widetilde{H}_\mu[X; q, t]$. Since $\widetilde{H}_\mu[X; 0, t]$ recovers the modified Hall–Littlewood polynomials, one would expect there to be a nice bijection between the monomials $\mathbf{x}^{\mathcal{D}_\mu}$ and the terms that arise in the combinatorial formula due to Haglund–Haiman–Loehr [7]. However, Carlsson–Chou state that there does not seem to be a "reasonable" weight-preserving bijection between the indexing sets of the two.

4 Charge monomial basis of R_{μ^t}

Our main contribution is that we construct sets of charge monomials that are monomial bases of Garsia–Procesi rings. Our construction involves charge words of permutations whose insertion tableaux satisfy a catabolizability condition.

Theorem 4.1. *Let $\mathcal{C}_\mu := \{\mathrm{c}(w) \mid w \in S_n,\ \mathrm{ctype}(P(w)^t) \trianglerighteq \mu^t\}$. The set $\mathbf{x}^{\mathcal{C}_\mu} = \{\mathbf{x}^\alpha \mid \alpha \in \mathcal{C}_\mu\}$ is a monomial basis of R_{μ^t}. In fact, it coincides with the basis $\mathbf{x}^{\mathcal{D}_\mu}$ given by Carlsson–Chou [3], ie., $\mathcal{C}_\mu = \mathcal{D}_\mu$.*

From Lemma 8, we know $\mathcal{C}_\mu \subset D_n$, thus this set of monomials is indeed a subset of the descent basis of the coinvariant ring. Though the two sets coincide, the definitions of $\mathcal{C}_\mu$ and $\mathcal{D}_\mu$ are independent, which we can see through the difference in the combinatorics used in the constructions.

Example 2. Consider $\mu = (3, 1)$. There are five standard tableaux S such that $\mathrm{ctype}(S^t) \trianglerighteq (2, 1, 1) = \mu^t$. We list them in Figure 1 along with all words w such that $P(w) = S$ and their charge monomials. The resulting set of charge monomials is

$$\{x_3x_4^2,\ x_2x_4^2,\ x_2x_3^2,\ x_3x_4,\ x_2x_4,\ x_1x_4,\ x_1x_3,\ x_2x_3,\ x_4,\ x_3,\ x_2,\ 1\},$$

which is the same as in Example 2.

Before we prove Theorem 4.1, we point out some key connections between our construction of $\mathbf{x}^{\mathcal{C}_\mu}$ and the combinatorics of $\widetilde{H}_\mu[X; q]$. Using the combinatorial formula (1.2) for the modified Hall–Littlewood polynomials and Proposition 9, we obtain the following expression for $\widetilde{H}_{\mu^t}[X; q]$ with respect to charge:

$$\widetilde{H}_{\mu^t}[X; q] = \sum_{\substack{T \in \mathrm{SYT}_n \\ \mathrm{ctype}(T^t) \trianglerighteq \mu^t}} q^{\mathrm{charge}(T)} s_{\mathrm{shape}(T^t)}(X). \tag{4.1}$$

S	$\{w \mid P(w) = S\}$	$\{x^{\mathrm{c}(w)} \mid P(w) = S\}$
$\begin{array}{lll} 2 & & \\ 1 & 3 & 4 \end{array}$	$\{2134,\ 2314,\ 2341\}$	$\{x_3x_4^2,\ x_2x_4^2,\ x_2x_3^2\}$
$\begin{array}{ll} 2 & 4 \\ 1 & 3 \end{array}$	$\{2143\ ,\ 2413\}$	$\{x_3x_4,\ x_2x_4\}$
$\begin{array}{ll} 4 & \\ 2 & \\ 1 & 3 \end{array}$	$\{4213,\ 4231,\ 2431\}$	$\{x_1x_4,\ x_1x_3,\ x_2x_3\}$
$\begin{array}{ll} 3 & \\ 2 & \\ 1 & 4 \end{array}$	$\{3214,\ 3241\ ,\ 3421\}$	$\{x_4,\ x_3,\ x_2\}$
$\begin{array}{l} 4 \\ 3 \\ 2 \\ 1 \end{array}$	$\{4321\}$	$\{1\}$

