Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #21, 12 pp.

Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

A Combinatorial Model for Affine Demazure Crystals of Levels Zero and One

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Abstract. Certain affine crystals, known as Kirillov–Reshetikhin (KR) crystals, were realized in a uniform way (for all untwisted affine types) in terms of the quantum alcove model, and their graded characters were shown to coincide with the specialization of symmetric Macdonald polynomials at $t = 0$. We generalize these results by introducing a "non-symmetric" quantum alcove model for certain level zero Demazure-type crystals inside tensor products of KR crystals, whose graded characters coincide with non-symmetric Macdonald polynomials at $t = 0$. Moreover, in type A, we construct an explicit affine crystal isomorphism between the non-symmetric quantum alcove model and the semi-standard key tabloid model for level one affine Demazure crystals due to Assaf and González.

Keywords: Kirillov–Reshetinkhin crystals, Demazure crystals, Macdonald polynomials, quantum alcove model, semistandard key tabloids.

1 Introduction

Certain representations of Lie algebras $\mathfrak{g}$ (including the highest weight ones) possess *crystal bases*, when viewed as representations of the quantum algebra $U_q(\mathfrak{g})$ [10]. In the limit $q \to 0$, the structure of a crystal basis is encoded in a colored directed graph on the basis elements, called a *crystal graph*. The directed i-edges give the action of the *Kashiwara operators* (analogues of the Chevalley generators) as $q \to 0$. Crystal graphs have various combinatorial models, and are useful tools in representation theory.

Kirillov–Reshetikhin (KR) modules [12] are finite-dimensional modules $W^{r,s}$ for affine Lie algebras, not of highest weight, labeled by a positive integer multiple $s\omega_r$ of the corresponding finite-type fundamental weight ω_r. We refer to the KR modules $W^{r,1}$ as single-column ones. In most cases, $W^{r,s}$ was shown to have a crystal basis, and the crystal graph is denoted $B^{r,s}$. The latter are building blocks for the highest weight crystals, constructed as infinite tensor products of certain $B^{r,s}$.

In [17] it was shown that the *quantum alcove model*, constructed in [15], is a uniform model for tensor products of single-column KR crystals, for all untwisted affine types.

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The quantum alcove model generalizes the alcove model in [14], which describes the highest weight crystals of symmetrizable Kac–Moody algebras. These models are based on enumeration of certain paths in the quantum Bruhat graph (resp. the Hasse diagram of the Bruhat order) for the finite Weyl group.

In classical types there are (type-specific) models for tensor products of KR crystals $B^{r,1}$ based on concatenations of fillings of columns called *Kashiwara–Nakashima (KN) columns* [11]. While these tableau models are simpler than the alcove models, they have less easily accessible information, so it is hard to use them in computations, for instance of the energy function, and the combinatorial R-matrix (see Section 2.2). As these computations are much simpler in the quantum alcove model [17], an alternative is to relate them to the tableau models, via affine crystal isomorphisms from the former model to the latter one. Such maps, to be called "alcoves-to-fillings" maps, were constructed in types A, C in [15], and in types B, D in [4].

A corollary of the realization of tensor products of KR crystals in terms of the quantum alcove model is their connection with Macdonald polynomials (of the corresponding untwisted affine type). The symmetric Macdonald polynomials $P_\lambda(x; q, t)$, indexed by dominant weights λ, are Weyl group invariant polynomials with rational function coefficients in parameters q, t [18]. They generalize the irreducible characters of the corresponding simple Lie algebras, which are recovered for $q = t = 0$. A uniform combinatorial formula for $P_\lambda(x; q, t)$ was given by Ram and Yip [19] in terms of *alcove walks*. By comparing with the quantum alcove model, it was shown in [17] that the specialization $P_\lambda(x; q, t = 0)$ coincides with the graded character of a tensor product of single-column KR modules, graded by the energy function.

There are also non-symmetric versions of Macdonald polynomials $E_\mu(x; q, t)$, where μ is an arbitrary weight [18], as well as a Ram–Yip formula for them. Specialized non-symmetric Macdonald polynomials are related to Demazure characters of affine Kac–Moody algebras, as discussed in [9]. Later it was shown that $E_\mu(x; q, t = 0)$ coincides with the graded character of a Demazure-type submodule of a tensor product of single-column KR modules [13]. In the mentioned paper, the latter can be thought of as a Demazure submodule of a certain quotient of a *level zero extremal weight module*. The corresponding crystal is called a *DARK crystal* (**K**irillov–**R**eshetikhin **A**ffine **D**emazure) in [3]. Thus, we expect a combinatorial model for the mentioned DARK crystal based on the alcove walks in the non-symmetric Ram–Yip formula specialized at $t = 0$. This is the first main result of the paper, a "non-symmetric version" of the quantum alcove model.

