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A hidden symmetry of refined canonical stable Grothendieck polynomials

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Abstract. Refined canonical stable Grothendieck polynomials were introduced by Hwang, Jang, Kim, Song, and Song. There exist two combinatorial models for these polynomials: one using hook-valued tableaux and the other using pairs of a semistandard Young tableau and (what we call) an exquisite tableau. An uncrowding algorithm on hook-valued tableaux was introduced by Pan, Pappe, Poh, and Schilling. In this paper, we discover a novel connection between the two models via the uncrowding and Goulden–Greene's jeu de taquin algorithms, using a classical result of Benkart, Sottile, and Stroomer on tableau switching. This connection reveals a hidden symmetry of the uncrowding algorithm defined on hook-valued tableaux. As a corollary, we obtain another combinatorial model for the refined canonical stable Grothendieck polynomials in terms of biflagged tableaux, which naturally appear in the characterization of the image of the uncrowding map.

Keywords: Grothendieck polynomials, hook-valued tableaux, flagged tableaux, uncrowding algorithm, tableau switching, jeu de taquin

1 Introduction

Refined canonical stable Grothendieck polynomials were introduced in [8], generalizing and unifying many of the previous variants of Grothendieck polynomials. They encompass the Grassmannian Grothendieck polynomials introduced by Lascoux and Schützenberger [13], the stable Grassmannian β-Grothendieck polynomials of Fomin and Kirillov [5], the canonical stable Grothendieck polynomials of Yeliussizov [19], flagged Grothendieck polynomials of Matsumura [15], and the refined stable Grothendieck polynomials of Chan and Pflueger [4].

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In analogy to Schur functions, which are generating functions of semistandard Young tableaux, Buch [3] showed that stable Grothendieck polynomials are generating functions of semistandard set-valued tableaux. Each cell in a set-valued tableau contains a set instead of just a natural number. Combinatorial models for the (refined) canonical stable Grothendieck polynomials are described in terms of hook-valued tableaux [19, 9], which contain a semistandard Young tableau of a hook shape in each cell. These tableaux models are intimately related to the monomial expansions of the different versions of the Grothendieck polynomials.

An important question is to find the Schur expansion of the various Grothendieck polynomials. Lenart [14] gave the Schur expansion of the symmetric stable Grothendieck polynomials, whose monomial expansion is given in terms of set-valued tableaux.

Buch [3, Theorem 6.11] developed an uncrowding algorithm on set-valued tableaux to give a bijective proof of Lenart's Schur expansion. The uncrowding algorithm on a set-valued tableau produces a semistandard Young tableau (using the RSK bumping algorithm to uncrowd cells that contain more than one integer) and a flagged increasing tableau [14] (also known as an elegant filling [12, 1, 17]), which serves as a recording tableau. This uncrowding algorithm was generalized by Chan and Pflueger [4], by Reiner, Tenner and Yong [18], and by Pan, Pappe, Poh, and Schilling [16].

Hwang et al. [8] found the Schur expansion for refined canonical Grothendieck polynomials, which we rephrase in terms of "exquisite" tableaux. Hence, it is a natural question to relate the combinatorial model for the refined canonical Grothendieck polynomials in terms of hook-valued tableaux with the combinatorial model in terms of exquisite tableaux by giving a bijection between hook-valued tableaux and pairs of a semistandard tableau and an exquisite tableau. In this paper, we find such a bijection (see Theorem 4.14) using two types of uncrowding maps by combining the uncrowding algorithm due to Pan et al. [16] with Goulden–Greene's jeu de taquin [6].

The uncrowding algorithm on hook-valued tableaux in [16] uncrowds the entries in the arms of the hooks and yields a set-valued tableau and a column-flagged increasing tableau. Subsequently applying the uncrowding algorithm by Buch [3] on the set-valued tableau yields a semistandard Young tableau and a recording tableau. It was proved in [16] that this uncrowding operator intertwines with the crystal operators of Hawkes and Scrimshaw [7]. Let us denote this sequence of uncrowding operations by $\mathcal{U}_{\mathcal{L}^\infty\mathcal{A}^\infty}$, which indicates that first arm and then leg uncrowding is performed. In this paper, we also consider other orderings of leg and arm uncrowding, in particular $\mathcal{U}_{\mathcal{A}^\infty\mathcal{L}^\infty}$ which first performs leg and then arm uncrowding. We relate the two orderings using tableau switching in the sense of Benkart, Sottile and Stroomer [2]. This yields a characterization of the recording tableaux under the uncrowding algorithm in terms of biflagged tableaux (see Corollary 4.16). To connect to the combinatorial model of [8] in terms of exquisite tableaux we use the jeu de taquin algorithm due to Goulden and Greene [6], which we call GG-jdt. This map was further studied by Krattenthaler [11]. We show that GG-jdt is a partial tableau switching procedure (see Proposition 4.8). This yields the bijection between the combinatorial models for the refined canonical stable Grothendieck polynomials (see Theorem 4.14). Our proof reveals a hidden symmetry of the uncrowding algorithm on hook-valued tableaux when interchanging the order of arm and leg uncrowding (see Theorem 4.3). This shows the equivalence of three combinatorial models for the refined canonical stable Grothendieck polynomials (see Corollary 4.17).