Figure 1: Charge monomials $\mathbf{x}^{\mathrm{c}(w)}$ for $w \in S_n$ with $\mathrm{ctype}(P(w)^t) \trianglerighteq (2,1,1)$

Using (4.1) we can express $\mathrm{Hilb}_q(R_{\mu^t})$ as a sum over certain permutations:

$$\mathrm{Hilb}_q(R_{\mu^t}) = \sum_{\substack{w \in S_n \\ \mathrm{ctype}(P(w)^t) \trianglerighteq \mu^t}} q^{\mathrm{charge}(w)}. \tag{4.2}$$

By our definition of $\mathcal{C}_\mu$, the number of monomials in $\mathbf{x}^{\mathcal{C}_\mu}$ of degree d is exactly the number of permutations that satisfy $\mathrm{ctype}(P(w)^t) \trianglerighteq \mu^t$ and $\mathrm{charge}(w) = d$. Comparing this observation with (4.2), it is evident $\mathbf{x}^{\mathcal{C}_\mu}$ has the correct cardinality in each degree to be a basis of R_{μ^t}. This is analogous to the result that we had for the descent basis and the coinvariant ring that we discussed in Section 1. This is also in contrast to the Carlsson–Chou construction where it was nontrivial to show the ungraded version of this equality ($|\mathcal{D}_\mu| = \dim(R_{\mu^t})$). Our construction also gives a direct way to determine for which $w \in S_n$ we have $g_w(\mathbf{x}) \in \mathbf{x}^{\mathcal{C}_\mu}$. In particular, we have $g_w(\mathbf{x}) \in \mathbf{x}^{\mathcal{C}_\mu}$ for $w = \mathrm{rev}(\sigma^{-1})$ where $\mathrm{ctype}(P(\sigma)^t) \trianglerighteq \mu^t$.

This construction also highlights a natural connection to the $q = 0$ specialization of the modified Macdonald polynomials $\widetilde{H}_\mu[X;q;t]$ that was difficult to see from the Carlsson–Chou construction. In [7], the authors give a bijection between the fillings τ of μ that satisfy $\mathrm{inv}(\tau) = 0$, which index the terms in the combinatorial formula of $\widetilde{H}_\mu[X;0;t]$ and pairs (S,T) where $S \in \mathrm{SSYT}, T \in \mathrm{SYT}$ are tableaux of the same shape. We can easily extend this bijection to $\mathcal{C}_\mu$ using Lascoux's cocharge preserving standardization.

We prove Theorem 4.1 by showing that $\mathcal{C}_\mu = \mathcal{D}_\mu$, which implies that the set $\mathbf{x}^{\mathcal{C}_\mu}$

is a basis of R_{μ^t}. Since we know $|\mathcal{C}_\mu| = \dim(R_{\mu^t})$ from the observation above and $\dim(R_{\mu^t}) = |\mathcal{D}_\mu|$ from the results of Carlsson–Chou, it suffices to show that $\mathcal{C}_\mu \subset \mathcal{D}_\mu$. This containment is almost immediate using the algorithm we defined in Section 2.3.

Proposition 3. *Let μ be a partition of n of length l and w be a permutation of n such that* $\mathrm{ctype}(\mathrm{rev}(w)) \trianglerighteq \mu^t$. *Then* $\mathrm{c}(w) \in \mathcal{D}_\mu$.

If $\mathrm{ctype}(\mathrm{rev}(w)) = \mu^t$, this follows from Proposition 13, which gives a decomposition of $\mathrm{c}(w)$ into l disjoint subwords $\mathrm{c}(w)^{(1)}, \ldots, \mathrm{c}(w)^{(l)}$. Note that $\mathrm{c}(w)^{(j)} = \mathrm{c}(u)$ for some $u \in S_{\mu_j}$, hence $\mathrm{c}(w)^{(j)} \in D_{\mu_j}$.

