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A generalization of Deodhar's defect statistic for Iwahori–Hecke algebras of type BC

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Abstract. Let H be the Iwahori–Hecke algebra corresponding to any Coxeter group. Deodhar's *defect* statistic [*Geom. Dedicata* **36**, no. 1 (1990)] allows one to expand products of simple Kazhdan–Lusztig basis elements of H in the natural basis of H. Billey and Warrington [*J. Algebraic Combin.* **13**, no. 2 (2001)] provided a graphical interpretation of the type-A case of this formula. Clearwater and the third author [*Ann. Comb.* **25**, no. 3 (2021)] extended the graphical type-A case of this formula to combinatorially expand products of Kazhdan–Lusztig basis elements indexed by "smooth" elements of the symmetric group. We similarly extend the type-BC case of Deodhar's result to combinatorially expand products of Kazhdan–Lusztig basis elements indexed by hyperoctahedral group elements which are "simultaneously smooth" in types B and C.

Keywords: Hecke algebra, Kazhdan–Lusztig basis, hyperoctahedral group

1 Introduction

Let (W, S) be a Coxeter system with generating set $S = \{s_1, \ldots, s_m\}$, let $\leq$ denote the Bruhat order on W, let H be the Iwahori–Hecke algebra corresponding to W, and let $\{T_w \,|\, w \in W\}$ be the *natural basis* of H as a $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-module. (See, e.g., [4].) A second basis $\{C'_w(q) \,|\, w \in W\}$ introduced by Kazhdan and Lusztig [12], sometimes rescaled as $\{\widetilde{C}_w(q) \,|\, w \in W\}$ with $\widetilde{C}_w(q) := q^{\ell(w)/2}C'_w(q)$, is important in many areas of mathematics. Products of these elements expand nonnegatively in the natural and Kazhdan–Lusztig bases, and have appeared in intersection homology [1], [16], combinatorial description of Kazhdan–Lusztig basis elements themselves [2, 7], Schubert varieties [2], total nonnegativity [9, 14, 17, 18], trace evaluations [5, 6, 10, 11, 15], and chromatic symmetric functions [5, 15].

We will focus on Deodhar's result [7, Proposition 3.5], where he considered sequences $(s_{i_1}, \ldots, s_{i_k})$ of generators in S, products of the corresponding Kazhdan–Lusztig basis elements $\widetilde{C}_{s_{i_j}}(q) = T_e + T_{s_{i_j}}$ of H, and their natural expansions

$$\widetilde{C}_{s_{i_1}}(q) \cdots \widetilde{C}_{s_{i_k}}(q) = \sum_{w \in W} a_w T_w. \tag{1.1}$$

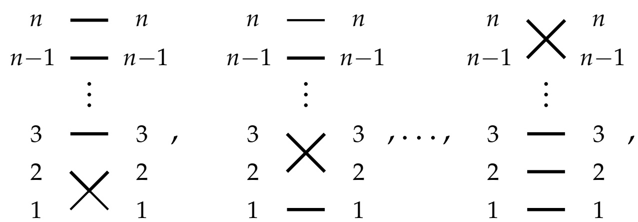
Deodhar described the coefficients $\{a_w \,|\, w \in W\} \subset \mathbb{Z}[q]$ in terms of *subexpressions* of $(s_{i_1}, \ldots, s_{i_k})$: sequences $\sigma = (\sigma_1, \ldots, \sigma_k)$ of elements of S with $\sigma_j \in \{e, s_{i_j}\}$ for $j = 1, \ldots, k$.

Call index j a *defect* of σ if $\sigma_1 \cdots \sigma_{j-1} s_{i_j} < \sigma_1 \cdots \sigma_{j-1}$, and let $\mathrm{dfct}(\sigma)$ denote the number of defects of σ. Then each coefficient on the right-hand side of (1.1) is given by

$$a_w = \sum_{\sigma} q^{\mathrm{dfct}(\sigma)}, \tag{1.2}$$

where the sum is over all subexpressions σ of $(s_{i_1}, \ldots, s_{i_k})$ satisfying $\sigma_1 \cdots \sigma_k = w$.

Billey and Warrington [2, Remark 6] observed that when W and H are the symmetric group $\mathfrak{S}_n$ and type-A Iwahori–Hecke algebra $H_n^{\mathsf{A}}(q)$, Deodhar's defect statistic has the following simple graphical interpretation. Let $F = F_{s_{i_1}} \circ \cdots \circ F_{s_{i_k}}$ be the wiring diagram corresponding to $(s_{i_1}, \ldots, s_{i_k})$, where $\circ$ denotes concatenation, and the factor wiring diagrams $F_{s_1}, \ldots, F_{s_{n-1}}$ are the planar networks