On the other hand, in type A, $E_\mu(x; q, t = 0)$ can be computed in terms of *semistandard key tabloids*. These were introduced in [2] as special cases of the *non-attacking skyline fillings* in the *Haglund–Haiman–Loehr formula* for non-symmetric Macdonald polynomials [7]. They were given an affine crystal structure in [1], which realizes the crystals of level one Demazure modules for the affine Lie algebra $\widehat{\mathfrak{sl}_n}$. The second main result of our paper is an explicit affine crystal isomorphism between the non-symmetric quantum alcove model in type A and semistandard key tabloids, i.e., an "alcoves-to-fillings" map in the type A Demazure case.

The last result is expected to generalize to the other classical types B, C, and D. We will construct alcoves-to-fillings correspondences which would extend the maps in [4, 15] mentioned above to the Demazure case. These constructions will combine those in [4, 15] with the ones in the current paper. In particular, we will derive the type B, C, and D analogues of the semistandard key tabloids, which will be Demazure-type analogues of the corresponding concatenations of KN columns.

2 Background

2.1 Root systems

Let $\mathfrak{g}$ be a complex simple Lie algebra, and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. Let $\Phi^+ \subset \Phi \subset \mathfrak{h}^*_{\mathbb{R}}$ be the corresponding irreducible root system and choice of positive roots. For a root $\alpha \in \Phi^+$, we use the notation $\alpha > 0 \Leftrightarrow \operatorname{sgn}(\alpha) = 1$. Let α_i be a choice of *simple roots* for i in some indexing set I, and let θ be the highest root with respect to the simple roots. We denote by $\langle \cdot, \cdot \rangle$ the nondegenerate scalar product on $\mathfrak{h}^*_{\mathbb{R}}$ induced by the Killing form. Given a root α, we have the *coroot* $\alpha^\vee := 2\alpha/\langle \alpha, \alpha \rangle$ and reflection s_α. The *weight lattice* P is generated by the *fundamental weights* ω_i for $i \in I$, which satisfy $\langle \omega_j, \alpha_i^\vee \rangle = \delta_{i,j}$. The set of dominant weights is denoted P^+. Define $\rho := \sum_{i \in I} \omega_i$, and for a root α, set $|\alpha| := \operatorname{sgn}(\alpha)\alpha$.

Let W be the finite Weyl group, generated by $s_i := s_{\alpha_i}$. With respect to these generators, W comes with a length function $\ell(\cdot)$ and *longest element* $w_\circ$. The Bruhat order on W is defined as the transitive closure of its covers: $w \lessdot ws_\alpha$ if $\ell(w) + 1 = \ell(ws_\alpha)$.

For a root $\alpha \in \Phi^+$ and an integer $k \in \mathbb{Z}$, consider the reflection $s_{\alpha,k}$ in the affine hyperplane $H_{\alpha,k} := \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} | \langle \lambda, \alpha^\vee \rangle = k\}$. The *affine Weyl group* W_{aff} for the dual root system $\Phi^\vee$ is generated by such reflections. The hyperplanes $H_{\alpha,k}$ divide $\mathfrak{h}^*_{\mathbb{R}}$ into connected components called *alcoves*. The *fundamental alcove* is denoted $A_\circ$. An important object in defining the quantum alcove model is a certain directed graph on W.

Definition 2.1. *The* quantum Bruhat graph *of W is the directed graph with vertex set W and edges $w \xrightarrow{\beta} ws_\beta$ labeled by $\beta \in \Phi^+$ if:*

(1) $w \lessdot ws_\beta$, *or*

(2) $\ell(ws_\beta) = \ell(w) - 2\langle \beta^\vee, \rho \rangle + 1$.

2.2 Kirillov–Reshetikhin crystals

Given a semisimple or affine Lie algebra $\mathfrak{g}$ with simple roots α_i for $i \in I$, a $\mathfrak{g}$-*crystal* is a nonempty set B together with maps $e_i, f_i : B \to B \sqcup \{0\}$ for $i \in I$ and $\operatorname{wt} : B \to P$

satisfying $f_i(b) = b'$ if and only if $e_i(b') = b$ for $b \in B$. The maps e_i, f_i are called crystal operators and they give B the structure of a colored directed graph with edges $b \xrightarrow{i} f_i(b)$. For $\mathfrak{g}$-crystals B_1 and B_2, the tensor product $B_1 \otimes B_2$ is defined on the set $B_1 \times B_2$ with a rule that determines on which factor e_i and f_i act [8].

