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RSK as a linear operator

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Abstract. The Robinson–Schensted–Knuth correspondence (RSK) is a bijection between nonnegative integer matrices and pairs of Young tableaux. We study it as a linear operator on the coordinate ring of matrices.

Keywords: Robinson–Schensted–Knuth correspondence, linear algebra, representation theory

1 Introduction

1.1 Background

This extended abstract of [11] is devoted to linear algebraic questions about the *Robinson–Schensted–Knuth correspondence* (RSK), an important combinatorial algorithm. RSK can be interpreted as the transition operator between the "representation-theoretic" and "obvious" bases of the vector space of polynomial functions on matrices. Examples of transition matrices between such bases of vector spaces include:

- *Kostka matrices* between the Schur and monomial bases of symmetric polynomials [10];
- *Symmetric group character tables* between the irreducible character basis and the indicator function basis of class functions [5]; and
- *Kazhdan–Lusztig matrices* between the Kazhdan–Lusztig basis and the standard basis of a Hecke algebra [8].

These matrices are of significant interest, and are all related to RSK.[1] Recognizing the centrality of RSK in combinatorial representation theory, we initiate a parallel study of the RSK transition matrix itself.

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[1] See Stanley [10, Section 7.12], Ram [9], and Ariki [1] for instances of the respective connections.

We begin with some basic definitions and notation. A *partition* $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell \geq 0)$ is a weakly decreasing sequence of ℓ nonnegative integers. Identify λ with its *Young diagram*, a configuration of ℓ rows of left-justified boxes with λ_i boxes in row i. A *semistandard Young tableau* is a filling of λ with positive integers that weakly increase, left-to-right, along rows and strictly increase, top-to-bottom, along columns. If $\lambda = (4, 2, 1)$, then $\begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 2 \\ \hline 2 & 3 \\ \cline{1-2} 3 \\ \cline{1-1}\end{array}$ is one such tableau (drawn in English notation). Let $\mathsf{SSYT}(\lambda, m)$ be the set of such tableaux taking values in $[m] := \{1, 2, \ldots, m\}$.

Fix $m, n \in \mathbb{N} := \{0, 1, 2, \ldots\}$ and let $\mathsf{Mat}_{m,n}(\mathbb{N})$ be the set of $m \times n$ matrices with entries from $\mathbb{N}$. RSK is usually described as a bare set bijection

$$\mathsf{RSK} : \mathsf{Mat}_{m,n}(\mathbb{N}) \to \bigcup_\lambda \mathsf{SSYT}(\lambda, m) \times \mathsf{SSYT}(\lambda, n),$$

where the union is over all partitions λ with at most $\min\{m, n\}$ rows. In Section 2.1 we recall one way to exhibit RSK via a combinatorial algorithm. The combinatorics of this bijection is well-studied, see, e.g., the books [5, 10] and references therein.

Our analysis of RSK is motivated by its equivalence to the first fundamental theorem of invariant theory for general linear groups (see [7]). Denote the coordinate ring of the space $\mathsf{Mat}_{m,n}(\mathbb{C})$ of $m \times n$ complex matrices by $R_{m,n} := \mathbb{C}[z_{ij}]_{1 \leq i \leq m, 1 \leq j \leq n}$. As a $\mathbb{C}$-vector space, $R_{m,n}$ has two bases of interest. One is the "obvious" *monomial basis*,

$$\left\{ z^\alpha := \prod_{i,j} z_{ij}^{\alpha_{i,j}} \;\middle|\; [\alpha_{i,j}] \in \mathsf{Mat}_{m,n}(\mathbb{N}) \right\},$$

where $\alpha = [\alpha_{i,j}]$ is an "exponent matrix". We will identify a monomial z^α with α. The second basis is the "representation-theoretic" *bitableau basis* of Doubilet–Rota–Stein [4]. It was used, by [4] and [3] respectively, to prove the first and second fundamental theorems of invariant theory for general linear groups over arbitrary commutative rings. Elements of the bitableaux basis are certain products of determinants $[P|Q]$ indexed by pairs $(P, Q) \in \mathsf{SSYT}(\lambda, m) \times \mathsf{SSYT}(\lambda, n)$; the definition is in Section 2.2.

Consequently, RSK may be interpreted as an operator $\mathsf{RSK} : R_{m,n} \to R_{m,n}$ by linearly extending the map $z^\alpha \mapsto [P|Q]$ (where $(P, Q) := \mathsf{RSK}(\alpha)$). Although RSK is an operator on an infinite-dimensional vector space, it decomposes as a direct sum of finite-dimensional operators. Let $R_{m,n,d}$ denote the vector space spanned by all degree-d monomials in $R_{m,n}$. Then $R_{m,n} = \bigoplus_{d \geq 0} R_{m,n,d}$. Since RSK is a degree-preserving operator, it splits as a direct sum of the restrictions $\mathsf{RSK}_{m,n,d}$ of RSK to $R_{m,n,d}$.