(1.3)

respectively. Edges of F are understood to be oriented from left to right, with n *source* vertices on the left, n *sink* vertices on the right, and k more *interior* vertices, one per factor. We can cover F with 2^k different path families of the form $\pi = (\pi_1, \ldots, \pi_n)$, if we allow two paths meeting at the interior vertex of a factor $F_{s_{i_j}}$ either to cross or not to cross there. Let $\Pi_w(F)$ be the subset of these path families having *type* w, i.e., for $i = 1, \ldots, n$, path π_i begins at source i and terminates at sink w_i. Call index j a *(type-*A*) defect* of π if the two paths meeting in $F_{s_{i_j}}$ cross an odd number of times in $F_{s_{i_1}} \circ \cdots \circ F_{s_{i_{j-1}}}$, and let $\mathrm{dfct}(\pi)$ be the number of defects in π. Then each coefficient on the right-hand side of (1.1) is given by

$$a_w = \sum_{\pi \in \Pi_w(F)} q^{\mathrm{dfct}(\pi)}. \tag{1.4}$$

Clearwater and the third author [6, Corollary 5.3] extended this type-A result to products of the form

$$\widetilde{C}_{v^{(1)}}(q) \cdots \widetilde{C}_{v^{(k)}}(q) = \sum_{w} a_w T_w \tag{1.5}$$

in $H_n^{\mathsf{A}}(q)$, where $v^{(1)}, \ldots, v^{(k)}$ are maximal elements of parabolic subgroups of $\mathfrak{S}_n$ (and more generally, 3412-avoiding, 4231-avoiding elements of $\mathfrak{S}_n$), and each factor

$$\widetilde{C}_{v^{(j)}}(q) := \sum_{u \leq v^{(j)}} T_u$$

belongs to the rescaled Kazhdan–Lusztig basis of $H_n^{\mathsf{A}}(q)$. Again we have (1.4), where F has the form $F_{v^{(1)}} \circ \cdots \circ F_{v^{(k)}}$, with factor networks generalizing those in (1.3) by allowing interior vertices to have indegree and outdegree greater than 2.

We extend this result further to analogous products of elements of the Kazhdan–Lusztig basis of the type-BC Iwahori–Hecke algebra $H_n^{\mathsf{BC}}(q)$ in Section 5. To do so, we review necessary background material on the hyperoctahedral group, $H_n^{\mathsf{BC}}(q)$, and type-BC planar networks in Sections 2 – 4.

2 The hyperoctahedral group

Recall that for a Coxeter group W with generating set S and an element $w \in W$, an expression $w = s_{i_1} \cdots s_{i_\ell}$ for w as a product of generators is called *reduced* if it is as short as possible, and $\ell = \ell(w)$ is called the *length* of w. We define the *Bruhat order* on W by declaring $v \leq w$ if some (equivalently, every) reduced expression for w contains a subsequence which is a reduced expression for v.

For a $J \subseteq S$ let W_J denote the *parabolic* subgroup of W generated by J. Each coset wW_J of W_J has a unique Bruhat-minimal element [4, Corollary 2.4.5]. Let W_-^J denote the set of minimal coset representatives. It is known that we have [4, Definition 2.4.2, Lemma. 2.4.3]

$$W_-^J = \{w \in W \,|\, ws > w \text{ for all } s \in J\} = \{w \in W \,|\, wv > v \text{ for all } v \in W_J\} \tag{2.1}$$

and that there is a bijection [4, Proposition 2.4.4]

$$W_-^J \times W_J \overset{1-1}{\longleftrightarrow} W \tag{2.2}$$

given by simple multiplication $(w, u) \mapsto wu$. The Bruhat order on W induces a related partial order on the set W/W_J of cosets $\{wW_J \,|\, w \in \mathfrak{B}_n\}$: we declare $vW_J \leq wW_J$ if the minimal element of vW_J is less than or equal to the minimal element of wW_J. (Equivalently, we may define $\leq$ by comparing arbitrary elements of the two cosets [8, Lemma 2.2].) We call this poset the *Bruhat order on W/W_J*.

The *hyperoctahedral group* $\mathfrak{B}_n$ is the Coxeter group of type $\mathsf{B}_n = \mathsf{C}_n$, generated by $S = \{s_0, \ldots, s_{n-1}\}$, subject to relations

$$\begin{aligned} {s_i}^2 &= e && \text{for } i = 0, \ldots, n-1, \\ s_0 s_1 s_0 s_1 &= s_1 s_0 s_1 s_0, && \\ s_i s_j &= s_j s_i && \text{for } i, j \geq 0 \text{ and } |i-j| \geq 2, \\ s_i s_j s_i &= s_j s_i s_j && \text{for } i, j \geq 1 \text{ and } |i-j| = 1. \end{aligned} \tag{2.3}$$

The parabolic subgroup of $\mathfrak{B}_n$ generated by $\{s_1, \ldots, s_{n-1}\}$ is the *symmetric group* $\mathfrak{S}_n$, the Coxeter group of type A_{n-1}.