Kirillov–Reshetikhin (KR) modules, denoted $W^{r,s}$, are finite-dimensional modules, not of highest weight, of affine Lie algebras. They are indexed by positive integer multiples of a fundamental weight $s\omega_r$. In classical types $s\omega_r$ corresponds to a rectangle of height r and length s. In the case $s = 1$, the modules are called "single-column" KR modules. In most cases, $W^{r,s}$ was shown to have a crystal basis, and the corresponding *Kirillov–Reshetikhin crystal* is denoted $B^{r,s}$. For a composition $\mathbf{a} = (a_1, a_2, \ldots, a_k)$, tensor products of column-shape KR crystals for untwisted types are defined as follows:

$$B = B^{\mathbf{a}} := \bigotimes_{i=1}^{k} B^{a_i,1} \,. \tag{2.1}$$

The crystal B is known to be connected as an affine crystal, but disconnected as a classical crystal. If $\mathbf{a'}$ is a permutation of $\mathbf{a}$, then there is a unique crystal isomorphism between $B^{\mathbf{a}}$ and $B^{\mathbf{a}'}$, called he *combinatorial R-matrix*.

Let $\widehat{\mathfrak{g}}$ be an affine Lie algebra and let $\lambda \in P^+$ be a dominant weight for $\mathfrak{g}$. Consider the *level zero extremal weight module* $V(\lambda)$ over the quantum group $U_q(\widehat{\mathfrak{g}})$, generated by a vector v_λ of weight λ. For a Weyl group element w, the *Demazure submodule* $V_w(\lambda)$ is given by $V_w(\lambda) := U_q^+(\mathfrak{g}) \cdot wv_\lambda$.

Beck and Nakajima introduced a subtle finite-dimensional quotient of $V_{w_\circ}(\lambda)$, denoted $U^+_{w_\circ}(\lambda)$ (see [13, Section 3.3]). As a $U_q(\mathfrak{g})$-module, $U^+_{w_\circ}(\lambda)$ is isomorphic to a tensor product of KR modules $W^{i,1}$. The image of the level zero Demazure module $V^+_w(\lambda)$ under the projection to $U^+_{w_\circ}(\lambda)$ is denoted $U^+_w(\lambda)$ and is a Demazure-type submodule.

2.3 The quantum alcove model

Fix a dominant weight $\lambda \in P^+$. The quantum alcove model depends on a sequence of roots $\Gamma := (\beta_1, \beta_2, \cdots, \beta_m)$ called a λ-chain [15]. This is equivalent to a shortest sequence of adjacent alcoves from the fundamental alcove $A_\circ$ to the translate $A_\circ - \lambda$. Define the reflection $r_i := s_{\beta_i}$.

Definition 2.2. *A subset $J = \{j_1, j_2, \ldots, j_s\} \subseteq [m]$ is called* admissible *if there is a path in the quantum Bruhat graph*

$$1 \to r_{j_1} \to r_{j_1} r_{j_2} \to \cdots \to r_{j_1} r_{j_2} \cdots r_{j_s} := end(J).$$

Denote by $\mathcal{A}(\Gamma)$ the set of admissible subsets of J.

Theorem 2.3 ([15, 17]). *Let $\lambda = \omega_{i_1} + \omega_{i_2} + \cdots + \omega_{i_k}$ be a dominant weight of a untwisted affine Lie algebra. Then $\mathcal{A}(\Gamma)$ with the proper crystal operators e_i, f_i is a model for the tensor product of KR crystals $B^\lambda := B^{i_1,1} \otimes B^{i_2,1} \otimes \cdots \otimes B^{i_k,1}$.*

Let $W_\lambda \subset W$ be the stabilizer of λ. Given a Weyl group element u, we denote by $\lfloor u \rfloor$ the lowest coset representative of uW_λ.