We were led to investigate the linear operator RSK by Bruns–Conca–Raicu–Varbaro's [2, Question 4.2.8], which asserts that little is known about it and asks, e.g., about its eigenvectors and eigenvalues. Our results concern the diagonalizability, eigenvalues, determinant, and trace of the matrices $\mathsf{RSK}_{m,n,d}$. The following statement summarizes some of our major conclusions:

Theorem 1.1. *Let $m, n, d \in \mathbb{N}$.*

(I) (Section 4.1) The matrix $\mathsf{RSK}_{m,n,d}$ is diagonalizable if and only if $d \leq 3$, $(m, n, d) \in \{(2, 3, 6), (3, 2, 6)\}$, or

$$\frac{1}{m} + \frac{1}{n} + \frac{1}{d} > 1.$$

(II) (Section 4.1) The characteristic polynomial of $\mathsf{RSK}_{m,n,d}$ is not solvable by radicals whenever $m, n \geq 3$ and $d \geq 4$.

(III) (Theorem 4.2) Fix d and let r be minimal such that $2^r > d$. The function $\det \mathsf{RSK}_{m,n,d}$ has period 2^r in both m and n, i.e.,

$$\det \mathsf{RSK}_{m,n,d} = \det \mathsf{RSK}_{m+2^r,n,d} = \det \mathsf{RSK}_{m,n+2^r,d}.$$

(IV) (Theorem 4.5) For fixed d, the trace of $\mathsf{RSK}_{m,n,d}$ is a polynomial in $O(m^d n^d)$.

Theorem 1.1(I) can be rephrased using Dynkin diagrams. Let $\mathcal{G}_{m,n,d}$ be the graph consisting of three paths of lengths m, n and d adjoined at one node in a "$\perp$" shape (so $|\mathcal{G}_{m,n,d}| = m + n + d - 2$). Theorem 1.1(I) states that $\mathsf{RSK}_{m,n,d}$ is diagonalizable if and only if $d \leq 3$ or $\mathcal{G}_{m,n,d}$ is a Dynkin diagram of type A_k, D_k, E_6, E_7, E_8, or E_9 (see Figure 1).

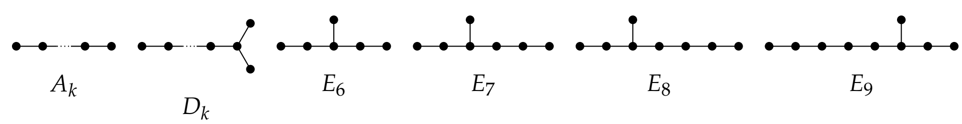

Figure 1: The Dynkin diagrams corresponding to diagonalizable matrices $\mathsf{RSK}_{m,n,d}$.

Our proofs use Theorem 1.9, which concerns the further restriction of RSK to *weight spaces* described below. The weight space arguments also yield formulas for the determinant and trace of $\mathsf{RSK}_{m,n,d}$ that are more efficient than the naïve algorithms.

1.2 Weight spaces and RSK-commuting maps

A pair $(\sigma, \pi) \in \mathbb{N}^m \times \mathbb{N}^n$ has *degree d* if $d = |\sigma| = |\pi|$, where $|\sigma| := \sum_{i=1}^m \sigma_i$. The *weight space* $R_{m,n,\sigma,\pi} \subseteq R_{m,n,d}$ is the subspace spanned by degree-d monomials z^α such that

$$\sum_j \alpha_{i,j} = \sigma_i,\ 1 \leq i \leq m \ \text{ and } \ \sum_i \alpha_{i,j} = \pi_j,\ 1 \leq j \leq n. \tag{1.1}$$

Equivalently, α is a *contingency table* with row margins σ and column margins π. Now,

$$R_{m,n,d} = \bigoplus_{\sigma,\pi : |\sigma| = |\pi| = d} R_{m,n,\sigma,\pi}. \tag{1.2}$$

Although $R_{m,n}$ and $R_{m,n,d}$ are both $GL := GL_m \times GL_n$ representations, the individual weight spaces are only representations of the maximal torus $T_m \times T_n \subseteq GL$. Our usage of the term "weight space" is consistent with that in Lie theory.

The *content* of a Young tableau T is the vector $(c_1, c_2, \ldots)$ such that T contains c_i i's. The *standard bitableaux* $[P|Q]$, where P has content σ and Q has content π, form a linear basis of $R_{m,n,\sigma,\pi}$. Thus the restriction $\mathsf{RSK}_{m,n,\sigma,\pi}$ of RSK to $R_{m,n,\sigma,\pi}$ is well-defined. After reordering the basis, the matrix $\mathsf{RSK}_{m,n,d}$ is block diagonal with each block a matrix $\mathsf{RSK}_{m,n,\sigma,\pi}$. Hence, it suffices to study $\mathsf{RSK}_{m,n,\sigma,\pi}$.

Example 1.2. Let $m = n = 2$ and $\sigma = \pi = (1,1)$. $R_{2,2,\sigma,\pi}$ is two-dimensional, spanned by the monomials $\{z_{11}z_{22}, z_{12}z_{21}\}$. The standard bitableaux spanning this weight space are

$$\left\{ \left[\, \boxed{1}\boxed{2} \;\middle|\; \boxed{1}\boxed{2} \,\right], \left[\begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \;\middle|\; \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline \end{array} \right] \right\} = \left\{ z_{11}z_{22}, \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} \right\}.$$

Now, $\mathsf{RSK}(z_{11}z_{22}) = z_{11}z_{22}$ and $\mathsf{RSK}(z_{12}z_{21}) = \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} = z_{11}z_{22} - z_{12}z_{21}$. Thus $\mathsf{RSK}_{2,2,\sigma,\pi}$ is represented by $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$. A basis of eigenvectors is $\{z_{11}z_{22}, z_{11}z_{22} - 2z_{12}z_{21}\}$, with eigenvalues 1 and -1 respectively.