Like elements of $\mathfrak{S}_n$, elements of $\mathfrak{B}_n$ naturally correspond to permutations of letters belonging to a certain alphabet. Specifically, we define the alphabet

$$[\overline{n}, n] := \{-n, \ldots, n\} \smallsetminus \{0\}$$

with notation $\overline{a} := -a$ for all $a \in [\overline{n}, n]$, and we consider permutations $w_{\overline{n}} \cdots w_{\overline{1}} w_1 \cdots w_n$ of $[\overline{n}, n]$ which satisfy $w_{\overline{i}} = \overline{w_i}$ for all $i \in [\overline{n}, n]$. We define a correspondence between $\mathfrak{B}_n$ and such permutations via the (left) action of $\mathfrak{B}_n$ on these permutations: s_0 swaps the letters in positions $\overline{1}$ and 1; s_i ($i = 1, \ldots, n-1$) simultaneously swaps the letters in positions $i, i+1$ with each other, and the letters in positions $\overline{i}, \overline{i+1}$ with each other. Then for $w = s_{i_1} \cdots s_{i_r} \in \mathfrak{B}_n$, we define the *(long) one-line notation* of w to be the permutation

$$w_{\overline{n}} \cdots w_{\overline{1}} w_1 \cdots w_n = s_{i_1}(s_{i_2}(\cdots (s_{i_r}(\overline{n} \cdots \overline{1} 1 \cdots n)) \cdots)). \tag{2.4}$$

For example, when $n = 4$, the element $s_0 s_1 \in \mathfrak{B}_4$ has long one-line notation

$$s_0(s_1(\overline{4}\overline{3}\overline{2}\overline{1}1234)) = s_0(\overline{4}\overline{3}\overline{1}\overline{2}2134) = \overline{4}\overline{3}1\overline{2}\overline{2}134. \tag{2.5}$$

It follows that w_i^{-1} is the index j satisfying $w_j = i$. By counting certain inversions in the long one-line notation of w, we can compute $\ell(w)$.

Lemma 2.1. *The length of $w \in \mathfrak{B}_n$ equals the number of pairs (i, j) with $|i| \leq j$ and j appearing earlier than i in $w_{\overline{n}} \cdots w_{\overline{1}} w_1 \cdots w_n$.*

Proof. Omitted. □

The condition $w_{\overline{i}} = \overline{w_i}$ implies that each element (2.4) is completely determined by the subword $w_1 \cdots w_n$, called the *short one-line notation* of w. The set of short one-line notations of elements of $\mathfrak{B}_n$ is the set of *signed permutations* of $[1, n]$: words $w_1 \cdots w_n$ with letters in the alphabet $[\overline{n}, n]$ with no repeated absolute values.

Certain elements of the hyperoctahedral group are most easily defined in terms of subintervals of $[\overline{n}, n]$, where we declare any subset $[a, b] := \{a, \ldots, b\} \smallsetminus \{0\}$ of $[\overline{n}, n]$ to be an *interval*, even if $a < 0 < b$. Roughly speaking, we define a *reversal* of $\mathfrak{B}_n$ to be an element $s_{[a,b]}$ obtained from the identity by reversing letters $[a, b]$ in positions $[a, b]$, and ensuring that the resulting permutation belongs to $\mathfrak{B}_n$. To be precise, we describe such elements using three cases: $a = b$, $a = \overline{b}$ ($b > 0$), and $0 < a < b$. When $a = b$, we have the trivial reversal $s_{[b,b]} = s_{\emptyset} = e$. For $b > 0$, the reversal $s_{[\overline{b},b]}$ is the element having one-line notation

$$\overline{n} \cdots \overline{(b+1)} \cdot b \cdots 1 \cdot \overline{1} \cdots \overline{b} \cdot (b+1) \cdots n, \tag{2.6}$$

and equal to the product of generators $s_0(s_1 s_0 s_1)(s_2 s_1 s_0 s_1 s_2) \cdots (s_{b-1} \ldots s_1 s_0 s_1 \ldots s_{b-1})$. For $0 < a < b$ the reversal $s_{[a,b]}$ is the element having one-line notation

$$\overline{n} \cdots \overline{(b+1)} \cdot \overline{a} \cdots \overline{b} \cdot \overline{(a-1)} \cdots \overline{1} \cdot 1 \cdots (a-1) \cdot b \cdots a \cdot (b+1) \cdots n,$$

and equal to the product of generators $s_a(s_{a+1}s_a)(s_{a+2}s_{a+1}s_a) \cdots (s_{b-1} \cdots s_a)$. Each reversal $s_{[a,b]}$ is the unique element of maximum length in a parabolic subgroup of $\mathfrak{B}_n$, generated by

$$J_{[a,b]} := \begin{cases} \{s_0, \ldots, s_{b-1}\} & \text{if } a = \overline{b} \ (b > 0), \\ \{s_a, \ldots, s_{b-1}\} & \text{if } 0 < a < b. \end{cases} \tag{2.7}$$

The first equality in (2.1) implies that when $J = J_{[a,b]}$, the subset W^J_- of $\mathfrak{B}_n$ may be characterized in terms of long one-line notation as follows.

Lemma 2.2. *Given reversal $s_{[a,b]} \in \mathfrak{B}_n$ and corresponding generators $J = J_{[a,b]}$, each minimum-length coset representative $w \in W^J_-$ satisfies*
(1) $w_i^{-1} < w_{i+1}^{-1}$ *for* $i = \max\{1, a\}, \ldots, b-1$, *and*
(2) $w_{\overline{1}}^{-1} < w_1^{-1}$ *if* $a = \overline{b}$.