For the long version containing all proofs, see [10].

2 Preliminaries

2.1 Refined canonical stable Grothendieck polynomials

Yeliussizov [19] introduced the *canonical stable Grothendieck polynomial* $G_\lambda^{(\alpha,\beta)}(\boldsymbol{x})$ indexed by a partition λ and two parameters α and β. It is a formal power series generalizing the *Grassmannian Grothendieck polynomial* $G_\lambda^{(\beta)}(\boldsymbol{x})$ with the property

$$\omega(G_\lambda^{(\alpha,\beta)}(\boldsymbol{x})) = G_{\lambda'}^{(\beta,\alpha)}(\boldsymbol{x}),$$

where ω is the involution that sends the Schur function $s_\lambda(\boldsymbol{x})$ to $s_{\lambda'}(\boldsymbol{x})$ indexed by the transpose partition λ'. The canonical stable Grothendieck polynomials have recently been studied in [7, 16].

The *refined canonical stable Grothendieck polynomials* $G_\lambda(\boldsymbol{x},\boldsymbol{\alpha},\boldsymbol{\beta})$, introduced by Hwang, Jang, Kim, Song, and Song [8], are refinements of $G_\lambda^{(\alpha,\beta)}(\boldsymbol{x})$ with infinite sets of parameters $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots)$ and $\boldsymbol{\beta} = (\beta_1, \beta_2, \dots)$. Here, we replaced every β_i with $-\beta_i$ in the original definition [8, Definition 1.1] in order to make $G_\lambda(\boldsymbol{x},\boldsymbol{\alpha},\boldsymbol{\beta})$ a formal power series with positive coefficients. We can set $\alpha_i = \alpha$ and $\beta_i = \beta$, for all i, in the refined version $G_\lambda(\boldsymbol{x},\boldsymbol{\alpha},\boldsymbol{\beta})$ to get the original version $G_\lambda^{(\alpha,\beta)}(\boldsymbol{x})$. Combinatorially, both $G_\lambda^{(\alpha,\beta)}(\boldsymbol{x})$ and $G_\lambda(\boldsymbol{x},\boldsymbol{\alpha},\boldsymbol{\beta})$ are the generating functions for hook-valued tableaux [19, 9].

We use French notation for partitions and tableaux throughout the paper.

Definition 2.1. A *hook-valued tableau* is a filling of a partition shape satisfying:

1. Each box contains a semistandard Young tableau of hook shape, i.e.,

$$\begin{array}{|c|cccc}\cline{1-1} L_\ell \\ \cline{1-1} \vdots \\ \cline{1-1} L_1 \\ \hline h & \multicolumn{1}{|c|}{A_1} & \multicolumn{1}{c|}{\cdots} & \multicolumn{1}{c|}{A_k} \\ \hline \end{array}$$

   where $h < L_1 < \cdots < L_\ell$ and $h \leqslant A_1 \leqslant \cdots \leqslant A_k$ are positive integers. The entry h is called the *hook entry*, the L_i's the *leg entries* and the A_i's the *arm entries*.

2. Each row is weakly increasing, i.e., any entry in a box is weakly smaller than any entry in the box directly to the right of it.

3. Each column is strictly increasing, i.e., any entry in a box is strictly smaller than any entry in the box directly above it.

We denote by $\mathsf{HVT}(\lambda)$ the set of hook-valued tableaux of shape λ. We write HVT for the set of hook-valued tableaux of any shape. The *weight* $\mathsf{wt}(T)$ of $T \in \mathsf{HVT}$ is

$$\mathsf{wt}(T) = \prod_{i \geqslant 1} \alpha_i^{(\text{\# of arm entries in column } i)} \beta_i^{(\text{\# of leg entries in row } i)} x_i^{(\text{\# of } i\text{'s in } T)}.$$

Example 2.2.

$$\text{Let} \quad T_1 = \begin{array}{|c|c|c|} \cline{1-2} \begin{matrix}6\\4\\335\end{matrix} & \begin{matrix}\\ \\67\end{matrix} & \multicolumn{1}{c}{} \\ \hline \begin{matrix}2\\11\end{matrix} & \begin{matrix}4\\334\end{matrix} & \begin{matrix}9\\445\end{matrix} \\ \hline \end{array} \quad \text{and} \quad T_2 = \begin{array}{|c|c|c|c|} \cline{1-2} \begin{matrix}7\\445\end{matrix} & \begin{matrix}\\789\end{matrix} & \multicolumn{2}{c}{} \\ \hline \begin{matrix}4\\3\\122\end{matrix} & \begin{matrix}\\5\\345\end{matrix} & \begin{matrix}8\\7\\567\end{matrix} & \begin{matrix}\\ \\7\end{matrix} \\ \hline \end{array}.$$

The tableau T_1 is a hook-valued tableau of shape $(3, 2)$, and T_2 is not a hook-valued tableau because the first row of T_2 is not weakly increasing (the first column of T_2 is not strictly increasing either). Furthermore, $\mathsf{wt}(T_1) = \alpha_1^3\alpha_2^3\alpha_3^2\beta_1^3\beta_2^2x_1^2x_2x_3^4x_4^5x_5^2x_6^2x_7x_9$.