We prove Theorem 1.1 by understanding weight pairs (σ, π) and $(\widetilde{\sigma}, \widetilde{\pi})$ such that the matrices $\mathsf{RSK}_{m,n,\sigma,\pi}$ and $\mathsf{RSK}_{\widetilde{m},\widetilde{n},\widetilde{\sigma},\widetilde{\pi}}$ are similar. Define a linear map $\psi : R_{m,n,\sigma,\pi} \to R_{\widetilde{m},\widetilde{n},\widetilde{\sigma},\widetilde{\pi}}$ to be *RSK-commuting* if

$$\psi \cdot \mathsf{RSK}_{m,n,\sigma,\pi} = \mathsf{RSK}_{\widetilde{m},\widetilde{n},\widetilde{\sigma},\widetilde{\pi}} \cdot \psi.$$

An RSK-commuting isomorphism ψ exists if and only if $\mathsf{RSK}_{m,n,\sigma,\pi} \sim \mathsf{RSK}_{\widetilde{m},\widetilde{n},\widetilde{\sigma},\widetilde{\pi}}$. The next two results summarize the RSK-commuting isomorphisms we use. Both are proved by combinatorial analysis of the given maps on the monomial basis, checking their commutation with the insertion algorithm RSK. Proposition 1.3 is basic; it shows that $\mathsf{RSK}_{m,n,\sigma,\pi}$ is determined by (σ, π) alone. We henceforth drop the m and n in the notation. The proof of part (I) is trivial, while (II) follows from the symmetry of RSK.

Proposition 1.3. *(I) Let $\sigma^+ = (\sigma_1, \ldots, \sigma_k, 0, \sigma_{k+1}, \ldots, \sigma_m) \in \mathbb{N}^{m+1}$. Then $\mathsf{RSK}_{m,n,\sigma,\pi} = \mathsf{RSK}_{m+1,n,\sigma^+,\pi}$.*

(II) The transposition map $z_{ij} \mapsto z_{ji}$ is an RSK-commuting isomorphism $R_{m,n,\sigma,\pi} \to R_{n,m,\pi,\sigma}$.

Theorem 1.4 below is the technical core of our paper [11]. Its proof (which we omit here) is more subtle and requires deeper analysis of RSK than Proposition 1.3.

Theorem 1.4. *Let (σ, π) be a degree-d weight pair. Then multiplication by $z_{k\ell}$ is an RSK-commuting isomorphism $R_{m,n,\sigma,\pi} \to R_{m,n,\sigma+\vec{e}_k,\pi+\vec{e}_\ell}$ if and only if $\sigma_k + \pi_\ell \geq d$.*

For the next definition, let $\ell(\sigma)$ denote the number of entries in σ.

Definition 1.5. A degree-d weight pair (σ, π) is *nice* if both σ and π have only nonzero entries, $\ell(\sigma) \leq \ell(\pi)$, and if $\ell(\sigma) = \ell(\pi)$ then σ is ordered before π lexicographically. A nice pair (σ, π) is *reduced* if it also satisfies $\max_i\{\sigma_i\} + \max_j\{\pi_j\} \leq d$.

Corollary 1.6. *Every weight pair (σ, π) is equivalent to a unique reduced pair, its* reduction *$(\sigma^{red}, \pi^{red})$ via the RSK-commuting isomorphisms of Theorem 1.4 and Proposition 1.3.*

Definition 1.7. The *growth potential* of a degree-d reduced pair (σ, π) is

$$g_{\sigma,\pi} := |\{(k, \ell) : \sigma_k + \pi_\ell = d\}|.$$

Lemma 1.8. *Fix $0 < d' \leq d$ and let (σ, π) be a degree-d' reduced pair. Let $A_{\sigma,\pi}(d)$ denote the number of degree-d nice pairs whose reduction is (σ, π).*

(I) If $\ell(\pi) \geq 3$ then $A_{\sigma,\pi}(d) = \begin{cases} \binom{(d-d') + (g_{\sigma,\pi} - 1)}{g_{\sigma,\pi} - 1} & \text{if } g_{\sigma,\pi} \geq 1, \\ \delta_{d,d'} & \text{if } g_{\sigma,\pi} = 0. \end{cases}$

(II) If $\ell(\sigma) = \ell(\pi) = 2$ then $A_{\sigma,\pi}(d) = 4(d - d') + \delta_{d,d'}$.

Combining Lemma 1.8 and Corollary 1.6 with a count of the degree-d weight pairs equivalent to a given nice pair via the RSK-commuting isomorphisms of Proposition 1.3 yields the following block matrix decomposition of $\mathsf{RSK}_{m,n,d}$.

