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Extending the Science Fiction and the Loehr–Warrington Formula

Donghyun Kim*[1] and Jaeseong Oh †

[1]*Department of Mathematical Sciences, Seoul National University, Seoul 151-247, Korea*
[2]*June E Huh Center for Mathematical Challenges, Korea Institute for Advanced Study, Seoul 02455, Korea*

Abstract. We introduce the Macdonald piece polynomial $\mathrm{I}_{\mu,\lambda,k}[X;q,t]$, which is a vast generalization of the Macdonald intersection polynomial in the science fiction conjecture by Bergeron and Garsia. We demonstrate a remarkable connection between $\mathrm{I}_{\mu,\lambda,k}$, ∇s_λ, and the Loehr–Warrington formula LW_λ, thereby obtaining the Loehr–Warrington conjecture as a corollary. We also present an extension of the science fiction conjecture and the Macdonald positivity by exploiting $\mathrm{I}_{\mu,\lambda,k}$.

Keywords: Science fiction conjecture, Macdonald piece polynomials, Macdonald positivity, Loehr–Warrington conjecture, Garsia–Haiman–Tesler plethystic formula, Haglund–Haiman–Loehr formula, P-tableau

1 Introduction

An important question posed by Macdonald was the *Macdonald positivity conjecture* [18] which asserts that the *modified Macdonald polynomial* $\widetilde{H}_\mu[X;q,t]$ is Schur positive. To address this conjecture, Garsia and Haiman introduced the bigraded $\mathfrak{S}_n$ module R_μ, now called the *Garsia–Haiman module* [6].

In his groundbreaking work [12], Haiman utilized the geometry of the Hilbert scheme of n points in the plane $\mathbb{C}^2$ to prove that $\mathrm{grFrob}\left(R_\mu;q,t\right) = \widetilde{H}_\mu[X;q,t]$, thereby settling the Macdonald positivity conjecture. Despite Haiman's resolution of Schur positivity, an explicit combinatorial formula for the Schur coefficients remains elusive.

Problem 1.1. Find a combinatorial description for the Schur coefficient $\widetilde{K}_{\lambda,\mu}(q,t)$ of the Macdonald polynomials: $\widetilde{H}_\mu[X;q,t] = \sum_\lambda \widetilde{K}_{\lambda,\mu}(q,t)s_\lambda$, where s_λ is the Schur function.

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†jsoh@kias.re.kr. Jaeseong Oh was partially supported by KIAS Individual Grant (HP083401) at June E Huh Center for Mathematical Challenges in Korea Institute for Advanced Study.[2]

One approach to resolving Problem 1.1 is to decompose $\widetilde{H}_\mu$ into Schur-positive pieces and find a combinatorial description of their Schur coefficients. The HHL formula [10] decomposes $\widetilde{H}_\mu$ into LLT polynomials, which are Schur positive [8], though describing their Schur coefficients remains challenging. Another approach uses k-Schur functions [16], with explicit Schur coefficients described in [3], but it's unclear if $\widetilde{H}_\mu$ can be positively expressed in terms of k-Schur functions. In this paper, we propose an alternative method, generalizing the science fiction conjecture of Bergeron and Garsia [1].

Let μ be a partition with n removable corners $\{c_1, \dots, c_n\}$ and let $1 \le k \le n-1$. For a subset $S \in \binom{[n]}{k}$ of $[n]$ of size k, let μ^S be the partition obtained from μ by deleting corners $\{c_i : i \in S\}$. Let $R(n,k) := ((n-k)^k)$ be the rectangular partition, and let λ be a partition inside $R(n,k)$. We define the *Macdonald piece polynomial* $\mathrm{I}_{\mu,\lambda,k}$ as

$$\mathrm{I}_{\mu,\lambda,k}[X;q,t] := (-1)^{|\lambda|} \sum_{S \in \binom{[n]}{k}} \frac{s_\lambda[z_S] \prod_{j \in S^c} z_j}{\prod_{\substack{i \in S \\ j \in S^c}} (z_j - z_i)} \widetilde{H}_{\mu^S}[X]. \tag{1.1}$$

Here, we let $z_i = q^{-\operatorname{coarm}(c_i)} t^{-\operatorname{coleg}(c_i)}$, and $f[z_S]$ denotes the polynomial f in the z_i's where i runs through S. For $k = 1$ and $\lambda = \emptyset$, $\mathrm{I}_{\mu,\lambda,k}$ recovers the Macdonald intersection polynomial $\mathrm{I}_{\mu^{(1)},\dots,\mu^{(n)}}$ defined in [1] and studied in [13, 14]. Using the novel symmetric function $\mathrm{I}_{\mu,\lambda,k}$, we define another symmetric function $\widetilde{H}^{\lambda}_{\mu^S}$. We conjecture that it is the graded Frobenius characteristic of the intersection of certain Garsia–Haiman modules (see Conjecture 2.3), largely generalizing the science fiction conjecture.

For a partition $\lambda \subseteq R(n,k)$, let $\tilde{\lambda}$ be the conjugate of $(n-k-\lambda_k, \dots, n-k-\lambda_1)$, where $\lambda_i = 0$ if $i > \ell(\lambda)$. The notation $\tilde{\lambda}$ depends on $R(n,k)$, which will be clear from context. Now, we state our main results.

Theorem 1.2. *Let μ be a partition with n corners, and $1 \le k < n$. Then for a partition λ inside a rectangle $R(n,k)$ the symmetric function $\mathrm{I}_{\mu,\lambda,k}$ satisfies the following:*

(a) For $N > |\mu| - |\tilde{\lambda}| - k$, we have $e_N^\perp \mathrm{I}_{\mu,\lambda,k} = 0$.

(b) We have $\frac{1}{T_{\mu^{[n]}}} e^\perp_{|\mu|-|\tilde{\lambda}|-k} \mathrm{I}_{\mu,\lambda,k} = \nabla s_{\tilde{\lambda}}$. In particular, $e^\perp_{|\mu|+|\tilde{\lambda}|-k} \mathrm{I}_{\mu,\lambda,k}$ does not depend on the partition μ (up to a constant).