Theorem 2.3. *[9] We have*

$$G_\lambda(\boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{T \in \mathsf{HVT}(\lambda)} \mathsf{wt}(T).$$

2.2 Various mixed tableaux

Definition 2.4. A *mixed tableau* of shape μ/λ is a filling T of the cells of μ/λ with elements in $\{\alpha_k \mid k \in \mathbb{Z}_{>0}\} \cup \{\beta_k \mid k \in \mathbb{Z}\}$. The *weight* $\mathsf{wt}(T)$ of a mixed tableau T is the product of its entries.

Definition 2.5. A *flagged-mixed tableau* is a mixed tableau satisfying the following conditions: if $T(i, j) = \alpha_k$, then $0 < k < j$, and if $T(i, j) = \beta_k$, then $0 < k < i$. Here, $T(i, j)$ denotes the entry in the i^{th} row and j^{th} column. We denote by FMT the set of all flagged-mixed tableaux.

Definition 2.6. Let T be a mixed tableau. For $\gamma \in \{\alpha, \beta\}$, we say that T is *γ-column-strict* (resp. *γ-row-strict*) if the following conditions hold:

1. If γ_i and γ_j are entries in T such that γ_i is weakly southwest of γ_j, then $i \geqslant j$.

2. If γ_i and γ_j are entries in T in the same column (resp. row), then $i \neq j$.

We also say that T is *totally column-strict* if the following conditions hold:

1. All indices of α and β in each column are strictly decreasing. More precisely, if $T(i,j) = \gamma_r$ and $T(i+1,j) = \delta_s$ with $\gamma, \delta \in \{\alpha, \beta\}$, then $r > s$.
2. All indices of α and β in each row are weakly decreasing. More precisely, if $T(i,j) = \gamma_r$ and $T(i,j+1) = \delta_s$ with $\gamma, \delta \in \{\alpha, \beta\}$, then $r \geqslant s$.

Definition 2.7. Let T be a mixed tableau of shape μ/λ. Let A (resp. B) be the set of cells in T containing α_k (resp. β_k) for any $k \in \mathbb{Z}$. We say that T is *(α, β)-sorted* if A and B form skew shapes ν/λ and μ/ν, respectively, for some partition ν with $\lambda \subseteq \nu \subseteq \mu$. Similarly, we say that T is *(β, α)-sorted* if B and A form skew shapes ν/λ and μ/ν, respectively, for some partition ν with $\lambda \subseteq \nu \subseteq \mu$.

2.3 Exquisite tableaux

The *content* $c(i,j)$ of the cell (i,j) is defined by $c(i,j) = j - i$.

Definition 2.8. Let T be a mixed tableau. We define $c^+_\beta(T)$ (resp. $c^-_\beta(T)$) to be the tableau obtained from T by replacing every β_r by β_{r+c} (resp. β_{r-c}), where c is the content of the cell containing β_r.

Definition 2.9. An *exquisite tableau* is a flagged-mixed tableau E such that $c^+_\beta(E)$ totally column-strict. Let $\mathsf{EXQ}(\mu/\lambda)$ denote the set of all exquisite tableaux of shape μ/λ.

Example 2.10. Let $\lambda = (2,1)$ and $\mu = (3,3,1)$. Then

$$\mathsf{EXQ}(\mu/\lambda) = \left\{ \begin{array}{|c|c|c|} \cline{1-1} \beta_2 \\ \hline \blacksquare & \beta_1 & \alpha_1 \\ \hline \blacksquare & \blacksquare & \alpha_2 \\ \hline \end{array}, \begin{array}{|c|c|c|} \cline{1-1} \beta_1 \\ \hline \blacksquare & \beta_1 & \alpha_1 \\ \hline \blacksquare & \blacksquare & \alpha_2 \\ \hline \end{array}, \begin{array}{|c|c|c|} \cline{1-1} \beta_2 \\ \hline \blacksquare & \alpha_1 & \alpha_1 \\ \hline \blacksquare & \blacksquare & \alpha_2 \\ \hline \end{array}, \begin{array}{|c|c|c|} \cline{1-1} \beta_1 \\ \hline \blacksquare & \alpha_1 & \alpha_1 \\ \hline \blacksquare & \blacksquare & \alpha_2 \\ \hline \end{array} \right\}.$$

Theorem 2.11. *[8, Corollary 4.5] We have*

$$G_\lambda(\boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{\mu \supseteq \lambda} s_\mu(\boldsymbol{x}) \sum_{E \in \mathsf{EXQ}(\mu/\lambda)} \mathsf{wt}(E).$$

3 Uncrowding algorithm and tableau switching

3.1 Uncrowding algorithms for hook-valued tableaux

We define the uncrowding algorithms for hook-valued tableaux following [16].