Theorem 1.9 (Block Decomposition Theorem)**.** *Fix $m, n, d \in \mathbb{N}$. Then*

$$\mathsf{RSK}_{m,n,d} = \left(\mathrm{Id}_1^{\oplus N_0(m,n,d)}\right) \oplus \left(\bigoplus_{(\sigma,\pi)} \mathsf{RSK}_{\sigma,\pi}^{\oplus N_{\sigma,\pi}(m,n,d)}\right),$$

where the sum is over all nonzero reduced weight pairs (σ, π) of degree $d' \leq d$,

$$N_{\sigma,\pi}(m,n,d) = \begin{cases} A_{\sigma,\pi}(d)\left(\binom{m}{\ell(\sigma)}\binom{n}{\ell(\pi)} + \binom{m}{\ell(\pi)}\binom{n}{\ell(\sigma)}\right) & \text{if } \sigma \neq \pi, \\ A_{\sigma,\pi}(d)\binom{m}{\ell(\sigma)}\binom{n}{\ell(\pi)} & \text{if } \sigma = \pi, \end{cases}$$

and

$$N_0(m,n,d) = \binom{d+n-1}{d}m + \binom{d+m-1}{d}n - mn.$$

Theorem 1.9 is central to the proofs of all parts of Theorem 1.1.

2 Preliminaries

2.1 RSK

We recall the RSK correspondence, following the standard treatment found in [10, Section 7.11] with one difference of convention. We record biwords from matrices differently from [5] and [10], so our insertion tableau is their recording tableau and vice versa.

Given a semistandard tableau P of shape λ, the row insertion of an integer $p \geq 1$, denoted $P \leftarrow p$, is defined as follows. Write $P = (P_1, \ldots, P_{\ell(\lambda)})$, where P_i is the ith row of P. If p is larger than all labels in P_1, then $P \leftarrow p$ is the same as P with $\boxed{p}$ adjoined to the end of P_1. Otherwise, consider the smallest $p' > p$ appearing in P_1. Let P_1^* be P_1 with that $\boxed{p'}$ replaced by $\boxed{p}$ and define $P \leftarrow p$ to be $(P_1^*, \overline{P} \leftarrow p')$, where $\overline{P} = (P_2, P_3, \ldots, P_{\ell(\lambda)})$.

Next, we define insertion of a *biletter* $(p|q)$ (an ordered pair of integers $p, q \geq 1$) into a pair of semistandard tableaux (P, Q) of common shape λ. We denote this operation by $(P, Q) \leftarrow (p|q)$. First we compute $P \leftarrow p$, whose shape is the same as P except with a new corner box added. Then define $Q^{\uparrow}$ to be Q with $\boxed{q}$ placed in that same corner. Now $(P, Q) \leftarrow (p|q)$ is defined to be $(P \leftarrow p, Q^{\uparrow})$.

Next, suppose $\alpha \in \mathsf{Mat}_{m,n}(\mathbb{N})$. We record a sequence of biletters by reading the entries of α down the columns from left to right. We record each $\alpha_{i,j}$ as $\left(\underbrace{ii\ldots i}_{\alpha_{i,j}} \middle| \underbrace{jj\ldots j}_{\alpha_{i,j}}\right)$. The *biword* of α, denoted $\mathsf{biword}(\alpha)$, is the concatenation of all these biletters (written with extraneous brackets and commas removed). For example, if

$$\alpha = \begin{bmatrix} 0 & 3 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}, \text{ then } \mathsf{biword}(\alpha) = (233111221133|111222223333).$$

RSK(α) is the result of inserting the biletters of $\mathsf{biword}(\alpha) = (p_1 p_2 \ldots p_d | q_1 q_2 \ldots q_d)$ successively starting with $(\emptyset, \emptyset)$. That is, we compute

$$(P, Q) = (\cdots(((\emptyset, \emptyset) \leftarrow (p_1|q_1)) \leftarrow (p_2|q_2)) \leftarrow (p_3|q_3) \cdots).$$

The reader can check that in our running example,

$$\mathrm{RSK}(\alpha) = \left(\begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 3 & 3 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 & 3 \\ \cline{1-2} \end{array}, \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 2 & 2 & 3 & 3 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 & 3 \\ \cline{1-2} \end{array} \right).$$

2.2 Bitableaux and straightening

We recall the bitableau basis of $R_{m,n}$ referenced in the introduction. Let $\Delta_1, \ldots, \Delta_N$ be a sequence of minors of the generic $m \times n$ matrix $Z = [z_{ij}]_{1 \leq i \leq m, 1 \leq j \leq n}$. We may assume that the respective sizes of the minors are weakly decreasing. We encode the product $\Delta_1 \ldots \Delta_N \in R_{m,n}$ as a pair of (not necessarily semistandard) Young tableaux (P, Q), where the c-th columns (from the left) of P and Q are filled by the row and column indices of Δ_c respectively. When P and Q are both semistandard, we call the corresponding product of minors a *standard bitableau* and denote it $[P|Q]$.