(c) We have $\frac{1}{T_{\mu^{[n]}}} e^\perp_{|\mu|-|\tilde{\lambda}|-k} \mathrm{I}_{\mu,\lambda,k} = (-1)^{\operatorname{adj}(\tilde{\lambda})} \operatorname{LW}_{\tilde{\lambda}}$.

Here, $T_\mu := \prod_{c \in \mu} q^{\operatorname{coarm}(c)} t^{\operatorname{coleg}(c)}$, and definitions for $\operatorname{LW}_\lambda$ and $\operatorname{adj}(\lambda)$ is given in Section 4.

Note that the straightforward implication of Theorem 1.2 (b) and (c) is the Loehr–Warrington conjecture, which was very recently proved in [2][1].

Corollary 1.3 (The Loehr–Warrington conjecture)**.** *We have $\nabla s_\lambda = (-1)^{\operatorname{adj}(\lambda)} \operatorname{LW}_\lambda$.*

[1]There is also an unpublished paper by Carlsson and Mellit proving the Loehr–Warrington conjecture.

2 Extending the Science Fiction conjecture with Macdonald piece polynomials

A *diagram* is a collection of cells in the first quadrant. A partition can be viewed as a left-justified diagram with row lengths weakly decreasing from bottom to top. A cell in the i-th row and j-th column is denoted by (i, j). We often represent a diagram as $D = [D^{(1)}, D^{(2)}, \ldots]$, where $D^{(j)} = \{i : (i, j) \in D\}$ is the set of row indices in the j-th column. The *size* $|D|$ is the number of cells in D.

We fix a partition μ with n corners $\{c_1, c_2, \ldots, c_n\}$ and let $\mu^{(i)}$ be a partition obtained by removing a corner c_i. Recall that the science fiction conjecture [1] suggests that

$$\mathrm{I}_{\mu^{(1)},\ldots,\mu^{(n)}}[X; q, t] := \sum_{i=1}^{n} \left(\prod_{j \neq i} \frac{z_j}{z_j - z_i} \right) \widetilde{H}_{\mu^{(i)}} = \mathrm{grFrob}\left(\cap_{i=1}^{n} R_{\mu^{(i)}} \right), \tag{2.1}$$

where $z_i = T_{\mu^{(i)}}/T_\mu$. Note that $\mathrm{I}_{\mu^{(1)},\ldots,\mu^{(n)}}[X; q, t]$ is a special case of the Macdonald piece polynomial $\mathrm{I}_{\mu,\lambda,k}$ obtained by letting $\lambda = \varnothing$ and $k = 1$. Our goal is to further generalize (2.1) and refine the Macdonald positivity as well (Conjecture 2.3).

Note that $\mathrm{I}_{\mu,\lambda,k}$ can be regarded as a linear combination of $\widetilde{H}_{\mu^S}$. We present the converse by representing each $\widetilde{H}_{\mu^S}$ as a linear combination of $\mathrm{I}_{\mu,\lambda,k}$.

Lemma 2.1. *We have*

$$\widetilde{H}_{\mu^S} = \sum_{\lambda \subseteq R(n,k)} \frac{s_{\tilde{\lambda}}[z_{S^c}]}{\prod_{j \in S^c} z_j} \mathrm{I}_{\mu,\lambda,k}$$

where $z_i = T_{\mu_{\{i\}}}/T_\mu = q^{-\operatorname{coarm}(c_i)} t^{-\operatorname{coleg}(c_i)}$.

From now on we regard $z_1, z_2, \ldots, z_n$ as indeterminates and let

$$\widetilde{H}_{\mu^S}[X; q, t, z] := \sum_{\lambda \subseteq R(n,k)} \frac{s_{\tilde{\lambda}}[z_{S^c}]}{\prod_{j \in S^c} z_j} \mathrm{I}_{\mu,\lambda,k}\,.$$

Define Φ to be a specialization map on functions in $z_1, z_2, \ldots, z_n$ given by letting $z_i = T_{\mu^{\{i\}}}/T_\mu$. By Lemma 2.1, we trivially have $\Phi(\widetilde{H}_{\mu^S}[X; q, t, z]) = \widetilde{H}_{\mu^S}$.

Let $\pi_{i,j}$[2] be the operator acting on functions in $z_1, z_2, \ldots, z_n$ defined by

$$\pi_{i,j} f(z_1, z_2, \cdots, z_n) = \frac{z_j f - z_i f|_{z_i \leftrightarrow z_j}}{z_j - z_i}.$$

Denoting $S = \{i_1 < i_2 < \cdots < i_k\}$ and $S^c = \{j_1 < j_2 < \cdots < j_{n-k}\}$, for a partition $\lambda \subseteq R(n, k)$ we define $\pi_{\lambda,S}$ to be a sequence of operators given by $\pi_{\lambda,S} = \prod_{r=1}^{\ell(\lambda)} \prod_{s=1}^{\lambda_r} \pi_{i_r, j_s}$.

[2] This operator is motivated by Butler's symmetric function [4] and for $j = i + 1$, $\pi_{i,j}$ coincides with the *Demazure operator* π_i.

We may consider $\widetilde{H}_{\mu^S}[X;q,t,z]$ as a function in $z_1, z_2, \ldots, z_n$ regarding $\mathrm{I}_{\mu,\lambda,k}$ as coefficients. In this sense the operators $\pi_{i,j}$ act on $\widetilde{H}_{\mu^S}[X;q,t,z]$ by

$$\pi_{\lambda,S}(\widetilde{H}_{\mu^S}[X;q,t,z]) = \sum_{\nu \subseteq R(n,k)} \left(\pi_{\lambda,S} \frac{s_{\tilde{\nu}}[z_{S^c}]}{\prod_{j\in S^c} z_j} \right) \mathrm{I}_{\mu,\nu,k}\,.$$

Finally, we define

$$\widetilde{H}^{\lambda}_{\mu^S} := \Phi\left(\pi_{\lambda,S}(\widetilde{H}_{\mu^S}[X;q,t,z]) \right). \tag{2.2}$$

Now we give a conjectural module theoretic interpretation of $\widetilde{H}^{\lambda}_{\mu^S}$.