Definition 3.1. The *arm-uncrowding bumping* $\mathcal{A}_b\colon \mathsf{HVT} \to \mathsf{HVT}$ is defined in [16, Definition 3.2]. The *leg-uncrowding bumping* $\mathcal{L}_b\colon \mathsf{HVT} \to \mathsf{HVT}$ is defined in [16, Definition 3.33].

Definition 3.2. The *single-arm-uncrowding map* $\mathcal{A}\colon \mathsf{HVT} \to \mathsf{HVT}$ is defined as follows. Let $T \in \mathsf{HVT}$. If T has no arms, or equivalently, if $\mathcal{A}_b(T) = T$, then define $\mathcal{A}(T) = T$. Otherwise, we define $\mathcal{A}(T) = \mathcal{A}_b^m(T)$, where m is the smallest integer such that the shape of $\mathcal{A}_b^m(T)$ is larger than that of T.

Definition 3.3. The *single-leg-uncrowding map* $\mathcal{L}\colon \mathsf{HVT} \to \mathsf{HVT}$ is defined as follows. Let $T \in \mathsf{HVT}$. If T has no legs, then $\mathcal{L}(T) = T$. Otherwise, we define $\mathcal{L}(T) = \mathcal{L}_b^m(T)$, where m is the smallest integer such that the shape of $\mathcal{L}_b^m(T)$ is larger than that of T.

Definition 3.4. Let $\mathcal{U} = \mathcal{U}_{f_n \cdots f_1}$, where $f_n \cdots f_1$ is a word in the alphabet $\{\mathcal{A}, \mathcal{L}\}$. Then the *uncrowding map* $\mathcal{U}\colon \mathsf{HVT} \to \mathsf{HVT} \times \mathsf{FMT}$ is defined as follows.

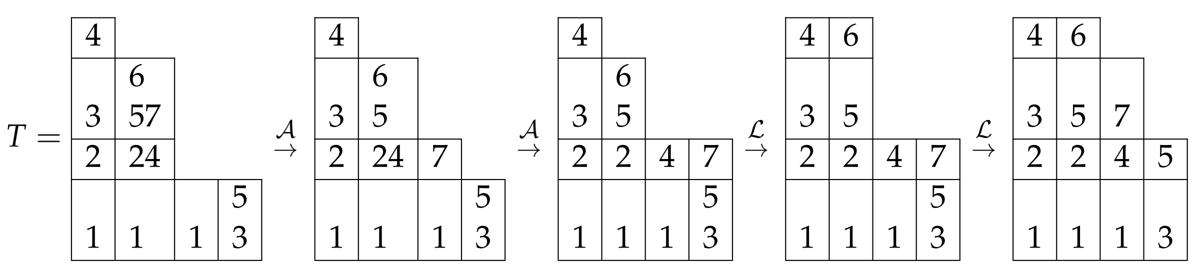
Let $T \in \mathsf{HVT}$. We construct two tableaux P and Q. For each $i = 0, 1, 2, \dots, n$, let $T_i = f_i \circ \cdots \circ f_1(T)$, where $T_0 = T$, and let $\lambda^{(i)}$ be the shape of T_i. First, we define $P = T_n$. Now we define a flagged-mixed tableau Q of shape $\lambda^{(n)}/\lambda^{(0)}$ as follows. For each $i = 1, 2, \dots, n$, there are two cases.

Case 1 $f_i = \mathcal{A}$. Then $T_i = \mathcal{A}(T_{i-1})$. If T_{i-1} has arms, suppose that (r, c) is the cell that contains the largest arm entry in the rightmost column containing an arm. Then fill the unique cell $\lambda^{(i)}/\lambda^{(i-1)}$ in Q with α_c. (If T_{i-1} has no arms, nothing happens.)

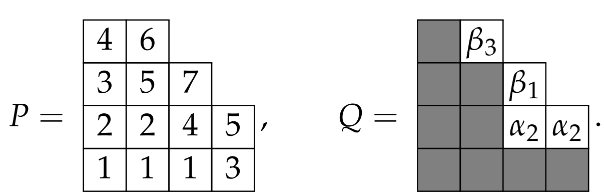
Case 2 $f_i = \mathcal{L}$. Then $T_i = \mathcal{L}(T_{i-1})$. If T_{i-1} has legs, suppose that (r, c) is the cell that contains the largest leg entry in the topmost row containing a leg. Then fill the unique cell $\lambda^{(i)}/\lambda^{(i-1)}$ in Q with β_r. (If T_{i-1} has no legs, nothing happens.)

Finally, we define $\mathcal{U}(T) = (P, Q)$. We call P and Q the *insertion tableau* and *recording tableau* of $\mathcal{U}(T)$, respectively.