Example 2.1. The following product of minors is a standard bitableau in $R_{4,4}$:

$$\begin{vmatrix} z_{11} & z_{12} & z_{14} \\ z_{21} & z_{22} & z_{24} \\ z_{41} & z_{42} & z_{44} \end{vmatrix} \begin{vmatrix} z_{12} & z_{13} \\ z_{32} & z_{33} \end{vmatrix} z_{22} z_{23} = \left[\ \ytableaushort{1122,23,4} \ \middle| \ \ytableaushort{1223,23,4} \ \right].$$

The simplest product of minors that is not standard is $z_{21} z_{12} \leftrightarrow \left(\ytableaushort{21}, \ytableaushort{12} \right)$.

The straightening law of [4] can be used to show that the standard bitableaux $[P|Q]$ form a $\mathbb{C}$-linear basis of $R_{m,n}$, and that any product of minors can be expressed as a $\mathbb{Z}$-linear combination of standard bitableaux via an explicit algorithm [2, Theorem 3.2.1]. In particular, the straightening law expresses any monomial z^α as a $\mathbb{Z}$-linear combination of standard bitableaux by viewing each variable z_{ij} as a 1×1 minor.

2.3 Notational conventions and examples

Recall that a *weight pair* of *degree d* is a tuple $(\sigma, \pi) \in \mathbb{N}^m \times \mathbb{N}^n$ such that $d = |\sigma| = |\pi|$. We often write (σ, π) in the abbreviated form $(\sigma_1\sigma_2 \ldots \sigma_m, \pi_1\pi_2 \ldots \pi_n)$. For example, $(21, 111)$ is shorthand for $((2, 1), (1, 1, 1))$. The *length* $\ell(\sigma)$ of σ is the number of entries it contains. Lowercase Greek letters generally denote nonnegative integer tuples: σ and π are weight vectors; λ is a partition; α and β are exponent matrices. Two exceptions are the minimal polynomial $\mu_M(t)$ of a matrix M and the Kronecker delta function $\delta_{i,j}$.

Since we index monomials z^α in $R_{\sigma,\pi}$ by their exponent matrices α, we also use contingency tables to index the rows and columns of $\mathsf{RSK}_{\sigma,\pi}$. To be fully explicit, the entry $\mathsf{RSK}_{\sigma,\pi}(\beta, \alpha)$ is defined to be $[z^\beta]\mathsf{RSK}(z^\alpha)$, the coefficient of z^β in the bitableau associated to z^α by RSK. We order the exponent matrices of monomials in $R_{\sigma,\pi}$ lexicographically:

Definition 2.2. Let $\mathsf{Cont}_{\sigma,\pi}$ denote the set of all contingency tables with row margins σ and column margins π; see (1.1). We order $\mathsf{Cont}_{\sigma,\pi}$ by placing α before α' if $z^\alpha > z^{\alpha'}$ in the lexicographic ordering where $z_{11} > z_{21} > \cdots > z_{m1} > z_{12} > \cdots > z_{mn}$.

The following three examples illustrate key concepts from the introduction: the definition of RSK, RSK-commuting isomorphisms, and Theorem 1.9.

Example 2.3. Let $(\sigma, \pi) = (111, 111)$. The vector space $R_{\sigma,\pi}$ is six-dimensional, with ordered monomial basis $\{z_{11}z_{22}z_{33}, z_{11}z_{32}z_{23}, z_{21}z_{12}z_{33}, z_{21}z_{32}z_{13}, z_{31}z_{12}z_{23}, z_{31}z_{22}z_{13}\}$. The bitableau basis of $R_{\sigma,\pi}$ is

$$\left\{ \left(\boxed{1}\boxed{2}\boxed{3}, \boxed{1}\boxed{2}\boxed{3}\right), \left(\begin{smallmatrix}\boxed{1}\boxed{2}\\ \boxed{3}\end{smallmatrix}, \begin{smallmatrix}\boxed{1}\boxed{2}\\ \boxed{3}\end{smallmatrix}\right), \left(\begin{smallmatrix}\boxed{1}\boxed{3}\\ \boxed{2}\end{smallmatrix}, \begin{smallmatrix}\boxed{1}\boxed{3}\\ \boxed{2}\end{smallmatrix}\right), \left(\begin{smallmatrix}\boxed{1}\boxed{3}\\ \boxed{2}\end{smallmatrix}, \begin{smallmatrix}\boxed{1}\boxed{2}\\ \boxed{3}\end{smallmatrix}\right), \left(\begin{smallmatrix}\boxed{1}\boxed{2}\\ \boxed{3}\end{smallmatrix}, \begin{smallmatrix}\boxed{1}\boxed{3}\\ \boxed{2}\end{smallmatrix}\right), \left(\begin{smallmatrix}\boxed{1}\\ \boxed{2}\\ \boxed{3}\end{smallmatrix}, \begin{smallmatrix}\boxed{1}\\ \boxed{2}\\ \boxed{3}\end{smallmatrix}\right) \right\}$$

$$= \left\{ z_{11}z_{22}z_{33}, \begin{vmatrix} z_{11} & z_{13} \\ z_{31} & z_{33} \end{vmatrix} z_{22}, \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix} z_{33}, \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix} z_{32}, \begin{vmatrix} z_{11} & z_{12} \\ z_{31} & z_{32} \end{vmatrix} z_{23}, \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\}.$$

The basis sets above are ordered such that RSK preserves the ordering. The reader can verify that, with respect to this ordered basis,

$$\mathsf{RSK}_{111,111} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 0 & 0 & -1 \end{bmatrix}.$$