Definition 2.2. For $S \in \binom{[n]}{k}$ and a partition $\lambda \in R(n,k)$, denote by $S = \{i_1 < i_2 < \cdots < i_k\}, S^c = \{j_1 < j_2 < \cdots < j_{n-k}\}$. We define $S(\lambda)$ to be a set of $S' \in \binom{[n]}{k}$ satisfying the following that if we denote $S \setminus S' = \{i_{a_1} < i_{a_2} < \cdots < i_{a_r}\}, S' \cap S^c = \{j_{b_1} > j_{b_2} > \cdots > j_{b_r}\}$, then we have $b_s \leq \lambda_{a_s}$ for $1 \leq s \leq r$.

Conjecture 2.3. We have $\widetilde{H}^{\lambda}_{\mu^S} = \mathrm{grFrob}\left(\cap_{S' \in S(\lambda)} R_{\mu^{S'}} \right)$.

Letting λ to be a full rectangle $R(n,k)$ the right-hand side is the Frobenius characteristic of the full intersection. Moreover, setting $k = 1$ and $\lambda = R(n,1)$, Conjecture 2.3 simply reduces to (2.1).

For partitions $\lambda^{(1)} \subset \lambda^{(2)}$, we trivially have $S(\lambda^{(1)}) \subset S(\lambda^{(2)})$, and Conjecture 2.3 implies that $\widetilde{H}^{\lambda^{(1)}}_{\mu^S} - \widetilde{H}^{\lambda^{(2)}}_{\mu^S}$ is Schur positive. Given a sequence $\varnothing = \lambda^{(1)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(\ell)} = R(n,k)$, we may obtain a sequence of symmetric functions that grows from $\widetilde{H}^{R(n,k)}_{\mu^S}$ to $\widetilde{H}^{\varnothing}_{\mu^S} = \widetilde{H}_{\mu^S}$ in a Schur positive sense. This refines the Macdonald positivity.

3 Proof of Theorem 1.2 (a) and (b)

3.1 Technical Lemmas

We begin by recalling several technical lemmas. To state the first lemma, we review the following notations: $M = (1-q)(1-t), B_\mu = \sum_{c\in\mu} q^{\mathrm{coarm}_\mu(c)} t^{\mathrm{coleg}_\mu(c)}, D_\mu = MB_\mu - 1, \tilde{h}_\mu = \prod_{c\in\mu}(q^{\mathrm{arm}_\mu(c)} - t^{\mathrm{leg}_\mu(c)+1}), \tilde{h}'_\mu = \prod_{c\in\mu}(t^{\mathrm{leg}_\mu(c)} - q^{\mathrm{arm}_\mu(c)+1})$. We will use brackets to denote plethystic substitutions. Furthermore, for a symmetric function f, define Π'_f as $\Pi'_f[X;q,t] = \nabla^{-1} f[X-\epsilon]$, and $\mathrm{rev}(f)$ as the q,t-reversal $\mathrm{rev}(f) = f\big|_{q\mapsto q^{-1}, t\mapsto t^{-1}}$. The following lemma [14, Equation (3.4)] provides a formula for the Macdonald polynomial skewed by an elementary symmetric function. The proof of the lemma uses Garsia, Haiman, and Tesler's plethystic formula [5, Theorem I.2].

Lemma 3.1 ([14]). *For a partition $\mu \vdash n$, we have*

$$e_{n-m}^{\perp}\widetilde{H}_{\mu} = (qt)^{m} T_{\mu} \sum_{\lambda \vdash m} \mathrm{rev}\left(\Pi'_{\widetilde{H}_{\lambda}}[D_{\mu}; q, t]\right) \frac{T_{\lambda}\widetilde{H}_{\lambda}}{\tilde{h}_{\lambda}\tilde{h}'_{\lambda}}.$$

We need one more lemma, which follows from the Laplace expansion of the determinant using complementary minors and Jacobi's bi-alternant formula for Schur functions.

Lemma 3.2. *Let $z_1, \ldots, z_n$ be variables. Let λ be a partition contained within the rectangle $R(n,k)$. For a partition μ of size $|\mu| \leq k(n-k) - |\lambda|$,*

$$\sum_{S \in \binom{[n]}{k}} \frac{s_{\lambda}[z_S] s_{\mu}[z_{S^c}]}{\prod_{i \in S, j \in S^c}(z_j - z_i)} = (-1)^{|\lambda|} \delta_{\tilde{\lambda}, \mu}. \tag{3.1}$$

3.2 Proof of Theorem 1.2 (a) and (b)

Let μ be a partition of N with n corners $\{c_1, \ldots, c_n\}$. Fix $0 \leq k \leq n$ and let λ be a partition inside the rectangular partition $R(n,k)$. Let $m = k(n-k) - |\lambda| = |\tilde{\lambda}|$. By the definition of $\mathrm{I}_{\mu,\lambda,k}$, for an integer ℓ,

$$e_{N-k-\ell}^{\perp} \mathrm{I}_{\mu,\lambda,k} = \sum_{S \in \binom{[n]}{k}} \frac{s_{\lambda}[z_S] \prod_{j \in S^c} z_j}{\prod_{i \in S, j \in S^c}(z_j - z_i)} e_{N-k-\ell}^{\perp} \widetilde{H}_{\mu^S}. \tag{3.2}$$

By Lemma 3.1, we have

$$e_{N-k-\ell}^{\perp} \mathrm{I}_{\mu,\lambda,k} = (qt)^{\ell} T_{\mu^{[n]}} \sum_{\nu \vdash \ell} \sum_{S \in \binom{[n]}{k}} \frac{s_{\lambda}[z_S]}{\prod_{i \in S, j \in S^c}(z_j - z_i)} \mathrm{rev}(\Pi'_{\widetilde{H}_{\nu}}[D_{\mu^S}(q,t); q, t]) \frac{T_{\nu}\widetilde{H}_{\nu}}{\tilde{h}_{\nu}\tilde{h}'_{\nu}}. \tag{3.3}$$

Here, we used $T_{\mu^{[n]}} = T_{\mu^S} \prod_{j \in S^c} z_j$. Note that

$$\mathrm{rev}\left(\Pi'_{\widetilde{H}_{\nu}}[D_{\mu^S}(q,t); q, t]\right) = T_{\nu}\widetilde{H}_{\nu}\left[D_{\mu^{[n]}}(q^{-1}, t^{-1}) - \epsilon + \frac{M}{qt} z_{S^c}; q^{-1}, t^{-1}\right]$$

is a polynomial in z_{S^c} of degree $|\nu| = \ell$. If, on the right-hand side of (3.3), the degree of the polynomial in z_{S^c} is less than m, or equivalently if $\ell < m$, then by Lemma 3.2, it vanishes. This proves Theorem 1.2 (a).