Proof. Omitted. □

3 The Hecke algebra of the hyperoctahedral group

Given Coxeter group W with generator set S, define the *Hecke algebra* $H = H(W)$ of W to be the $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-span of $\{T_w \,|\, w \in W\}$ with unit T_e and multiplication defined by

$$T_w T_s = \begin{cases} qT_{ws} + (q-1)T_w & \text{if } ws < w, \\ T_{ws} & \text{if } ws > w, \end{cases} \tag{3.1}$$

where $s \in S$ and $w \in W$. This formula guarantees that for $w \in W$ and any reduced expression $s_{i_1} \cdots s_{i_\ell}$ for w, we have $T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}$. Call $\{T_w \,|\, w \in W\}$ the *natural basis* of H. It is easy to see that the specialization of H at $q^{\frac{1}{2}} = 1$ is isomorphic to $\mathbb{Z}[W]$. A second basis [12] of H is the (rescaled) *Kazhdan–Lusztig basis* $\{\widetilde{C}_w(q) \,|\, w \in W\}$, related to the natural basis by

$$\widetilde{C}_w(q) = \sum_{v \le w} P_{v,w}(q) T_v, \tag{3.2}$$

where $\{P_{v,w}(q) \,|\, v, w \in W\} \subseteq \mathbb{Z}[q]$ are the *Kazhdan–Lusztig polynomials*, whose recursive definition appears in [12]. Coefficients of these polynomials may be interpreted in terms of intersection cohomology $\mathrm{IH}^*(\Omega_w)$ [13], where Ω_w is a certain *Schubert variety* indexed by w. (See, e.g., [3].) In particular, when Ω_w is rationally smooth, all polynomials $\{P_{v,w}(q) \,|\, v \le w\}$ are identically 1 [12, Theorem A.2].

For each subset J of generators of W, one forms a natural $\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$-submodule of H by taking the span of sums

$$T_{uW_J} := \sum_{v \in uW_J} T_v \tag{3.3}$$

of natural basis elements of H with each sum corresponding to a coset of W_J. This submodule

$$H_J := \mathrm{span}_{\mathbb{Z}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]} \{T_{uW_J} \,|\, u \in W^J_-\} \tag{3.4}$$

in fact forms a left H-module. A nice formula for the action of H on H_J was given by Douglass [8, Proposition 2.3].

Proposition 3.1. *Let s be a generator of W, and let J be some subset of the generators. Then for all $w \in W$, we have*

$$T_s T_{wW_J} = \begin{cases} qT_{swW_J} + (q-1)T_{wW_J} & \text{if } swW_J < wW_J, \\ T_{swW_J} & \text{if } swW_J > wW_J, \\ qT_{wW_J} & \text{if } swW_J = wW_J. \end{cases} \tag{3.5}$$

Corollary 3.2. *If $v \in W_J$, then $T_v T_{W_J} = q^{\ell(v)} T_{W_J}$.*

Let us consider the module $H_J^{\mathsf{BC}} := H_J$ in the special case that $H = H_n^{\mathsf{BC}}(q)$ and $J = J_{[a,b]}$, and its relation to Kazhdan–Lusztig basis elements. By [3, Section 13.3.7], rational smoothness of Schubert varieties of types A, B, C is implied by pattern avoidance. In particular, when $w \in \mathfrak{B}_n$ avoids the patterns 3412 and 4231, we have the stronger property that type-B and C Schubert varieties indexed by w are both smooth. Since a reversal in $\mathfrak{B}_n$ avoids both of these patterns, we have the following.

Proposition 3.3. *For each reversal $s_{[a,b]} \in \mathfrak{B}_n$, we have*

$$\widetilde{C}_{s_{[a,b]}}(q) = \sum_{v \leq s_{[a,b]}} T_v.$$

Thus each Kazhdan–Lusztig basis element indexed by a reversal $s_{[a,b]}$ is itself a coset sum (3.3) for the parabolic subgroup generated by $J = J_{[a.b]}$:

$$T_{W_J} = \widetilde{C}_{s_{[a,b]}}(q). \tag{3.6}$$

By (2.2) and (3.5), other defining basis elements (3.4) of H_J^{BC} can be written as

$$T_{uW_J} = T_u \widetilde{C}_{s_{[a,b]}}(q), \tag{3.7}$$

for $u \in W_-^J$. It follows that each product of the form

$$\widetilde{C}_{s_{[a_1,b_1]}}(q) \cdots \widetilde{C}_{s_{[a_k,b_k]}}(q) \tag{3.8}$$

belongs to $H^{\mathsf{BC}}_{J_{[a_k,b_k]}}$ and we can expand it in the defining basis of this module as follows.

Proposition 3.4. *Suppose that the product of the first $k-1$ factors of (3.8) expands in the natural basis of $H_n^{\mathsf{BC}}(q)$ as*

$$\widetilde{C}_{s_{[a_1,b_1]}}(q) \cdots \widetilde{C}_{s_{[a_{k-1},b_{k-1}]}}(q) = \sum_{v \in \mathfrak{B}_n} c_v T_v, \tag{3.9}$$

for some polynomials $\{c_v = c_v(q) \,|\, v \in \mathfrak{B}_n\}$ in $\mathbb{Z}[q]$, and define $J = J_{[a_k,b_k]}$. Then the full product (3.8) expands in the defining basis of H_J^{BC} as

$$\sum_{w \in W_-^J} \Big(\sum_{u \in W_J} q^{\ell(u)} c_{wu} \Big) T_w \widetilde{C}_{s_{[a_k,b_k]}}(q). \tag{3.10}$$