Theorem 2.4 ([13]). *The set of admissible subsets $J \in \mathcal{A}(\Gamma)$ satisfying the condition $end(J) \leq \lfloor w \rfloor$ is a model for the DARK crystal corresponding to the Demazure-type module $U_w^+(\lambda)$.*

Based on the quantum alcove model, the following is derived in [13, 17]:

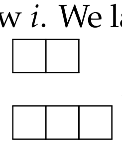
Theorem 2.5. *Let $\widehat{\mathfrak{g}}$ be an untwisted affine Lie algebra and $\lambda \in P^+$ a dominant weight of $\mathfrak{g}$.*

1. *The symmetric Macdonald polynomial $P_\lambda(x; q, t = 0)$ is a graded character of the tensor product of column-shape KR crystals B^λ, so it is expressed by $\mathcal{A}(\Gamma)$.*

2. *The nonsymmetric Macdonald polynomial $E_{w\lambda}(x; q, t = 0)$ is a graded character of the Demazure-type crystal corresponding to $U_w^+(\lambda)$, so it is expressed by the subset of $\mathcal{A}(\Gamma)$ defined in Theorem 2.4.*

2.4 Semistandard key tabloids and their crystal structure

In this section we describe the model defined in [1] for level one affine Demazure crystals in type A_{n-1}. The objects are fillings of diagrams subject to conditions on pairs and triples of entries. A *diagram* dg(μ) of the weak composition $\mu = (\mu_1, \mu_2, \ldots \mu_n)$ consists of μ_i boxes left-justified in row i. We label the box in row i column j as (i, j). For example, if $\mu = (3, 0, 2)$ then $dg(\mu) =$



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Definition 2.6. *A* filling *is an assignment of a positive integer $m_{(i,j)} \leq n$ to each box (i, j).*

Definition 2.7. *Given a diagram dg(μ), we say two boxes are* attacking *if they are in the same column or they are in adjacent columns with the left box strictly higher than the right box. A filling is* non-attacking *if every pair of attacking boxes have distinct entries and no entry in the first column exceeds its row index.*

Definition 2.8. *A* triple *is a collection of three boxes with two row adjacent and either (Type I) the third cell is above the left and the lower row is strictly longer, or (Type II) the third cell is below the right and the higher row is weakly longer. The orientation of a triple is determined by reading the entries of the boxes from smallest to largest. A* coinversion triple *is a Type I triple oriented counterclockwise or a Type II triple oriented clockwise.*

Definition 2.9 ([1])**.** *The* semi-standard key tabloids *of shape μ, denoted* SSKD(μ)*, are the non-attacking fillings of $dg(\mu)$ with no coinversion triples.*

Example 2.10. *The filling*

$$\begin{array}{|c|c|c} \hline 3 & 2 & \multicolumn{1}{|c}{} \\ \hline \\ \hline 1 & 1 & \multicolumn{1}{|c|}{3} \\ \hline \end{array}$$

is a semi-standard key tabloid of shape $\mu = (3, 0, 2)$.

The *weight* of a tabloid T, denoted $\mathrm{wt}(T)$, is defined as the weak composition with i-th part equal to the number of entries i in T. For example $\mathrm{wt}(T) = (2, 1, 2)$ in the example above. Assaf and González defined raising and lowering operators e_i, f_i for $i = 0, 1, \cdots, n-1$, which act on T by swapping certain entries i and $i+1$ (1 and n for $i = 0$). The affected entries are determined by a pairing rule [1, Definitions 3.1-3.6]. The operators e_i, f_i, and the map wt give an affine Demazure crystal structure on SSKD(μ).

Theorem 2.11 ([1])**.** *The above crystal structure on semi-standard key tabloids of shape μ is isomorphic to a certain level one Demazure crystal of the affine Lie algebra $\widehat{\mathfrak{sl}}_n$.*

Theorem 2.12 ([7])**.** *The type A specialized non-symmetric Macdonald polynomial is given by*

$$E_\mu(x; q, t = 0) = \sum_{T \in SSKD(\mu)} q^{\mathrm{maj(T)}} x^{\mathrm{wt}(T)} \;, \tag{2.2}$$

where maj(T) is the sum of all legs of boxes (i, j) of T with $m_{(i,j)} < m_{(i,j+1)}$.

Remark 2.13. The level zero and level one Demazure crystals discussed above are known to be isomorphic. [5].

3 Nonsymmetric quantum alcove model

We introduce a new model for the crystal of the Demazure-type module $U_w^+(\lambda)$. The objects of the model are non-symmetric analogues of admissible subsets as in Definition 2.2. This model has two benefits over the original quantum alcove model: 1) of removing the condition $\mathrm{end}(J) \leq \lfloor w \rfloor$ in Theorem 2.4, and 2) of being compatible with a tableau model for a type A level one affine Demazure crystal, to be discussed in Section 4.