Example 3.5. Consider the following hook-valued tableaux $T, \mathcal{A}(T), \mathcal{A} \circ \mathcal{A}(T), \mathcal{L} \circ \mathcal{A} \circ \mathcal{A}(T)$, and $\mathcal{L} \circ \mathcal{L} \circ \mathcal{A} \circ \mathcal{A}(T)$:



This shows that $\mathcal{U}_{\mathcal{L}\mathcal{L}\mathcal{A}\mathcal{A}}(T) = (P, Q)$, where



Note that if a hook-valued tableau T has a arms and ℓ legs, then $\mathcal{A}^a(T) = \mathcal{A}^{a+1}(T) = \cdots$ and $\mathcal{L}^\ell(T) = \mathcal{L}^{\ell+1}(T) = \cdots$. Thus, $\mathcal{U}_{\mathcal{L}^N \mathcal{A}^M}(T)$ is the same for all $N \geqslant \ell$ and $M \geqslant a$. We will write the result as $\mathcal{U}_{\mathcal{L}^\infty \mathcal{A}^\infty}(T)$. We define $\mathcal{U}_{\mathcal{A}^\infty \mathcal{L}^\infty}(T)$ similarly.

3.2 Tableau switching

We recall results from a paper by Benkart, Sottile, and Stroomer [2] on tableau switching.

Definition 3.6. Let T be an α-column-strict and β-row-strict mixed tableau. Let $u = T(i,j)$ be an entry of T and let v be the entry $T(i,j+1)$ or $T(i+1,j)$. Suppose $u = \alpha_r$ and $v = \beta_s$. Let T' be the mixed tableau obtained from T by interchanging u and v. If T' is α-column-strict and β-row-strict, such a process is called a *switch*. If there is no possible switch, we say that T is *fully switched*.

Note that in a switch, we always move an α-entry to the north or east, and a β-entry to the south or west. Hence, if we keep applying switches to a tableau, it eventually becomes fully switched.

Theorem 3.7. *[2, Theorem 2.2] Let T be an α-column-strict, β-row-strict, and (α, β)-sorted mixed tableau. We apply switches to T until it is fully switched. Then, the resulting tableau is α-column-strict, β-row-strict, and (β, α)-sorted. Furthermore, it is independent of the sequence of switches that produced it.*

Corollary 3.8.

1. *Let T be an α-column-strict, β-row-strict, and (α, β)-sorted mixed tableau. Let X_1 and X_2 be fully switched tableaux obtained from T by some sequences of switches. Then $X_1 = X_2$.*

2. *Let T_1 and T_2 be α-column-strict, β-row-strict, and (α, β)-sorted mixed tableaux. Suppose that X is a fully switched tableau obtained from both T_1 and T_2 by some sequences of switches. Then $T_1 = T_2$.*

4 Main results

4.1 Tableau switching on the recording tableau

We study the effects of first performing leg-uncrowding and then arm-uncrowding on a hook-valued tableau in comparison to [16]. To characterize the changes to the recording tableaux, we define a particular sequence of tableau switches on an (α, β)-tableau.

Definition 4.1. Let Q be a flagged-mixed tableau that is α-column-strict, β-row-strict, and (α, β)-sorted. The (jdt) *shuffle* of Q, denoted $\mathsf{shuff}(Q)$, is the tableau obtained by:

1. Find elements with weight α having an element of weight β above or to the right.
2. Among them, choose the rightmost element that has the smallest index, say α_i.
3. Continue the following 'switching process' until the cells above and to the right of α_i have a weight α or are empty. If only one of the cells directly above or to the right of α_i has weight β, then switch α_i with this element. Otherwise, let β_k and β_j be the entries above and to the right of α_i, respectively, and perform one of the following switches

$$\begin{array}{|c|c}\cline{1-1} \beta_k & \\ \hline \alpha_i & \multicolumn{1}{c|}{\beta_j} \\ \hline\end{array} \mapsto \begin{array}{|c|c}\cline{1-1} \alpha_i & \\ \hline \beta_k & \multicolumn{1}{c|}{\beta_j} \\ \hline\end{array} \quad \text{if } k > j, \qquad \begin{array}{|c|c}\cline{1-1} \beta_k & \\ \hline \alpha_i & \multicolumn{1}{c|}{\beta_j} \\ \hline\end{array} \mapsto \begin{array}{|c|c}\cline{1-1} \beta_k & \\ \hline \beta_j & \multicolumn{1}{c|}{\alpha_i} \\ \hline\end{array} \quad \text{if } k \leqslant j.$$

4. Repeat steps (1)-(3) until there is no cell having weight α with an element of weight β directly above or to its right.