Proof. Using (2.2) to factor each element v on the right-hand side of (3.9) as $v = wu$ with $w \in W_{-}^{J}$, $u \in W_J$, we may express the product (3.8) as

$$\sum_{w \in W_{-}^{J}} \Bigg(\sum_{u \in W_J} c_{wu} T_w T_u \Bigg) \widetilde{C}_{s_{[a_k,b_k]}}(q).$$

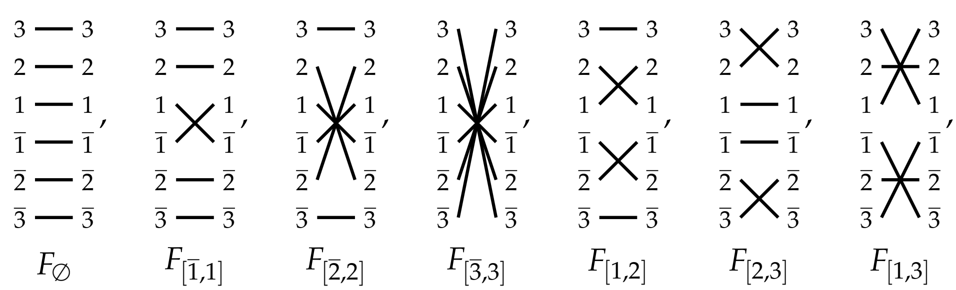
Then by Corollary 3.2, we have $T_u \widetilde{C}_{s_{[a_k,b_k]}}(q) = q^{\ell(u)} \widetilde{C}_{s_{[a_k,b_k]}}(q)$ and the claimed formula. □

4 Type-BC star networks

To graphically represent products of the form (3.8), we extend the idea of type-BC wiring diagrams to include planar networks in which interior vertices may have indegrees and outdegrees greater than 2. In particular, we associate to each reversal $s_{[a,b]} \in \mathfrak{B}_n$ a *type-BC simple star network* $F_{[a,b]}$ having $2n$ source vertices on the left and $2n$ sink vertices on the right, both labeled $\overline{n}, \ldots, n$ from bottom to top. For the three cases $a = b$, $a = \overline{b}$ $(b > 0)$, $0 < a < b$ of reversals, we include edges and 0, 1, or 2 additional interior vertices as follows.

1. When $F_{[a,a]} = F_{\emptyset}$ has a directed edge from source i to sink i, for $i = \overline{n}, \ldots, n$.
2. When $a = \overline{b}$ $(b > 0)$, $F_{[a,b]}$ has one interior vertex. For $i = a, \ldots, b$, a directed edge begins at source i and terminates at the interior vertex, and another directed edge begins at the interior vertex and terminates at sink i.
3. When $0 < a < b$, $F_{[a,b]}$ has two interior vertices. For $i = a, \ldots, b$, a directed edge begins at source i and terminates at the upper interior vertex, and another directed edge begins at the upper interior vertex and terminates at sink i. For $i = \overline{b}, \ldots, \overline{a}$, a directed edge begins at source i and terminates at the lower interior vertex, and another directed edge begins at the lower interior vertex and terminates at sink i.

For example, the seven type-BC simple star networks



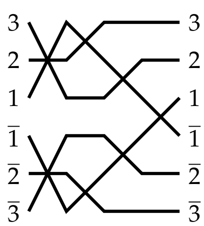
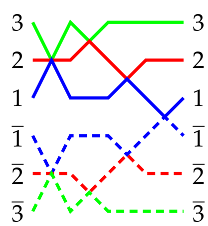
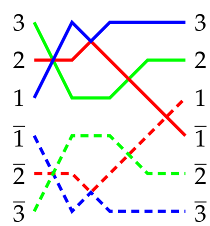
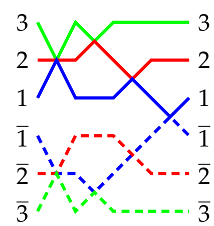
(4.1)

correspond to the reversals $s_\emptyset = e$, $s_{[\bar{1},1]}$, $s_{[\bar{2},2]}$, $s_{[\bar{3},3]}$, $s_{[1,2]}$, $s_{[2,3]}$, $s_{[1,3]}$ in $\mathfrak{B}_3$. Define $\mathcal{F}_n^{\mathsf{BC}}$ to be the set of all concatenations $F = F_{[a_1,b_1]} \circ \cdots \circ F_{[a_k,b_k]}$ of type-BC simple star networks and call these *type*-BC *star networks*. For instance, the type-BC star network $F = F_{[1,3]} \circ F_{[2,3]} \circ F_{[1,2]} \circ F_{[\bar{1},1]} \in \mathcal{F}_3^{\mathsf{BC}}$ is shown in (4.3).