Proceed to prove Theorem 1.2 (b). Now let $\ell = m$ in (3.3). Note that the leading term of $\mathrm{rev}\left(\Pi'_{\widetilde{H}_{\nu}}[D_{\mu^S}(q,t); q, t]\right)$ is equal to $T_{\nu}\widetilde{H}_{\nu}\left[\frac{M}{qt} z_{S^c}; q^{-1}, t^{-1}\right] = (qt)^{-m} \omega \widetilde{H}_{\nu}[M z_{S^c}; q, t]$, by the symmetry relation $\omega\widetilde{H}_\mu = T_\mu \operatorname{rev}(\widetilde{H}_\mu)$. Again, by Lemma 3.2, we only need to consider this leading term. Let us rewrite the right-hand side of (3.3) as

$$\begin{aligned} T_{\mu^{[n]}} \nabla & \left(\sum_{\nu\vdash\ell} \sum_{S\in\binom{[n]}{k}} \frac{s_\lambda[z_S]}{\prod_{i\in S, j\in S^c}(z_j - z_i)} \frac{\widetilde{H}_\nu \omega \widetilde{H}_\nu\,[Mz_{S^c}; q,t]}{\tilde{h}_\nu \tilde{h}'_\nu} \right) \\ &= T_{\mu^{[n]}} \nabla \left(\sum_{\rho\vdash\ell} \sum_{S\in\binom{[n]}{k}} \frac{s_\lambda[z_S] s_{\rho'}[z_{S^c}]}{\prod_{i\in S, j\in S^c}(z_j - z_i)} s_{\rho'} \right), \end{aligned}$$

where we used [9, (2.67)]: $s_{\rho'} = \sum_{\nu\vdash|\rho|} \dfrac{\tilde{K}_{\rho,\nu}(q,t)\widetilde{H}_\nu[MX;q,t]}{\tilde{h}_\nu \tilde{h}'_\nu}$. Now, applying Lemma 3.2 again we obtain $T_{\mu^{[n]}} \nabla \left(\sum_{\rho\vdash\ell} \sum_{S\in\binom{[n]}{k}} \frac{s_\lambda[z_S] s_{\rho'}[z_{S^c}]}{\prod_{i\in S, j\in S^c}(z_j - z_i)} s_{\rho'} \right) = T_{\mu^{[n]}} \nabla s_{\tilde{\lambda}}$.

4 Loehr–Warrington formula

4.1 The Loehr–Warrington formula

Throughout this section, we fix a poset $\mathbf{P}$ on $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$ defined as follows. For (a,b), $(c,d) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$ we say $(a,b) \prec_{\mathbf{P}} (c,d)$ in $\mathbf{P}$ if and only if $a+1 < c$ or $a+1 = c$ and $b \geq d$. Otherwise, we write $(a,b) \not\prec_{\mathbf{P}} (c,d)$. We also give a total ordering on $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$ by a (mixed) lexicographic ordering. Define $(a,b) <_{\text{lex}} (c,d)$ if $a < c$, or $a = c$ and $b > d$. Note that the ordering of the second coordinate is reversed.

Consider a tuple $L = (L_1, \ldots, L_r)$ such that each L_i is a finite sub(multi)set of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$. We define the *diagonal inversion* dinv of L by

$$\operatorname{dinv}(L) = \sum_{i<j} \sum_{\substack{(a,b)\in L_i \\ (a',b')\in L_j}} \chi((a,b) <_{\text{lex}} (a',b'))\chi((a,b) \not\prec_{\mathbf{P}} (a',b')).$$

For a diagram D, let T be a filling of D with elements from $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$. Denote $T(i,j)$ to be a filling in the cell $(i,j) \in D$. We use $T(i,j)_1$ (resp. $T(i,j)_2$) to refer to the first (resp. second) entry of $T(i,j)$. We define the area of T as $\operatorname{area}(T) = \sum_{(i,j)\in D} T(i,j)_1$. We use the notation $\operatorname{dinv}(T)$ to denote $\operatorname{dinv}(L_1, \ldots, L_r)$, where L_i represents the set of entries in the i-th column of T.

We fix a partition $\lambda \subseteq R(n,k)$. Let s be the size of the *Durfee square*, which is the maximal number such that $\lambda_s \geq s$. We define a *dinv adjustment* of λ denoted by $\operatorname{adj}(\lambda)$ as $\operatorname{adj}(\lambda) = \sum_{i=1}^{s}(\lambda_i - i)$. Consider a n-vector $(\lambda_1 + k - 1, \lambda_2 + k - 2, \ldots, \lambda_k, k, k+1, \ldots, n-1)$ where we regard $\lambda_i = 0$ if $i > \ell(\lambda)$, and define $v(\lambda)$ to be a vector obtained by sorting entries in the above vector in a weakly increasing order. Then we define the *bottom* of

λ, denoted by $\mathrm{bot}(\lambda)$, as an n-vector given by $\mathrm{bot}(\lambda) = (s+1, s+2, \ldots, s+n) - v(\lambda)$, where the subtraction is performed element-wise. We also define a *pivot* of λ, denoted by $\mathrm{piv}(\lambda)$, to be a vector $(a_1, a_2, \ldots, a_s)$ of length s where a_i is a number satisfying

$$v(\lambda)_{a_i} = v(\lambda)_{a_i+1} = \lambda_{s+1-i} + k - (s+1-i),$$

i.e., the indices where $v(\lambda)$ is non increasing. Lastly we associate a diagram $D(\lambda)$ to the partition λ as

$$D(\lambda) = [[\mathrm{bot}(\lambda)_1, s], [\mathrm{bot}(\lambda)_2, s], \ldots, [\mathrm{bot}(\lambda)_n, s]]$$

and define a set $\mathcal{T}(\lambda)$ consisting of filling T of $D(\lambda)$ satisfying the following conditions:

- $T(i+1, j) \succ_{\mathbf{P}} T(i, j)$ and $T(i, j-1) \not\succ_{\mathbf{P}} T(i, j)$
- for each $j > k - s$ if $j \in \mathrm{piv}(\lambda)$ we have $T(\mathrm{bot}(\lambda)_j, j)_1 > 0$
- for each $j > k - s$ if $j \notin \mathrm{piv}(\lambda)$ we have $T(\mathrm{bot}(\lambda)_j, j)_1 = 0$.