Example 4. Let $\mu = (3, 2)$. Consider $w = 52143$ from Example 15 with $\mathrm{c}(w) = 10011$ and $\mathrm{ctype}(\mathrm{rev}(w)) = (2, 2, 1) = \mu^t$. Using the subwords coming from the columns of T_w, we can write $\mathrm{c}(w) = 10\mathbf{0}\mathbf{1}1$ as a shuffle of $\mathrm{c}(w)^{(1)} = 101$ and $\mathrm{c}(w)^{(2)} = 01$ where the bold letters indicate those in $\mathrm{c}(w)^{(2)}$. Thus $\mathrm{c}(w) \in \mathrm{Sh}(101, 01)$ where $(101, 01) \in D_3 \times D_2$ since $101 = \mathrm{c}(312)$ and $01 = \mathrm{c}(12)$.

It remains to consider $\mathrm{ctype}(\mathrm{rev}(w)) \triangleright \mu^t$. Recall that for any partition $\lambda \triangleright \mu^t$, we can go from shape λ to μ^t by moving boxes to higher rows, which results in lowering the shape in the dominance ordering. Using this idea and Lemma 4, we define an algorithm that takes the filling T_w of shape λ and constructs a filling $T_w^{\mu^t}$ of shape μ^t by moving certain boxes to higher rows. By definition of the algorithm, the resulting columns of $T_w^{\mu^t}$ correspond to disjoint subwords of $\mathrm{c}(w)$ that are each charge words themselves as well. For details, see Section 4 of [8].

Theorem 4.1 follows from Propositions 1 and 3. Hence we have that our set $\mathbf{x}^{\mathcal{C}_\mu}$ is indeed a basis of R_{μ^t}. Note that this proof does not depend on the fact that $\mathrm{Frob}_q(R_\mu) = \widetilde{H}_\mu[X; q]$. It suffices to know that the ungraded Frobenius character of R_μ is given by h_μ, which can easily be identified directly from the structure of R_μ.

5 Antisymmetric part

We now use the construction of our basis to give a direct, elementary proof of the fact that $\mathrm{Frob}_q(R_{\mu^t}) = \widetilde{H}_{\mu^t}[X; q]$. We do this by showing that for any $\gamma = (\gamma_1, \ldots, \gamma_l) \vdash n$, we have $\mathrm{Hilb}_q(N_\gamma R_{\mu^t}) = \langle e_\gamma, \widetilde{H}_{\mu^t}[X; q] \rangle$.

Using (4.1), we can explicitly write out $\langle e_\gamma,\ \widetilde{H}_{\mu^t} \rangle$ as the following:

$$\langle e_\gamma,\ \widetilde{H}_{\mu^t} \rangle = \sum_{\substack{T \in \mathrm{SYT} \\ \mathrm{ctype}(T^t) \trianglerighteq \mu^t}} q^{\mathrm{charge}(T)} K_{\mathrm{shape}(T), \gamma} \tag{5.1}$$

$$= \sum_{\substack{w \in S_n \\ \mathrm{ctype}(P(w)^t) \trianglerighteq \mu^t \\ \mathrm{Des}(Q(w)) \subset \{\gamma_1, \gamma_1 + \gamma_2, \ldots, \gamma_1 + \cdots + \gamma_{l-1}\}}} q^{\mathrm{charge}(w)}. \tag{5.2}$$

Note that we use the interpretation of $K_{\lambda,\gamma}$ with respect to standard tableaux:

$$K_{\lambda,\gamma} = |\{w \in S_n \mid \text{shape}(P(w)) = \lambda, \text{Des}(Q(w)) \subset \{\gamma_1, \gamma_1+\gamma_2, \dots, \gamma_1+\cdots+\gamma_{l-1}\}\}|.$$

From this, proving $\text{Frob}_q(R_{\mu^t}) = \widetilde{H}_{\mu^t}[X;q]$ is equivalent to showing that for any $\gamma \vdash n$, the Hilbert series $\text{Hilb}_q(N_\gamma R_{\mu^t})$ is equal to (5.2). We show this by proving that the natural subset of charge monomials to expect from Equation (5.2) gives a basis of $N_\gamma R_{\mu^t}$ by antisymmetrization.