Following [2], [6], we consider families $\pi = (\pi_{\bar{n}}, \dots, \pi_{\bar{1}}, \pi_1, \dots, \pi_n)$ of paths *covering* a star network $F \in \mathcal{F}_n^{\mathsf{BC}}$, i.e., using all edges in F. Call π a BC-*path family* if for each factor $F_{[a,b]}$ of F and each index $i \in [1,n]$, there exist indices j, k, such that paths π_i and $\pi_{\bar{i}}$ enter $F_{[a,b]}$ via sources j, $\bar{j}$ and exit $F_{[a,b]}$ via sinks k, $\bar{k}$, respectively. That is, $\pi_{\bar{i}}$ must be the reflection of π_i. Define π to have *type* $u = u_{\bar{n}} \cdots u_{\bar{1}} u_1 \cdots u_n \in \mathfrak{B}_n$ if for all i, path π_i begins at source i and terminates at sink u_i. For $F \in \mathcal{F}_n^{\mathsf{BC}}$ and $u \in \mathfrak{B}_n$, define the sets

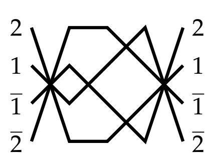
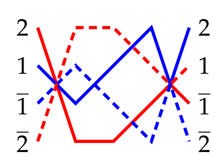
$$\begin{gathered} \Pi^{\mathsf{BC}}(F) = \{\pi \mid \pi \text{ a BC-path family covering } F\}, \\ \Pi_u^{\mathsf{BC}}(F) = \{\pi \in \Pi^{\mathsf{BC}}(F) \mid \mathrm{type}(\pi) = u\}. \end{gathered} \tag{4.2}$$

For example, the star network $F = F_{[1,3]} \circ F_{[2,3]} \circ F_{[1,2]} \circ F_{[\bar{1},1]} \in \mathcal{F}_3^{\mathsf{BC}}$ and path families

$F =$ , $\pi =$ , $\sigma =$ , $\tau =$ , (4.3)

satisfy $\pi \in \Pi_{123}^{\mathsf{BC}}(F)$, $\sigma \in \Pi_{3\bar{1}2}^{\mathsf{BC}}(F)$, $\tau \notin \Pi_{123}^{\mathsf{BC}}(F)$, with τ failing to be a BC-path family because τ_1, $\tau_{\bar{1}}$ in blue (also τ_2, $\tau_{\bar{2}}$ in red) are not reflections of one another.

Each element $F \in \mathcal{F}_n^{\mathsf{BC}}$ combinatorially interprets a product of Kazhdan–Lusztig basis elements in $H_n^{\mathsf{BC}}(q)$. To describe this interpretation, we extend the defect statistic [2], [6], [7] described in Section 1. Given a BC-path family π covering F, define a *type*-BC *defect* of π to be a triple (π_i, π_j, k) with $|i| \leq j$, and π_i, π_j meeting at one of the internal vertices of $F_{[c_k,d_k]}$ after having crossed an odd number of times. Let $\mathrm{dfct}^{\mathsf{BC}}(\pi)$ denote the number of type-BC defects of π. Observe that a single vertex can be the location of more than one defect. For example, consider the star network and BC-path family

$F_{[\bar{2},2]} \circ F_{[\bar{1},1]} \circ F_{[1,2]} \circ F_{[\bar{2},2]} =$ , $\pi =$ . (4.4)

We have $\mathrm{dfct}^{\mathsf{BC}}(\pi) = 4$: defects of π are $(\pi_{\bar{1}}, \pi_1, 2)$, $(\pi_{\bar{1}}, \pi_2, 3)$, $(\pi_1, \pi_2, 4)$, $(\pi_{\bar{2}}, \pi_2, 4)$. Refining the sets (4.2) by counting defects of path families in them, let us define

$$\Pi_{u,d}^{\mathsf{BC}}(F) = \{\pi \in \Pi_u^{\mathsf{BC}}(F) \mid \mathrm{dfct}^{\mathsf{BC}}(\pi) = d\}. \tag{4.5}$$

5 Main results

When one compares concatenations F' of $k-1$ simple star networks and F of k simple star networks, one sees bijections between certain sets of path families in F and F'.

Proposition 5.1. *Fix a sequence* $(s_{[a_1,b_1]},\ldots,s_{[a_k,b_k]})$ *of reversals in* $\mathfrak{B}_n$*, define the generator subset* $J = J_{[a_k,b_k]}$*, and choose an element* $w \in W_-^J$*. Consider two type-*BC *star networks*

$$F' = F_{[a_1,b_1]} \circ \cdots \circ F_{[a_{k-1},b_{k-1}]} \qquad \text{and} \qquad F = F' \circ F_{[a_k,b_k]}.$$

Then for each element $v \in W_J$ *and each* $d \geq 0$*, we have a bijection*

$$\Pi^{\mathsf{BC}}_{wv,d}(F) \quad \overset{1-1}{\longleftrightarrow} \quad \bigcup_{u \in W_J} \Pi^{\mathsf{BC}}_{wu,\, d-\ell(u)}(F'). \tag{5.1}$$