The characteristic and minimal polynomials are, respectively,

$$p_{\mathsf{RSK}_{111,111}}(t) = (t-1)(t+1)^2(t^3+2t^2+1), \; \mu_{\mathsf{RSK}_{111,111}}(t) = (t-1)(t+1)(t^3+2t^2+1).$$

The integer eigenvectors are $(z_{11}z_{22}z_{33}, z_{11}z_{22}z_{33} - 2z_{12}z_{21}z_{33}, z_{12}z_{23}z_{31} - z_{13}z_{21}z_{32})$, with eigenvalues $(1, -1, -1)$ respectively, but the other three eigenvectors have unpleasant coordinates. This basis of eigenvectors shows that $\mathsf{RSK}_{111,111}$ is diagonalizable.

Example 2.4. Natural linear isomorphisms $R_{\sigma,\pi} \to R_{\widetilde{\sigma},\widetilde{\pi}}$ may not be RSK-commuting. Consider the two matrices

$$\mathsf{RSK}_{21,111} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \text{ and } \mathsf{RSK}_{12,111} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Although swapping rows 1 and 2 of the contingency tables induces a linear isomorphism $\psi : R_{21,111} \to R_{12,111}$, this map is not RSK-commuting. Indeed, the matrices above are not similar: $\mathsf{RSK}_{21,111}$ has eigenvalues $(1, \frac{-1\pm i\sqrt{3}}{2})$, while $\mathsf{RSK}_{12,111}$ has eigenvalues $(1, -1, -1)$. Thus there is no RSK-commuting isomorphism $R_{21,111} \to R_{12,111}$.

Example 2.5. We use Theorem 1.9 to describe $\mathsf{RSK}_{2,2,d}$. The only reduced weights (σ, π) of degree $0 < d' \leq d$ are those of the form $\sigma = \pi = (a, a)$ for each $0 < a \leq \lfloor d/2 \rfloor$. One can then compute that $N_{aa,aa}(2,2,d) = A_{aa,aa}(d) = 4(d-2a) + \delta_{d,2a}$ and $N_0(2,2,d) = 4d$.

We therefore obtain a relatively simple block decomposition for $\mathsf{RSK}_{2,2,d}$:

$$\mathsf{RSK}_{2,2,d} = (\mathrm{Id}_1^{\oplus 4d}) \oplus \left(\bigoplus_{a=1}^{\lfloor d/2 \rfloor} \mathsf{RSK}_{aa,aa}^{\oplus 4(d-2a)+\delta_{d,2a}} \right).$$

Example 3.4 computes each matrix $\mathsf{RSK}_{aa,aa}$, making this decomposition fully explicit.

3 Two useful families of examples

In this section we give more explicit descriptions of $\mathsf{RSK}_{\sigma,\pi}$ for two infinite families of reduced pairs (σ, π). These pairs are used to establish results about $\mathsf{RSK}_{m,n,d}$ in Section 4.

3.1 Permutation weights

Let 1^d denote the weight vector $(1, \ldots, 1) \in \mathbb{N}^d$. Then $\mathsf{Cont}_{1^d,1^d}$ is the set of all $d \times d$ permutation matrices, and $\mathsf{RSK}_{1^d,1^d}$ is the matrix describing Schensted insertion as a linear operator. Let $\alpha, \beta \in \mathsf{Mat}_{m,n}(\mathbb{N})$ with $(P, Q) = \mathsf{RSK}(\alpha)$. Determining $\mathsf{RSK}_{\sigma,\pi}(\beta, \alpha) = [z^\beta][P|Q]$ is *a priori* difficult. When $\alpha, \beta \in \mathsf{Cont}_{1^d,1^d}$, however, the situation simplifies dramatically. Let $\beta_c(\alpha)$ be the submatrix of β using row indices from the c-th column of P and column indices from the c-th column of Q. The next proposition allows us to determine individual entries of $\mathsf{RSK}_{1^d,1^d}$ quickly, without computing the entire matrix.

Proposition 3.1. *Let $\alpha, \beta \in \mathsf{Cont}_{1^d,1^d}$. Then $\mathsf{RSK}_{1^d,1^d}(\beta, \alpha) = \prod_c \det \beta_c(\alpha)$.*

Proof. Let $\alpha \in \mathsf{Cont}_{1^d,1^d}$ and $(P, Q) = \mathsf{RSK}(\alpha)$. Since α is a permutation matrix, P and Q are both standard tableaux, i.e., each entry of $[d]$ appears exactly once in P and once in Q. Thus each variable z_{ij} appears in at most one minor of $[P|Q]$. For each $\beta \in \mathsf{Cont}_{1^d,1^d}$, we know $[z^\beta][P|Q] \neq 0$ if and only if some factor of z^β appears in the c-th minor of $[P|Q]$. This factor corresponds to a (necessarily unique) nonzero term in $\det \beta_c(\alpha)$. □

Section 4.3 uses Proposition 3.1 to study Tr $\mathsf{RSK}_{m,n,d}$. Examples 1.2 and 2.3 display $\mathsf{RSK}_{1^d,1^d}$ for $d = 2$ and $d = 3$ respectively.