3.1 w-admissible subsets

Given a weight $\mu \in P$, let $w \in W$ be the element of maximal length such that $\mu = w\lambda$ for a dominant weight λ. Consider a reduced alcove path Γ from $A_\circ$ to $ww_\circ A_\circ + \mu$:

$$\Gamma := A_\circ \xrightarrow{-\gamma_1} A_1 \xrightarrow{-\gamma_2} A_2 \xrightarrow{-\gamma_3} \cdots \xrightarrow{-\gamma_m} A_m = ww_\circ A_\circ + \mu. \tag{3.1}$$

The sequence of crossed hyperplanes is given by $H_{|\gamma_k|,m_k}$ where m_k is defined by this relation. To describe the objects of the model, we need another alcove path Γ' obtained from Γ by applying $-w^{-1}$ on the left:

$$\Gamma' = -w^{-1}\Gamma = -w^{-1}A_\circ \xrightarrow{-\beta_1} A'_1 \xrightarrow{-\beta_2} A'_2 \xrightarrow{-\beta_3} \cdots \xrightarrow{-\beta_m} A'_m = A_\circ - \lambda. \tag{3.2}$$

Note that in this case all the roots β_i are positive. Let $J = \{j_1 < j_2 < \cdots < j_s\} \subseteq [m]$ and define the reflections $r_{j_i} := s_{\gamma_{j_i}}$ for $i = 1, 2, \ldots, s$. We can interpret J as a "folding" of the alcove path Γ, resulting in a "folded" alcove path $\Gamma(J) := (\delta_1, \delta_2, \ldots, \delta_m)$, where

$$\delta_k := r_{j_1} r_{j_2} \cdots r_{j_i}(\gamma_k). \tag{3.3}$$

Here $i = i(k)$ is the largest index for which $j_i < k$. After folding Γ, the k-th hyperplane crossed in the path $\Gamma(J)$ is given by:

$$H_{|\delta_k|, l_k^J} := \widetilde{r}_{j_1} \cdots \widetilde{r}_{j_i} H_{|\gamma_k|,m_k} \tag{3.4}$$

where $\widetilde{r}_k := s_{\gamma_k,m_k}$ and the integers l_k^J are defined by this relation. The objects of our model are subsets $J \subset [m]$ satisfying the following condition:

Definition 3.1. *We say $J = \{j_1 < j_2 < \cdots < j_s\} \subset [m]$ is w-*admissible *and write $J \in \mathcal{A}(\Gamma')$ if there is a path ending at w in the quantum Bruhat graph:*

$$ws_{\beta_{j_1}} s_{\beta_{j_2}} \cdots s_{\beta_{j_s}} \to \cdots \to ws_{\beta_{j_1}} s_{\beta_{j_2}} \to ws_{\beta_{j_1}} \to w\ .$$

Define $\mathrm{wt}(J) := \widetilde{r}_{j_1}\widetilde{r}_{j_2} \cdots \widetilde{r}_{j_s}(\mu)$ *and* $\mathrm{ht}(J) := \sum_i \left(\langle \mu, \gamma_{j_i}\rangle - m_{j_i}\right)$ *where the sum is over all i such that the edge corresponding to β_{j_i} is a quantum edge, i.e. an edge of type (2) in Definition 2.1.*

Theorem 3.2 ([19])**.** *The above model computes the non-symmetric Macdonald polynomial*

$$E_\mu(x, q, t = 0) = \sum_{J \in \mathcal{A}(\Gamma')} q^{\mathrm{ht}(J)} x^{\mathrm{wt}(J)}\ . \tag{3.5}$$

3.2 Crystal operators and the first main result

Using the setup in the previous section, we define crystal operators e_i, f_i on $\mathcal{A}(\Gamma')$. For J a w-admissible subset, recall the "folded" alcove path $\Gamma(J) = (\delta_1, \delta_2, \ldots, \delta_m)$ from Equation (3.3). The action of the operators is determined by the levels l^J of the hyperplanes crossed by the folded path $\Gamma(J)$. Fix a root $\alpha \in \Phi$ and define:

$$I_\alpha(J) := \{i \in [m] \mid \delta_i = \pm\alpha\}\ , \qquad \widehat{I}_\alpha = \widehat{I}_\alpha(J) := I_\alpha \cup \{\infty\}\ ,$$

$$l_\alpha^\infty := \langle \mathrm{wt}(J), \mathrm{sgn}(\alpha)\alpha^\vee\rangle, \qquad \delta_\infty := r_{j_1} r_{j_2} \cdots r_{j_s} v\rho\ .$$

$$\widehat{I}_\alpha = \{i_1 < i_2 < \cdots < i_n < i_{n+1} = \infty\} \text{ and } \varepsilon_i := \begin{cases} 1 & \text{if } i \notin J \\ -1 & \text{if } i \in J \end{cases}.$$