Example 4.2. The following shows the process of the shuffle, where boxes containing an α are colored in yellow:

$$Q = \begin{array}{cccccc} \beta_8 & \beta_6 & \beta_5 & \beta_2 & & \\ \alpha_2 & \alpha_1 & \beta_6 & \beta_2 & \beta_1 & \\ \blacksquare & \alpha_2 & \alpha_2 & \alpha_1 & \beta_5 & \beta_1 \end{array} \Longrightarrow \begin{array}{cccccc} \beta_8 & \beta_6 & \beta_5 & \beta_2 & & \\ \alpha_2 & \alpha_1 & \beta_6 & \beta_2 & \beta_1 & \\ \blacksquare & \alpha_2 & \alpha_2 & \beta_5 & \beta_1 & \alpha_1 \end{array} \Longrightarrow \begin{array}{cccccc} \beta_8 & \beta_6 & \beta_2 & \alpha_1 & & \\ \alpha_2 & \beta_6 & \beta_5 & \beta_2 & \beta_1 & \\ \blacksquare & \alpha_2 & \alpha_2 & \beta_5 & \beta_1 & \alpha_1 \end{array}$$

$$\Longrightarrow \begin{array}{cccccc} \beta_8 & \beta_6 & \beta_2 & \alpha_1 & & \\ \alpha_2 & \beta_6 & \beta_5 & \beta_1 & \alpha_2 & \\ \blacksquare & \alpha_2 & \beta_5 & \beta_2 & \beta_1 & \alpha_1 \end{array} \Longrightarrow \begin{array}{cccccc} \beta_8 & \beta_2 & \alpha_2 & \alpha_1 & & \\ \alpha_2 & \beta_6 & \beta_5 & \beta_1 & \alpha_2 & \\ \blacksquare & \beta_6 & \beta_5 & \beta_2 & \beta_1 & \alpha_1 \end{array} \Longrightarrow \begin{array}{cccccc} \beta_2 & \alpha_2 & \alpha_2 & \alpha_1 & & \\ \beta_8 & \beta_6 & \beta_5 & \beta_1 & \alpha_2 & \\ \blacksquare & \beta_6 & \beta_5 & \beta_2 & \beta_1 & \alpha_1 \end{array} = \mathsf{shuff}(Q).$$

Theorem 4.3. *Let T be a hook-valued tableau and set $(P_1, Q_1) = \mathcal{U}_{\mathcal{L}^\infty\mathcal{A}^\infty}(T)$ and $(P_2, Q_2) = \mathcal{U}_{\mathcal{A}^\infty\mathcal{L}^\infty}(T)$. Then $P_2 = P_1$ and $Q_2 = \mathsf{shuff}(Q_1)$.*

The above theorem reveals a hidden symmetry of hook-valued tableaux which leads to a characterization of the image for the uncrowding algorithm and the equivalence of multiple combinatorial models of $G_\lambda(\boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{\beta})$, which is discussed in Section 4.3.

4.2 Goulden-Greene jeu de taquin

We define the Goulden–Greene jeu de taquin algorithm [6] (a process they refer to as the modified jeu de taquin) using our notation. This map was also studied by Krattenthaler [11]. We then show that it can be realized as a tableau switching.

Definition 4.4 (GG-jdt slides)**.** Let T be a α-column-strict and β-row-strict mixed tableau. We say that the entry $T(i, j)$ is *out of order* if at least one of the following conditions holds:

1. $T(i, j) = \alpha_r$ and $T(i, j + 1) = \beta_s$ for some r and s with $r < s + (j + 1) - i$.

2. $T(i,j) = \alpha_r$ and $T(i+1,j) = \beta_t$ for some r and t with $r \leqslant t + j - (i+1)$.

If Condition (1) (resp. Condition (2)) holds, we define the *horizontal slide* (resp. *vertical slide*) at (i,j) to be the operation that swaps α_r and β_s.

If $T(i,j)$ is out of order, the *GG-jdt slide* at (i,j) is the unique available operation between the horizontal slide and the vertical slide at (i,j) such that the resulting tableau is still β-row-strict. More precisely, if only Condition (1) (resp. (2)) holds, then the GG-jdt slide at (i,j) is the horizontal (resp. vertical) slide at (i,j). If both Conditions (1) and (2) hold, then $T(i,j) = \alpha_r$, $T(i,j+1) = \beta_s$, and $T(i+1,j) = \beta_t$ for some r, s, and t with $r < s + (j+1) - i$ and $r \leqslant t + j - (i+1)$. In this case, the GG-jdt slide at (i,j) is the horizontal slide at (i,j) if $t \leqslant s$ and the vertical slide at (i,j) if $t > s$:

$$\begin{array}{|c|c} \hline \beta_t & \\ \hline \alpha_r & \multicolumn{1}{c|}{\beta_s} \\ \hline \end{array} \mapsto \begin{array}{|c|c} \hline \beta_t & \\ \hline \beta_s & \multicolumn{1}{c|}{\alpha_r} \\ \hline \end{array} \quad \text{if } t \leqslant s, \qquad \begin{array}{|c|c} \hline \beta_t & \\ \hline \alpha_r & \multicolumn{1}{c|}{\beta_s} \\ \hline \end{array} \mapsto \begin{array}{|c|c} \hline \alpha_r & \\ \hline \beta_t & \multicolumn{1}{c|}{\beta_s} \\ \hline \end{array} \quad \text{if } t > s.$$

Definition 4.5 (GG-jdt map)**.** Let T be a mixed tableau that is α-column-strict, β-row-strict, and (α, β)-sorted. The *GG-jdt map* is the map $\mathsf{jdt}_{\mathrm{GG}}$ sending T to the tableau $\mathsf{jdt}_{\mathrm{GG}}(T)$ obtained as follows:

1. Find the smallest r such that α_r is out of order in T. Let (i,j) be the rightmost cell containing such an α_r.
2. Apply the GG-jdt slide to T at (i,j).
3. Repeat (1)-(2) until no entries are out of order.