Note that the first condition says that T (after taking appropriate reflection) is a **P**-tableau defined in [7]. The *Loehr–Warrington formula* LW_λ in [17] can be rewritten by

$$\mathrm{LW}_\lambda := q^{\mathrm{adj}(\lambda)} \sum_{T \in \mathcal{T}(\lambda)} q^{\mathrm{dinv}(T)} t^{\mathrm{area}(T)} x^T, \qquad \text{where } x^T = \prod x_{T(i,j)_2} \tag{4.1}$$

4.2 Jacobi–Trudi type formula for the Loehr–Warrington formula

In this section, we provide a Jacobi–Trudi type formula (Proposition 4.3) for LW_λ in terms of the operators $\mathfrak{h}_m$, $\bar{\mathfrak{h}}_m$, and $\hat{\mathfrak{h}}_m$, defined below.

Definition 4.1. We define $\mathbf{C}_m$ to be a set of all **P**-chains $\{(a_1, b_1) \prec_{\mathbf{P}} (a_2, b_2) \prec_{\mathbf{P}} \cdots \prec_{\mathbf{P}} (a_m, b_m)\}$ where each $(a_\ell, b_\ell) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$. Then $\bar{\mathbf{C}}_m$ (resp $\hat{\mathbf{C}}_m$) is a subset of $\mathbf{C}_m$ consisting of elements satisfying $a_1 = 0$ (resp $a_1 > 0$). Obviously, $\mathbf{C}_m = \bar{\mathbf{C}}_m \cup \hat{\mathbf{C}}_m$.

Let $\mathbf{y} = \{y_{i,j}\}_{i \in \mathbb{Z}_{\geq 0}, j \in \mathbb{Z}_{\geq 1}}$ be a set of indeterminates. For a multiset A of elements in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 1}$, let y^A be the monomial whose exponent of $y_{i,j}$ is given by the number of elements (i, j) in A. We define the operators $\mathfrak{h}_m$, $\bar{\mathfrak{h}}_m$, and $\hat{\mathfrak{h}}_m$ acting on a polynomial ring $\mathbb{F}[\mathbf{y}]$ ($\mathbb{F}$ is a ground field containing $\mathbb{C}(q)$) by describing their actions on a monomial as

$$\mathfrak{h}_m \cdot y^A = \sum_{L \in \mathbf{C}_m} q^{\mathrm{dinv}(L,A)} y^{(L,A)}, \quad \bar{\mathfrak{h}}_m \cdot y^A = \sum_{L \in \bar{\mathbf{C}}_m} q^{\mathrm{dinv}(L,A)} y^{(L,A)},$$

$$\hat{\mathfrak{h}}_m \cdot y^A = \sum_{L \in \hat{\mathbf{C}}_m} q^{\mathrm{dinv}(L,A)} y^{(L,A)},$$

where $y^{(L,A)} = y^L y^A$. If $m < 0$ then they are all zero operators.

Lemma 4.2. *We have* $\mathfrak{h}_n\mathfrak{h}_m = \mathfrak{h}_m\mathfrak{h}_n$, $\bar{\mathfrak{h}}_n\bar{\mathfrak{h}}_m = \bar{\mathfrak{h}}_m\bar{\mathfrak{h}}_n$, *and* $\hat{\mathfrak{h}}_n\hat{\mathfrak{h}}_m = \hat{\mathfrak{h}}_m\hat{\mathfrak{h}}_n$

Proof. The proof parallels with [19, Lemma 4.4, Theorem 4.5]. □

For an n by n matrix $W = (W_{i,j})$ whose entry is an operator acting on $\mathbb{F}[\mathbf{y}]$, we define $\det(W) := \sum_{\sigma\in\mathfrak{S}_n}(-1)^{\mathrm{sgn}(\sigma)}W_{\sigma(1),1}W_{\sigma(2),2}\cdots W_{\sigma(n),n}$.

Proposition 4.3. *For integers $k < n$ and a partition $\lambda \subseteq R(n,k)$ we associate an n by n matrix*

$$W(\lambda)_{i,j} = \begin{cases} \mathfrak{h}_{v(\lambda)_j-i+1} & \text{if } j \le k-s \\ \bar{\mathfrak{h}}_{v(\lambda)_j-i+1} & \text{if } j > k-s \text{ and } j \text{ is an entry of } \mathrm{piv}(\lambda) \\ \hat{\mathfrak{h}}_{v(\lambda)_j-i+1} & \text{if } j > k-s \text{ and } j \text{ is not an entry of } \mathrm{piv}(\lambda), \end{cases}$$

where s is the size of the Durfee square of λ. Then we have $q^{-\mathrm{adj}(\lambda)}\,\mathrm{LW}_\lambda = \det(W(\lambda))\cdot 1|_{y_{i,j}=t^i x_j}$

4.3 Reformulation of Loehr–Warrington formula

We reformulate the Loehr–Warrington formula LW_λ from Proposition 4.3. Our main tool is Lemma 4.4, whose proof is given in the full version [15] of this paper. For a matrix A, let $T_{ij}(A)$ be the matrix obtained by moving the j-th column to the position of the i-th column and shifting the ℓ-th column to the $(\ell+1)$-th position for $i < \ell < j$.

Lemma 4.4. *For an integer vector $v = (v_1, v_2, \ldots, v_n)$ of length $n \ge 2$, consider an $n \times n$ matrix $V = (V_{i,j})$ given by* $V_{i,j} = \begin{cases} \hat{\mathfrak{h}}_{v_i+j-1} & \text{if } j < n \\ \bar{\mathfrak{h}}_{v_i+n-1} & \text{if } j = n \end{cases}$. *Then we have* $(-q)^{n-1}\det V = \det T_{1,n}(V)$.