If $\alpha > 0$, we define the continuous piece-wise linear function $g_\alpha : [0, n + \frac{1}{2}] \to \mathbb{R}$ by

$$g_\alpha(0) = -\frac{1}{2}, \quad g'_\alpha(x) = \begin{cases} \mathrm{sgn}(\delta_{i_k}) & \text{if } x \in (k-1, k-\frac{1}{2}),\ k = 1, \ldots, n \\ \varepsilon_{i_k}\mathrm{sgn}(\delta_{i_k}) & \text{if } x \in (k-\frac{1}{2}, k),\ k = 1, \ldots, n \\ \mathrm{sgn}(\langle \delta_\infty, -\alpha^\vee \rangle) & \text{if } x \in (n, n+\frac{1}{2}). \end{cases} \tag{3.6}$$

If $\alpha < 0$, then g_α is the graph obtained by reflecting $g_{-\alpha}$ across the x-axis. For any α:

$$\mathrm{sgn}(\alpha) l^J_{i_k} = -g_\alpha\left(k - \frac{1}{2}\right) \text{ for } k = 1, \ldots, n \tag{3.7}$$

$$\mathrm{sgn}(\alpha) l^\infty_\alpha := -\langle \mathrm{wt}(J), \alpha^\vee \rangle = -g_\alpha\left(n + \frac{1}{2}\right). \tag{3.8}$$

Fix $i \in \{0, \ldots, n-1\}$. If $i > 0$ then α_i is a simple root, and if $i = 0$ then $\alpha_0 := -\theta$.

Define $M := \max(g_{\alpha_i})$. Assuming that $M > \langle \mathrm{wt}(J), \alpha_i^\vee \rangle$, let k be the maximum index j in I_{α_i} for which we have $\mathrm{sgn}(\alpha_i) l^J_j = -M$, and let m be the successor of k in $\widehat{I}_{\alpha_i}$. Assuming also that $M \geq \delta_{i,0}$, we have $k \in J$, and either $m \notin J$ or $m = \infty$. Define:

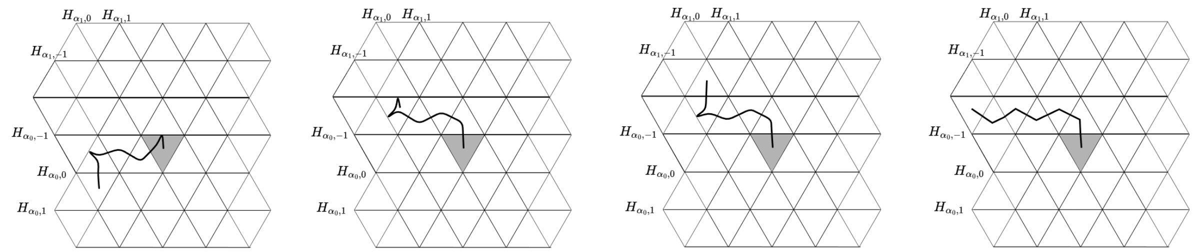
$$f_i(J) := \begin{cases} (J \setminus \{k\}) \cup \{m\} & \text{if } M > \langle \mu(J), \alpha_i^\vee \rangle \text{ and } M \geq \delta_{i,0} \\ \mathbf{0} & \text{otherwise.} \end{cases} \tag{3.9}$$

The operators e_i are defined similarly. We use the convention that $J \setminus \{\infty\} = J \cup \{\infty\} = J$. Using the properties of the quantum Bruhat graph, we show that the set $\mathcal{A}(\Gamma')$ is closed under the crystal operators f_i, e_i, so the model is well-defined. Based on this, and by using a similar approach to the one in [13, 17], we derive our first main result.

Theorem 3.3. *The set of w-admissible subsets $\mathcal{A}(\Gamma')$, together with the operators e_i, f_i and map* wt *defined above, is a model for the Demazure-type crystal corresponding to the module $U^+_w(\lambda)$.*

Remark 3.4. The entire crystal is constructed by repeated application of the operators e_i and f_i, starting from the lowest element $J = \emptyset$ ($f_i(J) = \mathbf{0}$ for all i). By comparison, the Demazure-type crystal from Theorem 2.4 is constructed by repeated application of only the operators f_i, starting from the highest element $J = \emptyset$ ($e_i(J) = \mathbf{0}$ for all i) but checking the condition $\mathrm{end}(J) \leq \lfloor w \rfloor$ at each step.