3.2 Triangular weights

Our other family of reduced pairs (σ, π) are called *triangular* because $\mathsf{RSK}_{\sigma,\pi}$ turns out to be upper triangular in the basis ordering of Definition 2.2.

Definition 3.2. A reduced weight pair (σ, π) is *triangular* if $\ell(\sigma) = 2$ and $\sigma_1 = \pi_1$.

Proposition 3.3. *If (σ, π) is a triangular pair, then $\mathsf{RSK}_{\sigma,\pi}$ is an upper triangular matrix. Moreover, $\mathsf{RSK}_{\sigma,\pi}$ is diagonalizable and all eigenvalues are ± 1.*

The proof of Proposition 3.3 gives an explicit formula for the RSK matrix indexed by a triangular pair (σ, π): each entry is a product of binomial coefficients. We omit the proof but present an example where these entries are particularly simple.

Example 3.4. Suppose (σ, π) is a reduced pair with $\ell(\sigma) = \ell(\pi) = 2$. Then $(\sigma, \pi) = (aa, aa)$ for some $a \in \mathbb{N}$. Thus (σ, π) is triangular. The (omitted) formula in the proof of Proposition 3.3 shows that elements of $\mathsf{Cont}_{aa,aa}$ are indexed by nonnegative integers $k \leq a$, and in fact $RSK_{aa,aa}(k, \ell) = (-1)^k \binom{\ell}{k}$.

4 Proofs of main results

With Theorem 1.9 and our various examples established, we can now sketch short proofs of all results stated in Theorem 1.1 and more.

4.1 Diagonalizability and eigenvalues of $\mathsf{RSK}_{m,n,d}$

Let us first prove Theorem 1.1(I) and (II). Recall, Theorem 1.1(I) classifies triples (m, n, d) such that $\mathsf{RSK}_{m,n,d}$ is diagonalizable.

Proof sketch of Theorem 1.1(I): We prove the Dynkin diagram version of the criterion. Proposition 1.3(II) justifies the assumption $m \leq n$. Computation using Theorem 1.9 shows that $\mathsf{RSK}_{m,n,d}$ is diagonalizable whenever $d \leq 3$, so we also assume $d > 3$:

(Type A) Then $m \leq 1$, so $\mathsf{RSK}_{m,n,d}$ is the identity matrix, hence diagonalizable.

(Type D) Then $m = n = 2$. Theorem 1.9 implies that each block in $\mathsf{RSK}_{2,2,d}$ is either the 1×1 identity matrix or a matrix $\mathsf{RSK}_{aa,aa}$ for some $1 \leq a \leq \lfloor d/2 \rfloor$ (see Example 2.5). Since each pair (aa, aa) is triangular, $\mathsf{RSK}_{2,2,d}$ is diagonalizable for all d by Proposition 3.3.

(Type E) Then $m = 2$ and $n = 3$. Computation shows that $\mathsf{RSK}_{43,223}$ is not diagonalizable, and $N_{43,223}(2, 3, d) > 0$ if and only if $d > 6$. Thus $\mathsf{RSK}_{2,3,d}$ is not diagonalizable for $d > 6$. A finite computation then shows that $\mathsf{RSK}_{2,3,d}$ is diagonalizable for $d \leq 6$, corresponding to the Dynkin diagrams E_k for $k \leq 9$.

If $\mathcal{G}_{m,n,d}$ is not of type A, D, or E, then either $n > 3$ or $m > 2$. In the $n > 3$ case, $\mathsf{RSK}_{22,1111}$ and $\mathsf{RSK}_{32,2111}$ are not diagonalizable, and at least one of them appears in $\mathsf{RSK}_{m,n,d}$ for $d > 3$ by Theorem 1.9. In the $m > 2$ case, $\mathsf{RSK}_{211,211}$ is not diagonalizable and appears in $\mathsf{RSK}_{m,n,d}$ for $d > 3$ by Theorem 1.9. This completes the proof. $\square$

Although Theorem 1.1(I) characterizes diagonalizability of $\mathsf{RSK}_{m,n,d}$, in general we do not know when an individual block $\mathsf{RSK}_{\sigma,\pi}$ is diagonalizable. The next result, Theorem 1.1(II), establishes a barrier to explicitly understanding the eigenvalues of $\mathsf{RSK}_{m,n,d}$.

Proof sketch of Theorem 1.1(II): By computation, $p_{\mathsf{RSK}_{211,121}}(t)$ contains an unsolvable quintic factor. Since $N_{211,121}(m, n, d) > 0$ whenever $m, n \geq 3$ and $d \geq 4$, the result follows. $\square$

We also present Theorem 4.1, which shows that all roots of unity occur infinitely often as eigenvalues of RSK.

Theorem 4.1. *If $m \geq 2$ and $n, d \geq k$, then all k-th roots of unity are eigenvalues for $\mathsf{RSK}_{m,n,d}$.*

Proof Sketch. Computation shows that $p_{\mathsf{RSK}_{\sigma,\pi}} = t^k - 1$ when $(\sigma, \pi) = ((k-1, 1), 1^k)$. Then one shows that for these pairs, $N_{\sigma,\pi}(m, n, d) > 0$ whenever $m \geq 2$ and $n, d \geq k$. $\square$

4.2 Determinant of $\mathsf{RSK}_{m,n,d}$

We next consider the determinant of $\mathsf{RSK}_{m,n,d}$, which is ± 1 because both RSK and RSK^{-1} are integer matrices. Theorem 1.1(III) is part (I) of Theorem 4.2 below.