Proposition 1. *Let $\mu \vdash n$ and $\gamma = (\gamma_1, \dots, \gamma_l) \vdash n$. The set*

$$\{N_\gamma \mathbf{x}^{\mathrm{c}(w)} \mid w \in S_n,\ \text{ctype}(P(w)^t) \trianglerighteq \mu^t,\ \text{Des}(Q(w)) \subset \{\gamma_1, \gamma_1+\gamma_2, \dots, \gamma_1+\cdots+\gamma_{l-1}\}\}.$$

is a basis of $N_\gamma R_{\mu^t}$.

Using the construction of our basis, we can translate questions about the structure of R_{μ^t} into questions about conditions on tableaux. In this case, determining the elements that are a basis of $N_\gamma R_{\mu^t}$ is equivalent to looking at pairs of standard tableaux (P, Q) of the same shape with conditions on both P and Q. We illustrate this with an example:

Example 2. Consider $\mu = (3,1)$ and $\gamma = (2,2)$. All standard tableaux P that satisfy $\text{ctype}(P^t) \trianglerighteq (2,1,1) = (3,1)^t$ are listed below:

$$\begin{array}{|c|c|c|}\hline 2 \\ \hline 1 & 3 & 4 \\ \hline\end{array}\ ,\quad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array}\ ,\quad \begin{array}{|c|c|}\hline 4 \\ \hline 2 \\ \hline 1 & 3 \\ \hline\end{array}\ ,\quad \begin{array}{|c|c|}\hline 3 \\ \hline 2 \\ \hline 1 & 4 \\ \hline\end{array}\ ,\quad \begin{array}{|c|}\hline 4 \\ \hline 3 \\ \hline 2 \\ \hline 1 \\ \hline\end{array}.$$

We also list all standard tableaux Q such that $\text{Des}(Q) \subset \{2\}$:

$$\begin{array}{|c|c|c|c|}\hline 1 & 2 & 3 & 4 \\ \hline\end{array}\ ,\quad \begin{array}{|c|c|c|}\hline 3 \\ \hline 1 & 2 & 4 \\ \hline\end{array}.$$

Thus the only pair (P, Q) of standard tableaux of the same shape that satisfy $\text{ctype}(P^t) \trianglerighteq (2,1,1)$ and $\text{Des}(Q) = (2,2)$ is the pair $\left(\begin{array}{|c|c|c|}\hline 2 \\ \hline 1 & 3 & 4 \\ \hline\end{array}, \begin{array}{|c|c|c|}\hline 3 \\ \hline 1 & 2 & 4 \\ \hline\end{array}\right)$, which corresponds to the permutation $w = 2314$. Since $\mathrm{c}(w) = 0102$, we conclude the basis of $N_\gamma R_{\mu^t}$ is given by the polynomial $N_\gamma(x_2x_4^2) = x_2x_4^2 - x_1x_4^2 - x_2x_3^2 + x_1x_3^2$.

Corollary 3. *For any partition μ of n we have* $\text{Frob}_q(R_{\mu^t}) = \widetilde{H}_{\mu^t}[X;q]$.

Though Garsia–Procesi [6] provide the first combinatorial proof of this result, their proof requires deriving recursive properties of $\widetilde{H}_\mu[X;q]$ that match the recursion occurring in the construction of their monomial basis of R_μ. Not only is our proof independent from theirs, it also relies solely on the combinatorial formula (4.1) for $\widetilde{H}_\mu[X;q]$, as well as some basic symmetric function theory, providing a direct connection between the Schur expansion of $\widetilde{H}_\mu[X;q]$ and $\text{Frob}_q(R_\mu)$.

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