Remark 3.5. Viewing J as a folding of the alcove path Γ, the crystal operator f_i acts by removing the fold along the hyperplane $H_{\alpha_i, -x}$ for $x \geq 0$ maximal, and adding a fold along the hyperplane $H_{\alpha_1, -(x+1)}$ (if such x exists). The value of x is determined by the maximum M of the function g_{α_i} defined above. The action of e_i is similar, this time moving folds in one level. See Figure 1.



$J = \{1,5\}$ $\quad f_0(J) = \{5,6\}$ $\quad f_0 f_0(J) = \{5\}$ $\quad f_1 f_0 f_0(J) = \emptyset$

Figure 1: Action of the composition $f_1 f_0 f_0$ on the folded alcove path $\Gamma(J)$ in type A_2.

4 Specializing the model to type A_{n-1}

4.1 A choice of μ-chain

Recall that the nonsymmetric quantum alcove model depends on a choice of alcove path Γ', i.e. a sequence of positive roots called a μ-chain. To relate the alcove model to the tabloid model, a particular sequence of roots is needed. In this section we explicitly give such a sequence.

To this end, consider the setup as in Section 3, now specialized to type A_{n-1}. Let λ' be the conjugate partition to λ and let dg(μ) be an empty diagram of shape μ. The alcove path Γ' will be constructed by concatenating smaller sequences of roots $\Gamma'(j)$, one for each column of dg(μ):

$$\Gamma' = \Gamma'(1)\Gamma'(2)\cdots\Gamma'(\lambda_1).$$

Suppose $\lambda'_j = k$. Rename and reorder $w(1), w(2), \cdots, w(k)$ as $i_1, i_2, \cdots, i_k$ with

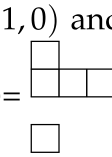
$$i_1 < i_2 < \cdots < i_k\,. \tag{4.1}$$

For $s \leq k$, define $m_s := \min(\{a \in \{k+1, \cdots, \lambda'_{j-1}\} | i_s > w(a)\} \cup \{\lambda'_{j-1}+1\})$ (by convention $\lambda'_0 = n$). Then $\Gamma'(j)$ is given by the following sequence of roots:

$$\Gamma'(j) = \begin{pmatrix} (w^{-1}(i_1), m_1) & (w^{-1}(i_1), m_1+1) & \dots & (w^{-1}(i_1), n) \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ (w^{-1}(i_k), m_k) & (w^{-1}(i_k), m_k+1) & \dots & (w^{-1}(i_k), n) \end{pmatrix}. \tag{4.2}$$

Proposition 4.1 ([6]). *The sequence of roots Γ', defined as the above concatenation, encodes an alcove path of the form* (3.2)*, with which it is identified.*

Example 4.2. Let $\mu = (1,0,3,1)$. Then $\mu = w\lambda$ for $\lambda = (3,1,1,0)$ and $w = w^{-1} = 3412$.

The corresponding diagram has the following shape: dg(μ)=  .

- The first column has 3 boxes: $i_1 = 1, m_1 = 5;\ \ i_2 = 3, m_2 = 4; i_3 = 4, m_3 = 4$.
- The second column has 1 box: $i_1 = 3, m_1 = 3$.
- The third column has 1 box: $i_1 = 3, m_1 = 2$.

The corresponding sequences of roots $\Gamma'(j)$ are:

$$\Gamma'(1) = ((1,4)(2,4)); \quad \Gamma'(2) = ((1,3)(1,4)); \quad \Gamma'(3) = ((1,2)(1,3)(1,4))\,.$$

Concatenating produces the alcove path Γ' with vertical bars separating each part.

$$\Gamma' = ((1,4)(2,4)|(1,3)(1,4)|(1,2)(1,3)(1,4))\,.$$

4.2 The filling map

We now describe a map from w-admissible subsets $\mathcal{A}(\Gamma')$ to semi-standard key tabloids $SSKD(\mu)$. Recall that a w-admissible subset selects a subsequence of roots within the alcove path Γ'. Let T_j be the sequence of transpositions corresponding to the roots in $\Gamma'(j)$ that are selected, noting that T_j could possibly be empty. Given a w-admissible subset J and a column j of the empty diagram $\mathrm{dg}(\mu)$, define:

$$u^j := wT_1T_2\cdots T_j$$

where the right hand side is the permutation obtained from w by consecutively multiplying by each transposition present in $T_1, T_2, \ldots, T_j$ from left to right.