Observe that the GG-jdt slide is exactly a switch defined in Definition 3.6 except that it does not require the resulting tableau to be α-column-strict and β-column-strict. However, we will see that this condition holds automatically. Hence, the GG-jdt map is a sequence of switches.

Example 4.6. The GG-jdt map applied to Q proceeds as follows, where we truncate the first two columns and the first six rows of the tableau:

$$Q = \begin{array}{|c|c|c|c|c|c|} \hline \beta_8 & \beta_6 & \beta_5 & \beta_2 & & \\ \hline \alpha_2 & \alpha_1 & \beta_6 & \beta_2 & \beta_1 & \\ \hline \blacksquare & \alpha_2 & \alpha_2 & \alpha_1 & \beta_5 & \beta_1 \\ \hline \end{array} \xrightarrow{\mathsf{jdt}_{\mathrm{GG}}} \begin{array}{|c|c|c|c|c|c|} \hline \alpha_2 & \beta_6 & \alpha_1 & \beta_2 & & \\ \hline \beta_8 & \alpha_2 & \beta_5 & \beta_2 & \beta_1 & \\ \hline \blacksquare & \beta_6 & \beta_5 & \alpha_2 & \beta_1 & \alpha_1 \\ \hline \end{array} \xrightarrow{c_\beta^+} \begin{array}{|c|c|c|c|c|c|} \hline \alpha_2 & \beta_1 & \alpha_1 & \beta_{-1} & & \\ \hline \beta_3 & \alpha_2 & \beta_2 & \beta_0 & \beta_0 & \\ \hline \blacksquare & \beta_3 & \beta_3 & \alpha_2 & \beta_1 & \alpha_1 \\ \hline \end{array} .$$

We can compare this with the tableau switching in Example 4.2. Note that this process is a sequence of switches. By the uniqueness of the fully switched tableau, see Corollary 3.8 (1), if we keep applying switches to $\mathsf{jdt}_{\mathrm{GG}}(Q)$, then we get the same resulting tableau $\mathsf{shuff}(Q)$ at the end of Example 4.2.

The following properties of the GG-jdt map were stated in [6, Section 3] without proof. Krattenthaler [11, Lemma 1] gave complete proofs of these results.

Proposition 4.7.

1. *The GG-jdt map is a bijection from the set of α-column-strict, β-row-strict, and (α, β)-sorted tableaux to the set of tableaux E such that $c^+_\beta(E)$ is a totally column-strict tableau.*

2. *Each iteration of the GG-jdt slide results in an α-column-strict and β-row-strict tableau.*

By Proposition 4.7 (2), the GG-jdt map can be formulated as a tableau switching procedure as defined in Section 3.2.

Proposition 4.8. *Let T be a flagged-mixed tableau that is α-column-strict, β-row-strict, and (α, β)-sorted. Then $\mathsf{jdt}_{\mathrm{GG}}(T)$ is obtained from T by a sequence of switches.*

4.3 Bijections and the image of the uncrowding algorithm

In this subsection, we find a bijection relating hook-valued tableaux, exquisite tableaux, and a new class of tableaux called biflagged tableaux. We also show that GG-jdt is a bijection between biflagged tableaux and exquisite tableaux. As a corollary, we characterize the image of the uncrowding algorithm $\mathcal{U}_{\mathcal{L}^\infty\mathcal{A}^\infty}$ defined on hook-valued tableaux.

Definition 4.9. A *biflagged tableau* is a mixed tableau T satisfying:

1. T is α-column-strict, β-row-strict, and (α, β)-sorted.

2. Both T and $\mathsf{shuff}(T)$ are flagged-mixed tableaux.

We denote by $\mathsf{BFT}(\mu/\lambda)$ the set of biflagged tableaux of shape μ/λ.