Theorem 4.2. *Let $m, n, d \in \mathbb{N}$.*

(I) Fix d and let r be the least positive integer such that $2^r > d$. Then $\det \mathsf{RSK}_{m,n,d}$ has period 2^r in both m and n, i.e., $\det \mathsf{RSK}_{m,n,d} = \det \mathsf{RSK}_{m+2^r,n,d} = \det \mathsf{RSK}_{m,n+2^r,d}$.

(II) In the case where $m = n$ we have the formula $\det \mathsf{RSK}_{m,n,d} = \prod_\sigma \det \mathsf{RSK}_{\sigma,\sigma}$. The product is over reduced pairs (σ, σ) of degree $d' \leq d$ such that $A_{\sigma,\sigma}(d)\binom{m}{\ell(\sigma)}$ is odd.

Proof Sketch. Both (I) and (II) are direct consequences of Theorem 1.9 and the parity of the constants $N_{\sigma,\pi}(m, n, d)$. Part (I) follows from Lucas's theorem on the parity of $\binom{a}{b}$, while part (II) follows from the fact that $N_{\sigma,\pi}(m, m, d) = 2A_{\sigma,\pi}(d)\binom{m}{\ell(\sigma)}\binom{m}{\ell(\pi)}$ when $\sigma \neq \pi$. □

Example 4.3. By Theorem 4.2, $\det \mathsf{RSK}_{2^k,2^k,d} = 1$ for all $d < 2^k$. Indeed, by Lucas's theorem, each $N_{\sigma,\sigma}(2^k, 2^k, d)$ for reduced (σ, σ) is even, since $\ell(\sigma) \leq d < 2^k$.

Remark 4.4. For a fixed d, the period of $\det \mathsf{RSK}_{m,n,d}$ given in Theorem 4.2(I) need not be minimal. For instance, the minimal period of $\det \mathsf{RSK}_{m,n,4}$ is 4 rather than 8.

For any fixed d, Theorem 4.2(II) allows one to in principle determine $\det \mathsf{RSK}_{m,m,d}$ by computing $\det \mathsf{RSK}_{\sigma,\sigma}$ for a finite collection of weights σ.

4.3 Trace of $\mathsf{RSK}_{m,n,d}$

The first sentence of Theorem 4.5 below is Theorem 1.1(IV).

Theorem 4.5. *For fixed d, $\operatorname{Tr} \mathsf{RSK}_{m,n,d}$ is a polynomial in $O(m^d n^d)$. More specifically,*

$$\operatorname{Tr} \mathsf{RSK}_{m,n,d} = N_0(m, n, d) + \sum_{(\sigma,\pi)} N_{\sigma,\pi}(m, n, d) \operatorname{Tr} \mathsf{RSK}_{\sigma,\pi}. \tag{4.1}$$

The sum is over nonzero reduced pairs (σ, π) of degree $d' \leq d$, and $N_{\sigma,\pi}(m, n, d)$ and $N_0(m, n, d)$ are as in Theorem 1.9. The lead term is $\dfrac{\operatorname{Tr} \mathsf{RSK}_{1^d,1^d}}{(d!)^2} m^d n^d$ whenever $\operatorname{Tr} \mathsf{RSK}_{1^d,1^d} \neq 0$.

Proof. The first formula is immediate from Theorem 1.9. The formula for $N_{\sigma,\pi}(m, n, d)$ in Theorem 1.9 shows that for fixed d, $N_{\sigma,\pi}(m, n, d)$ is a polynomial in m and n of total degree $\ell(\sigma) + \ell(\pi)$. Similarly, the expression for $N_0(m, n, d)$ is a polynomial of total degree $d + 1$. Hence $\operatorname{Tr} \mathsf{RSK}_{m,n,d}$ is polynomial for any fixed d. Moreover, the fastest-growing summand in (4.1) is $N_{1^d,1^d}(m, n, d)$, which has lead term $\frac{(mn)^d}{(d!)^2}$. □

Conjecture 4.6. $\operatorname{Tr} \mathsf{RSK}_{1^d,1^d} \neq 0$ *for $d \neq 2$, i.e., $\operatorname{Tr} \mathsf{RSK}_{m,n,d}$ has total degree $2d$ for $d \neq 2$.*

To investigate Conjecture 4.6, we record a practical formula that follows from Proposition 3.1. It allowed us to compute up to $d = 11$:

$$\{\operatorname{Tr} \mathsf{RSK}_{1^d,1^d}\}_{d\geq 1} = \{1, 0, -3, -5, 23, 96, -279, -3498, 124, 120819, 185838, \ldots\}.$$

Corollary 4.7. $\operatorname{Tr} \mathsf{RSK}_{1^d,1^d} = \sum_{\alpha \in \mathsf{Cont}_{1^d,1^d}} \prod_c \det \alpha_c(\alpha)$.

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