Definition 4.3. *The map* fill *is a map from subsets J (not necessarily w-admissible) to fillings of shape μ with entry in row i, column j given by*

$$\mathrm{fill}(J)(i,j) := u^j(w^{-1}(i)).$$

Theorem 4.4. *If J is w-admissible, then* $\mathrm{fill}(J)$ *is non-attacking and has no coinversion triples. Moreover, "*fill*" is a weight preserving and height preserving bijection between w-admissible subsets and semi-standard key tabloids of shape μ.*

Example 4.5. Continuing Example 4.2, let $J = \{2,5,6,7\}$. One can check that J is w-admissible. From the decomposition of $\Gamma' = ((1,4)(2,4)|(1,3)(1,4)|(1,2)(1,3)(1,4))$ into $\Gamma'(1)\Gamma'(2)\Gamma'(3)$, we have:

$$T_1 = (2,4); \quad T_2 = \emptyset; \quad T_3 = (1,2)(1,3)(1,4)\,.$$

Starting from $w = 3412$, we apply in order the transpositions above:

$$w = \begin{array}{|c|}\hline 2 \\ \hline 1 \\ \hline 4 \\ \hline 3 \\ \hline\end{array} \xrightarrow{(2,4)} \begin{array}{|c|}\hline 4 \\ \hline \mathbf{1} \\ \hline \mathbf{2} \\ \hline \mathbf{3} \\ \hline\end{array} = u^1 \ \Bigg| \ \begin{array}{|c|}\hline 4 \\ \hline 1 \\ \hline 2 \\ \hline \mathbf{3} \\ \hline\end{array} = u^2 \ \Bigg| \ \xrightarrow{(1,2)} \begin{array}{|c|}\hline 4 \\ \hline 1 \\ \hline 3 \\ \hline 2 \\ \hline\end{array} \xrightarrow{(1,3)} \begin{array}{|c|}\hline 4 \\ \hline 2 \\ \hline 3 \\ \hline 1 \\ \hline\end{array} \xrightarrow{(1,4)} \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline \mathbf{4} \\ \hline\end{array} = u^3$$

$$u^1 = 3412(2,4) = \mathbf{321}4; \quad u^2 = \mathbf{3}214; \quad u^3 = 3214(1,2)(1,3)(1,4) = \mathbf{4}321$$

Since $w^{-1} = 3412$, the entry in box (i,j) is $u^j(w^{-1}(i))$, so $\mathrm{fill}(J) = \begin{array}{|c|c|c|}\hline 2 & \multicolumn{2}{c}{} \\ \hline 3 & 3 & 4 \\ \hline \multicolumn{3}{c}{} \\ \hline 1 & \multicolumn{2}{c}{} \\ \cline{1-1} \end{array}$.

4.3 The inverse map

Given a semi-standard key tabloid T, to reconstruct a w-admissible subset J such that $T = \mathrm{fill}(J)$, a "greedy" algorithm is run once for each box of T, working up the columns then across the rows. The initial input is the permutation w. If the target box is in column j, row i and has entry b, we select every root of $\Gamma'(j)$ (see Equation (4.2)) of the form $(w^{-1}(i), x)$ that brings us closer to our target entry with respect to a clockwise order, and consecutively multiply the current permutation by each selected transposition.

Proposition 4.6. *Repeated application of the process described above terminates, produces a unique path in the quantum Bruhat graph, and is the inverse of the map* "fill"*.*

4.4 The second main result

Here we recap the main constructions and state our main result. For a dominant weight λ and a Weyl group element w, we construct a crystal structure on w-admissible subsets that realizes the crystal of the level zero Demazure-type module $U_w^+(\lambda)$. In type A, Assaf and González constructed a crystal on semi-standard key tabloids that realizes a certain level one affine Demazure crystal. Through the bijection fill, we show that the crystal operators on tabloids agree with the crystal operators on w-admissible subsets. Moreover, "fill" identifies the statistics "ht" and "maj", which compute the energy function [16].

Theorem 4.7. *The map* fill *is an affine crystal isomorphism from w-admissible subsets to semi-standard key tabloids. Moreover,* $\mathrm{ht}(J) = \mathrm{maj}(\mathrm{fill}(J))$.

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