Let s be the largest integer such that $\lambda_s \ge s$. Set $W^{(0)} = W(\lambda)$ as in Proposition 4.3, and construct new matrices recursively as follows:

(Step 1) Given $W^{(i)}$, let $V^{(i+1)} = T_{ab}(W^{(i)})$ where $a = k-s+i$ and $b = \mathrm{piv}(\lambda)_i$.

(Step 2) Add the $(\mathrm{piv}(\lambda)_i+1)$-th column to the $(k-s+i)$-th column in $V^{(i+1)}$ to obtain $W^{(i+1)}$.

By Lemma 4.4, We conclude that $(-q)^{\mathrm{adj}(\lambda)}\det(W(\lambda)) = \det(W^{(s)})$, and

$$W^{(s)}_{i,j} = \begin{cases} \mathfrak{h}_{\lambda_{k+1-j}+j-i} & \text{if } j \le k \\ \hat{\mathfrak{h}}_{j-i} & \text{if } j > k. \end{cases} \tag{4.2}$$

Finally, by Proposition 4.3, we conclude

$$\mathrm{LW}_\lambda = (-1)^{\mathrm{adj}(\lambda)}\det(W^{(s)})\cdot 1|_{y_{i,j}=t^i x_j}. \tag{4.3}$$

5 Proof of Theorem 1.2 (c)

5.1 Macdonald polynomials for filled diagrams and HHL formula

To deal with the combinatorics of Macdonald piece polynomial, we rely on the celebrated result of Haglund–Haiman–Loehr [10, 11]. For a diagram D, and cells $u = (i, j)$ and $v = (i', j')$ of D, we say that a pair (u, v) is an *attacking pair* if either

$$(1)\ i = i' \text{ and } j < j' \qquad \text{or} \qquad (2)\ i = i' + 1 \text{ and } j > j'.$$

For a word $\sigma \in \mathbb{Z}_{\geq 1}^{|D|}$, we consider a pair (u, v) of cells in D to be an *inversion pair* of σ if (u, v) is an attacking pair and $\sigma(u) > \sigma(v)$. We define $\mathrm{inv}_D(\sigma) := q^{|\mathrm{Inv}_D(\sigma)|}$, where $\mathrm{Inv}_D(\sigma)$ is the set of inversion pairs of σ. A cell u is a *descent* of σ if $\sigma(u) > \sigma(v)$, where the cell v is the cell right below u, i.e. $u = (i, j)$ and $v = (i - 1, j)$. Define $\mathrm{Des}_D(\sigma)$ to be the set of descents of σ.

A *filling* of D is a function $f : D \to \mathbb{F}$, where we take $\mathbb{F}$ as a field containing $\mathbb{C}(q, t)$. Then a *filled diagram* (D, f) is a pair of a diagram and a filling on it. We define $\mathrm{maj}_{(D,f)}(\sigma)$ as the product of $f(u)$ over all cells u which are descents of σ, i.e., $\mathrm{maj}_{(D,f)}(\sigma) := \prod_{u \in \mathrm{Des}_D(\sigma)} f(u)$. Finally, $\mathrm{stat}_{(D,f)}(w)$ is defined by the product of inv_D and $\mathrm{maj}_{(D,f)}$, $\mathrm{stat}_{(D,f)}(\sigma) := \mathrm{inv}_D(\sigma)\,\mathrm{maj}_{(D,f)}(\sigma)$. In [13], the *(generalized) modified Macdonald polynomial* of a filled diagram (D, f) is defined by $\widetilde{H}_{(D,f)} := \sum_{\sigma \in \mathbb{Z}_{\geq 1}^{|D|}} \mathrm{stat}_{(D,f)}(\sigma) x^{\sigma}$. Indeed, this generalizes the celebrated HHL formula [10], which gives the combinatorial formula for the modified Macdonald polynomial.

5.2 Deformation of filled diagrams

By the shape independence in Theorem 1.2 (b), to prove Theorem 1.2 (c), it suffices to consider a specific partition μ of n corners. We fix μ to be the $\delta = \delta_{n,N} := (n^N, n - 1, \ldots, 1)$ for a large enough N ($N > |\lambda|$ suffices).

Let the corners of δ be indexed by $c_1, c_2, \ldots, c_n$ from top to bottom. Additionally, for a k-subset $S = \{i_1 < i_2 < \cdots < i_k\}$ of $[n]$, let δ^S denote the partition obtained from δ by deleting the corners c_i's for $i \in S$. We rearrange columns of δ^S by moving the $i_1, i_2, \ldots, i_k$-th columns all the way to the left in this order and denote by β. We let f_β^{st} be the 'standard filling' in [11], or the filling obtained by the column exchange rule [13, Proposition 5.3]. Now we define $(\delta_S, f_S) := \mathrm{Cyc}^k(\beta, f_\beta^{\mathrm{st}})$ (the diagram δ_S should not be confused with δ^S). Here Cyc is the cycling operator (cf. [13, Lemma 3.6]). We may use (δ_S, f_S) for the computation of $\widetilde{H}_{\delta^S}$ i.e. $\widetilde{H}_{\delta^S} = \widetilde{H}_{(\delta_S, f_S)}$.

Now, we introduce indeterminates $z_1, z_2, \ldots$, and define (δ_S, f_S^z) which is a z-deformation of (δ_S, f_S). Let $S^c = \{j_1 < j_2 < \cdots < j_{n-k}\}$. Then the filling f_S^z is given by

$$f_S^z(a, b) = q\frac{z_{N+n+1-a}}{z_{j_b}} \quad \text{for } b \leq n - k,$$

$$f_S^z(a, n-k+b) = \begin{cases} \frac{z_{N+n+1-a}}{z_{i_b}} & \text{if } N+n+1-a \in S^c \\ q\frac{z_{N+n+1-a}}{z_{i_b}} & \text{otherwise} \end{cases} \qquad \text{for } b \leq k.$$

A straightforward calculation shows that under the specialization given by

$$z_j = \begin{cases} q^{1-j}t^{j+1-N-n} & \text{if } j \leq n \\ q^{-n}t^{j+1-N-n} & \text{if } j > n \end{cases} \tag{5.1}$$

(δ_S, f_S^z) recovers (δ_S, f_S). Lastly for a positive integer $m \leq N$, we define $(\mathrm{Rec}^{\leq m}, g_S^z)$ to be a sub filled diagram of (δ_S, f_S^z) obtained by restricting to the first m rows. Note that the diagram $\mathrm{Rec}^{\leq m}$ is independent of S.