Proof. First we demonstrate the bijection (5.1) in the case that $v = e$. Observe that each path family π in $\Pi^{\mathsf{BC}}_w(F)$ can be decomposed uniquely as the concatenation of its truncation π' to F' with its truncation π'' to $F_{[a_k,b_k]}$: $\pi = \pi' \circ \pi''$. These truncations satisfy $\mathrm{type}(\pi')\mathrm{type}(\pi'') = w$ and $\mathrm{type}(\pi'') \in W_J$. Thus we have $\pi' \in \Pi^{\mathsf{BC}}_{wu}(F')$ for some $u \in W_J$. Let us write the first truncation map $\pi \mapsto \pi'$ as $\mathrm{trunc} : \Pi(F) \to \Pi(F')$. Let ψ be the restriction of trunc to $\Pi^{\mathsf{BC}}_w(F)$. We claim that the map

$$\psi : \Pi^{\mathsf{BC}}_w(F) \to \bigcup_{u \in W_J} \Pi^{\mathsf{BC}}_{wu}(F') \tag{5.2}$$

is bijective. To see this, keep $w \in W_-^J$ fixed, and choose a path family τ belonging to $\cup_{u\in W_J} \Pi^{\mathsf{BC}}_{wu}(F')$ and define $y = y(\tau) = w^{-1}\mathrm{type}(\tau) \in W_J$. The set $\Pi^{\mathsf{BC}}_{y^{-1}}(F_{[a_k,b_k]})$ contains a unique element, call it $\sigma_{y^{-1}}$. It is easy to see that the map

$$\begin{aligned} \kappa : \bigcup_{u \in W_J} \Pi^{\mathsf{BC}}_{wu}(F') &\to \Pi^{\mathsf{BC}}_w(F) \\ \tau &\mapsto \tau \circ \sigma_{y(\tau)^{-1}} \end{aligned} \tag{5.3}$$

inverts ψ. Now we claim that for $\pi \in \Pi^{\mathsf{BC}}_w(F)$, the paths π, $\pi' = \psi(\pi)$ satisfy

$$\mathrm{dfct}^{\mathsf{BC}}(\pi') = \mathrm{dfct}^{\mathsf{BC}}(\pi) - \ell(u).$$

To see this, observe that Lemma 2.2 implies the paths of $\pi = \pi' \circ \sigma_{u^{-1}}$ terminating at certain sinks to have increasing indices: sinks $[\overline{b_k}, b_k]$ if $a_k = \overline{b_k}$, and sinks $[\overline{b_k}, \overline{a_k}]$ and $[a_k, b_k]$ if $0 < a_k < b_k$. Therefore by Lemma 2.1, the paths of π' terminating at the corresponding sinks of F' include $\ell(u^{-1}) = \ell(u)$ path pairs (π'_i, π'_j) with $|i| \leq j$ and with π'_j terminating at a sink with smaller index than that of π'_i. Each such pair has crossed an odd number of times in F' and crosses again, defectively, in $F_{[a_k,b_k]}$. Thus $\mathrm{dfct}^{\mathsf{BC}}(\pi') = \mathrm{dfct}^{\mathsf{BC}}(\pi) - \ell(u)$ and the bijection (5.2) restricts for all d to

$$\Pi^{\mathsf{BC}}_{w,d}(F) \xleftrightarrow{1-1} \bigcup_{u \in W_J} \Pi^{\mathsf{BC}}_{wu,d-\ell(u)}(F').$$

We complete the proof of the bijections (5.1) by demonstrating bijections between all pairs of sets $\Pi^{\mathsf{BC}}_{wv,d}(F)$ and $\Pi^{\mathsf{BC}}_{wy,d}(F)$ with $v, y \in W_J$. In particular, for fixed $v, y \in W_J$, define the map $\phi_{v,y} : \Pi^{\mathsf{BC}}_{wv}(F) \to \Pi^{\mathsf{BC}}_{wy}(F)$ by writing $\pi = \mathrm{trunc}(\pi) \circ \sigma_{z^{-1}}$ for some $z \in W_J$ and by mapping

$$\pi \mapsto \mathrm{trunc}(\pi) \circ \sigma_{z^{-1}v^{-1}y}. \tag{5.4}$$

To see that $\phi_{v,y}$ is well defined, observe that since the type of $\mathrm{trunc}(\pi)$ is wvz, the type of $\phi_{v,y}(\pi)$ is $wvzz^{-1}v^{-1}y = wy$. To see that $\phi_{v,y}$ is invertible, note that $\phi_{y,v}(\phi_{v,y}(\pi))$ equals

$$\phi_{y,v}(\mathrm{trunc}(\pi) \circ \sigma_{z^{-1}v^{-1}y}) = \mathrm{trunc}(\pi) \circ \sigma_{(z^{-1}v^{-1}y)y^{-1}v} = \mathrm{trunc}(\pi) \circ \sigma_{z^{-1}} = \pi.$$

Finally, we claim that $\phi_{v,y}$ restricts to a bijection from $\Pi^{\mathsf{BC}}_{wv,d}(F)$ to $\Pi^{\mathsf{BC}}_{wy,d}(F)$ for all d. To see this, observe that as z varies over W_J, the set of paths of $\mathrm{trunc}(\pi) \circ \sigma_{z^{-1}}$ which meet at the central vertex of $F_{[a_k,b_k]}$ is constant. □

This leads to our main result: combinatorial formulas for coefficients appearing in the natural expansions of certain products of Kazhdan–Lusztig basis elements.