Example 4.10. When $\lambda = (2, 1)$ and $\mu = (3, 3, 1)$, we have

$$\mathsf{BFT}(\mu/\lambda) = \left\{ \begin{array}{|c|cc}\hline \beta_2 & \multicolumn{2}{c}{} \\ \hline \blacksquare & \multicolumn{1}{|c|}{\alpha_1} & \multicolumn{1}{c|}{\beta_1} \\ \hline \blacksquare & \multicolumn{1}{|c|}{\blacksquare} & \multicolumn{1}{c|}{\alpha_2} \\ \hline \end{array}, \begin{array}{|c|cc}\hline \beta_1 & \multicolumn{2}{c}{} \\ \hline \blacksquare & \multicolumn{1}{|c|}{\alpha_1} & \multicolumn{1}{c|}{\beta_1} \\ \hline \blacksquare & \multicolumn{1}{|c|}{\blacksquare} & \multicolumn{1}{c|}{\alpha_2} \\ \hline \end{array}, \begin{array}{|c|cc}\hline \beta_2 & \multicolumn{2}{c}{} \\ \hline \blacksquare & \multicolumn{1}{|c|}{\alpha_1} & \multicolumn{1}{c|}{\alpha_1} \\ \hline \blacksquare & \multicolumn{1}{|c|}{\blacksquare} & \multicolumn{1}{c|}{\alpha_2} \\ \hline \end{array}, \begin{array}{|c|cc}\hline \beta_1 & \multicolumn{2}{c}{} \\ \hline \blacksquare & \multicolumn{1}{|c|}{\alpha_1} & \multicolumn{1}{c|}{\alpha_1} \\ \hline \blacksquare & \multicolumn{1}{|c|}{\blacksquare} & \multicolumn{1}{c|}{\alpha_2} \\ \hline \end{array} \right\}.$$

We will show that GG-jdt is a bijection from the set of biflagged tableaux $\mathsf{BFT}(\mu/\lambda)$ to the set of exquisite tableaux $\mathsf{EXQ}(\mu/\lambda)$ defined in Definition 2.9. We denote by $\mathsf{SSYT}(\mu)$ the set of semistandard Young tableaux of shape μ. As usual, the weight $\mathsf{wt}(P)$ of a semistandard Young tableau P is the product of x_i for all entries i in P.

Lemma 4.11. *Let $T \in \mathsf{HVT}(\lambda)$ and $\mathcal{U}_{\mathcal{L}^\infty\mathcal{A}^\infty}(T) = (P, Q)$. Then $P \in \mathsf{SSYT}(\mu)$ and $Q \in \mathsf{BFT}(\mu/\lambda)$ for a partition μ with $\lambda \subseteq \mu$. Furthermore, $\mathsf{wt}(T) = \mathsf{wt}(P)\mathsf{wt}(Q)$.*

Lemma 4.12. *Let* $Q \in \mathsf{BFT}(\mu/\lambda)$*. Then* $\mathsf{jdt}_{\mathsf{GG}}(Q) \in \mathsf{EXQ}(\mu/\lambda)$*.*

By Lemmas 4.11 and 4.12, we can define the following map.

Definition 4.13. We define the map

$$\Phi\colon \mathsf{HVT}(\lambda) \to \bigsqcup_{\mu \supseteq \lambda} (\mathsf{SSYT}(\mu) \times \mathsf{EXQ}(\mu/\lambda))$$

as the composition of the following two maps:

$$\mathsf{HVT}(\lambda) \xrightarrow{\mathcal{U}_{\mathcal{L}^\infty \mathcal{A}^\infty}} \bigsqcup_{\mu \supseteq \lambda} (\mathsf{SSYT}(\mu) \times \mathsf{BFT}(\mu/\lambda)) \xrightarrow{\mathrm{id} \times \mathsf{jdt}_{\mathsf{GG}}} \bigsqcup_{\mu \supseteq \lambda} (\mathsf{SSYT}(\mu) \times \mathsf{EXQ}(\mu/\lambda)) . \tag{4.1}$$

In other words, $\Phi(T) = (P, E)$, where $\mathcal{U}_{\mathcal{L}^\infty \mathcal{A}^\infty}(T) = (P, Q)$ and $E = \mathsf{jdt}_{\mathsf{GG}}(Q)$.

Theorem 4.14. *The following are weight-preserving bijections:*

$$\mathsf{jdt}_{\mathsf{GG}}\colon \mathsf{BFT}(\mu/\lambda) \to \mathsf{EXQ}(\mu/\lambda) \qquad \textit{and}$$

$$\Phi\colon \mathsf{HVT}(\lambda) \to \bigsqcup_{\mu \supseteq \lambda} (\mathsf{SSYT}(\mu) \times \mathsf{EXQ}(\mu/\lambda)) .$$

Example 4.15. The map $\mathsf{jdt}_{\mathsf{GG}}$ sends the biflagged tableaux in Example 4.10 to the exquisite tableaux in Example 2.10 in that order.

We now characterize the image of the uncrowding algorithm.

Corollary 4.16. *The following is a weight-preserving bijection:*

$$\mathcal{U}_{\mathcal{L}^\infty \mathcal{A}^\infty}\colon \mathsf{HVT}(\lambda) \to \bigsqcup_{\mu \supseteq \lambda} (\mathsf{SSYT}(\mu) \times \mathsf{BFT}(\mu/\lambda)) .$$

Corollary 4.17. *We have*

$$G_\lambda(\boldsymbol{x}, \boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{H \in \mathsf{HVT}(\lambda)} \mathsf{wt}(H) = \sum_{\mu \supseteq \lambda} s_\mu(\boldsymbol{x}) \sum_{E \in \mathsf{EXQ}(\mu/\lambda)} \mathsf{wt}(E) = \sum_{\mu \supseteq \lambda} s_\mu(\boldsymbol{x}) \sum_{Q \in \mathsf{BFT}(\mu/\lambda)} \mathsf{wt}(Q).$$

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