5.3 Reducing to the $\mathrm{Rec}^{\leq N}$

In this section we compute $\left(e^{\perp}_{|\delta|-k-|\lambda|} \mathrm{I}_{\delta,\tilde{\lambda},k}\right)$ starting from filled diagrams (δ_S, f_S) for the computation of $\widetilde{H}_{\delta^S}$. Given a diagram D and a positive integer $\ell \leq |D|$, we define $\mathrm{Sub}(D, \ell)$ to be the set of assignments of ℓ cells of D by positive integers. We think of $R \in \mathrm{Sub}(D, \ell)$ as a function $R : D' \to \mathbb{Z}_{\geq 1}$ for some subdiagram $D' \subseteq D$ with $|D'| = \ell$. We also define an induced full assignment $\bar{R} : D \to \mathbb{Z}_{\geq 1}$ given by assigning $1, 2, \ldots, |D| - \ell$ for cells in $D \setminus D'$ in a reverse reading order, and by letting $\bar{R}(c) = R(c) + |D| - \ell$ for cells $c \in D'$. We let $x^R = \prod_{c\in D'} x_{R(c)}$ and lastly for a filled diagram (D, f), we abuse the notation for $\mathrm{stat}_{(D,f)}(R)$ instead of $\mathrm{stat}_{(D,f)}(\bar{R})$.

Let $D = \mathrm{Rec}^{\leq m}$ for some $m \leq N$ and consider $R \in \mathrm{Sub}(D, \ell)$. We say that a cell $c = (i, j) \in D'$ is *bad* if $c' = (i+1, j) \in \mathrm{Des}_D(\bar{R})$ where $\bar{R}$ is the associated full assignments. We define $\gamma(R)$ to be a vector whose i-th entry is the number of bad cells in the i-th row. By employing delicate arguments, as outlined in the full version of this paper [15, Section 6.2], one can show the following.

Lemma 5.1. *The following quantity specializes to $e^{\perp}_{|\delta|-k-|\lambda|} \mathrm{I}_{\delta,\tilde{\lambda},k}$ under the specialization (5.1).*

$$(-1)^{|\tilde{\lambda}|} \frac{\mathrm{stat}_{(\delta_S, f_S^z)}(R)}{\mathrm{stat}_{(\mathrm{Rec}^{\leq N}, g_S^z)}(R)} \sum_{S\in\binom{[n]}{k}} \frac{s_{\tilde{\lambda}}[z_S]\prod_{j\in S^c} z_j}{\prod_{\substack{i\in S\\ j\in S^c}}(z_j - z_i)} \sum_{\substack{R\in\mathrm{Sub}(\mathrm{Rec}^{\leq N}, |\lambda|)\\ \gamma(R)=0}} \mathrm{stat}_{(\mathrm{Rec}^{\leq N}, g_S^z)}(R) x^R \tag{5.2}$$

5.4 Toward the Jacobi–Trudi type formula

Lemma 5.2. *Let $\lambda \subseteq R(n,k)$, and $P(z_1, \ldots, z_n)$ be a polynomial of degree $|\lambda|$ that is symmetric in variables $z_1, z_2, \ldots, z_{n-k}$ and symmetric in variables $z_{n-k+1}, z_{n-k+2}, \ldots, z_n$. Then we have*

$$\sum_{S\in\binom{[n]}{k}} \frac{s_{\tilde{\lambda}}[z_S]P(z_{S^c}, z_S)}{\prod_{\substack{i\in S\\ j\in S^c}}(z_j - z_i)} = (-1)^{|\tilde{\lambda}|} \sum_{\sigma\in\mathfrak{S}_n} (-1)^{\mathrm{sgn}(\sigma)} [\prod_{i=1}^{n} z_i^{V_{\sigma(i),i}}] P(z_1, z_2, \ldots, z_n)$$

where $[z_1^{c_1} z_2^{c_2} \cdots z_n^{c_n}] P(z_1, z_2, \ldots, z_n)$ *represents the operation of taking the coefficient and*

$$V_{i,j} = \begin{cases} (\lambda_{n-k+1-j}) + j - i & \text{if } j \le n-k \\ j - i & \text{if } j > n-k \end{cases}.$$

Note that we can write (5.2) as

$$(-1)^{|\tilde{\lambda}|} \operatorname{stat}_{(\delta_S, f_S^z)}(R) \times \sum_{S \in \binom{[n]}{k}} \frac{s_{\tilde{\lambda}}[z_S] B_S}{\prod_{\substack{i \in S \\ j \in S^c}} (z_j - z_i)}, \tag{5.3}$$

where $B_S = \frac{1}{\operatorname{stat}_{(\mathrm{Rec}^{\le N}, g_S^z)}(w_0)} \sum_{\substack{R \in \mathrm{Sub}(\mathrm{Rec}^{\le N}, |\lambda|) \\ \gamma(R)=0}} \operatorname{stat}_{(\mathrm{Rec}^{\le N}, g_S^z)}(R) x^R$. To apply Lemma 5.2 to (5.3), we now describe a monomial coefficient of $B = B_{[n-k+1,n]}$. For $R \in \mathrm{Sub}(\mathrm{Rec}^{\le N}, |\lambda|)$ denoted by $R : D \to \mathbb{Z}_{\ge 1}$, we associate $\eta(R) = (L_1, L_2, \ldots, L_n)$ where L_i is a subset of $\mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 1}$ given by: $(c, d) \in L_i$ if and only if $(c+1, i) \in D$ and $R(c+1, i) = d$.