Theorem 5.2. *Fix a sequence $(s_{[a_1,b_1]}, \ldots, s_{[a_k,b_k]})$ of reversals in $\mathfrak{B}_n$ and define the type-*BC *star network $F = F_{[a_1,b_1]} \circ \cdots \circ F_{[a_k,b_k]}$. Then we have*

$$\widetilde{C}_{s_{[a_1,b_1]}}(q) \cdots \widetilde{C}_{s_{[a_k,b_k]}}(q) = \sum_{y \in \mathfrak{B}_n} \sum_{d \geq 0} |\Pi^{\mathsf{BC}}_{y,d}(F)| q^d T_y. \tag{5.5}$$

Proof. (Idea) For any reversal $s_{[a,b]}$ and corresponding generator set $J = J_{[a,b]}$, we have by Proposition 3.3 that the coefficient of T_u in $\widetilde{C}_{s_{[a,b]}}(q)$ is 1 for $u \in W_J$ and is 0 otherwise. On the other hand, the star network $F_{[a,b]}$ satisfies

$$|\Pi^{\mathsf{BC}}_{u,d}(F_{[a,b]})| = \begin{cases} 1 & \text{if } u \in W_J \text{ and } d = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus the identity (5.5) holds for a single Kazhdan–Lusztig basis element indexed by a reversal. Now assume that the identity holds for products of $1, \ldots, k-1$ such elements and consider a product of k of them. Writing $F' = F_{[a_1,b_1]} \circ \cdots \circ F_{[a_{k-1},b_{k-1}]}$, we have

$$\widetilde{C}_{s_{[a_1,b_1]}}(q) \cdots \widetilde{C}_{s_{[a_{k-1},b_{k-1}]}}(q) = \sum_{v \in \mathfrak{B}_n} \sum_{d \geq 0} |\Pi^{\mathsf{BC}}_{v,d}(F')| q^d T_v. \tag{5.6}$$

Defining $J = J_{[a_k,b_k]}$ and applying Proposition 3.4 to (5.5) – (5.6) we have that the left-hand side of (5.5) equals

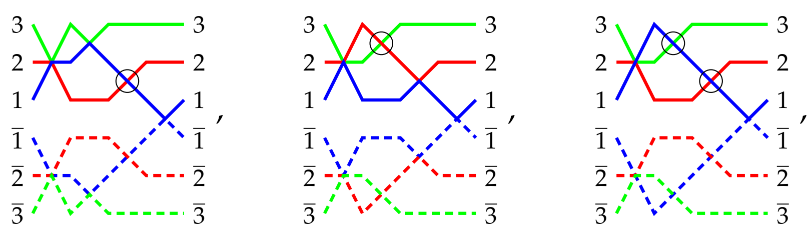
$$\sum_{w \in W_-^J} \Big(\sum_{u \in W_J} \sum_{d \geq 0} |\Pi^{\mathsf{BC}}_{wu,d}(F')| q^{\ell(u)+d} \Big) T_w \widetilde{C}_{s_{[a_k,b_k]}}(q). \tag{5.7}$$

Applying Proposition 5.1 and using (3.7), we may rewrite (5.7) as

$$\begin{aligned} \sum_{w \in W_-^J} \Big(\sum_{d \geq 0} q^d |\Pi^{\mathsf{BC}}_{w,d}(F)| \Big) T_w \widetilde{C}_{s_{[a_k,b_k]}}(q) &= \sum_{w \in W_-^J} \Big(\sum_{d \geq 0} q^d |\Pi^{\mathsf{BC}}_{w,d}(F)| \Big) \sum_{u \in W_J} T_{wu} \\ &= \sum_{w \in W_-^J} \sum_{u \in W_J} \sum_{d \geq 0} q^d |\Pi^{\mathsf{BC}}_{w,d}(F)| T_{wu} \\ &= \sum_{w \in W_-^J} \sum_{u \in W_J} \sum_{d \geq 0} q^d |\Pi^{\mathsf{BC}}_{wu,d}(F)| T_{wu}, \end{aligned} \tag{5.8}$$

which by (2.2) equals the right-hand side of (5.5). □

To illustrate Theorem 5.2, consider the coefficient of T_e in $\widetilde{C}_{s_{[1,3]}}(q)\widetilde{C}_{s_{[2,3]}}(q)\widetilde{C}_{s_{[1,2]}}(q)\widetilde{C}_{s_{[\overline{1},1]}}(q)$. Exactly four path families of type e cover the network $F = F_{[1,3]} \circ F_{[2,3]} \circ F_{[1,2]} \circ F_{[\overline{1},1]}$ in (4.3). These include the defect-free path family π in (4.3) and the three path families



with circled defects. The four families contribute $1, q, q, q^2$, respectively, to the coefficient of T_e. Thus the desired coefficient is $1 + 2q + q^2$.

As a corollary of Theorem 5.2, one may similarly interpret the natural expansion of products $\widetilde{C}_{w^{(1)}}(q) \cdots \widetilde{C}_{w^{(r)}}(q) \in H_n^{\mathsf{BC}}(q)$ when $w^{(1)}, \ldots, w^{(r)} \in \mathfrak{B}_n$ avoid the patterns 3412 and 4231. By [15, Theorem 5.21], each of these Kazhdan–Lusztig basis elements factors as on the left-hand side of (5.5), and by the comment preceding Proposition 3.3, each of these elements of $\mathfrak{B}_n$ corresponds to Schubert varieties which are simultaneously smooth in types B and C. It would be interesting to extend the theorem to Hecke algebras of Coxeter groups of type D.

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