Lemma 5.3. *For* $R \in \mathrm{Sub}(\mathrm{Rec}^{\le N}, |\lambda|)$ *with* $\gamma(R) = 0$, $\alpha = \mathrm{rsum}(R)$, *and* $\beta = \mathrm{csum}(R)$,

1. *denoting* $\eta(R) = (L_1, L_2, \ldots, L_n)$, *we have* $L_i \in \mathbf{C}_{\beta_i}$ *if* $i \le n-k$ *and* $L_i \in \hat{\mathbf{C}}_{\beta_i}$ *if* $i > n-k$
2. *we have* $\operatorname{stat}_{(\mathrm{Rec}^{\le N}, g_S^z)}(R) = \frac{\operatorname{stat}_{(\mathrm{Rec}^{\le N}, g_S^z)}(w_0)}{q^{n|\lambda|} \prod_{i=1}^{N-1} z_{N+n-i}^{\alpha_i}} q^{\operatorname{dinv}(\eta(R))} \prod_{i=1}^n z_i^{\beta_i}$ *for* $S = \{n-k+1, n-k+2, \ldots, n\}$.

We let Φ be an evaluation map given by: $y_{i,j} \to z_{N+n-i-1}^{-1} x_j$ if $0 \le i \le N-2$ and $y_{i,j} \to 0$ if $i > N-2$. By Lemma 5.3, we have $[\prod_{i=1}^n z_i^{\beta_i}] B = \frac{1}{q^{n|\lambda|}} \Phi\left(\prod_{i=1}^{n-k} \mathfrak{h}_{\beta_i} \prod_{i=n-k+1}^{n} \hat{\mathfrak{h}}_{\beta_i} \cdot 1\right)$. By Lemma 5.2, (5.3) equals to $\frac{\operatorname{stat}_{(\delta_S, f_S^z)}(R)}{q^{n|\lambda|}} \Phi\left(\det(W) \cdot 1\right)$, where $W = W^{(s)}$ (k is replaced by $n-k$) is defined in (4.2).

Proof of Theorem 1.2 (c). In the expansion $\det(W) \cdot 1$, a variable $y_{i,j}$ for $i > N-2$ does not appear as N is sufficiently large. We may only consider the specialization $y_{i,j} \to z_{N+n-i-1}^{-1} x_j$ and again specializing with (5.1), gives $y_{i,j} \to q^n t^i x_j$. Note that $\det(W) \cdot 1$ is of (homogenous) degree $|\lambda|$ in $\mathbf{y}$'s, comparing with (4.3), we have $\frac{\operatorname{stat}_{(\delta_S, f_S^z)}(R)}{q^{n|\lambda|}} \Phi\left(\det(W) \cdot 1\right)$ specialize by (5.1) $\frac{T_{\delta[n]} (q^n)^{|\lambda|}}{q^{n|\lambda|}} \left(\det(W) \cdot 1|_{y_{i,j} = t^i x_j}\right) = T_{\delta[n]} (-1)^{\operatorname{adj}(\lambda)} \mathrm{LW}_\lambda$. Together with Lemma 5.1, the proof is complete. □

References

[1] F. Bergeron and A. M. Garsia. "Science fiction and Macdonald's polynomials". CRM Proc. Lecture Notes **22** (1999), pp. 1–52. DOI.

[2] J. Blasiak, M. Haiman, J. Morse, A. Pun, and G. Seelinger. "Dens, nests and the Loehr-Warrington conjecture" (2021). arXiv:2112.07070.

[3] J. Blasiak, J. Morse, A. Pun, and D. Summers. "Catalan functions and k-Schur positivity". *J. Amer. Math. Soc.* **32**.4 (2019), pp. 921–963. DOI.

[4] L. M. Butler. "Subgroup lattices and symmetric functions". *Mem. Amer. Math. Soc.* **112**.539 (1994), pp. vi+160. DOI.

[5] A. M. Garsia, M. Haiman, and G. Tesler. "Explicit plethystic formulas for Macdonald q,t-Kostka coefficients". Vol. 42. The Andrews Festschrift (Maratea, 1998). 1999, Art. B42m, 45.

[6] A. M. Garsia and M. Haiman. "A graded representation model for Macdonald's polynomials". *Proc. Nat. Acad. Sci. U.S.A.* **90**.8 (1993), pp. 3607–3610. DOI.

[7] V. Gasharov. "Incomparability graphs of $(3+1)$-free posets are s-positive". *Discrete Math.* **157**.1-3 (1996), pp. 193–197. DOI.

[8] I. Grojnowski and M. Haiman. "Affine Hecke algebras and positivity of LLT and Macdonald polynomials". preprint. 2007.

[9] J. Haglund. "A proof of the q,t-Schröder conjecture". *Int. Math. Res. Not.* 11 (2004), pp. 525–560. DOI.

[10] J. Haglund, M. Haiman, and N. Loehr. "A combinatorial formula for Macdonald polynomials". *J. Amer. Math. Soc.* **18**.3 (2005), pp. 735–761. DOI.

[11] J. Haglund, M. Haiman, and N. Loehr. "A combinatorial formula for nonsymmetric Macdonald polynomials". *Amer. J. Math.* **130**.2 (2008), pp. 359–383. DOI.

[12] M. Haiman. "Hilbert schemes, polygraphs and the Macdonald positivity conjecture". *J. Amer. Math. Soc.* **14**.4 (2001), pp. 941–1006. DOI.

[13] D. Kim, S. J. Lee, and J. Oh. "Toward Butler's conjecture". 2022. arXiv:2212.09419.

[14] D. Kim, S. J. Lee, and J. Oh. "Shuffle formula in science fiction for Macdonald polynomials". 2023. arXiv:2306.14371.

[15] D. Kim and J. Oh. "Extending the science fiction and the Loehr–Warrington formula". 2024. arXiv:2409.01041.

[16] L. Lapointe, A. Lascoux, and J. Morse. "Tableau atoms and a new Macdonald positivity conjecture". *Duke Math. J.* **116**.1 (2003), pp. 103–146. DOI.

[17] N. A. Loehr and G. S. Warrington. "Nested quantum Dyck paths and $\nabla(s_\lambda)$". *Int. Math. Res. Not. IMRN* 5 (2008), Art. ID rnm 157, 29 pp. DOI.

[18] I. Macdonald. "A new class of symmetric functions." *Séminaire Lotharingien de Combinatoire* **20** (1988), B20a–41.

[19] J. Shareshian and M. L. Wachs. "Chromatic quasisymmetric functions". *Adv. Math.* **295** (2016), pp. 497–551. DOI.