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A new proof of an inverse Kostka matrix problem posed by Eğecioğlu and Remmel and related identities in Sym and NSym

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Abstract. Eğecioğlu and Remmel provide a combinatorial proof (using special rim hook tableaux) that the Kostka matrix times its inverse equals the identity matrix and pose the problem of proving the reverse identity (that the inverse Kostka matrix times the Kostka matrix equals the identity) combinatorially. Sagan and Lee prove a special case of this identity using overlapping special rim hook tableaux. Loehr and Mendes provide a full proof using bijective matrix algebra that relies on the Eğecioğlu–Remmel map. In this extended abstract, we solve the problem in full generality independent of the Eğecioğlu–Remmel bijection. To do this, we start by proving NSym versions of both Kostka matrix identities using the tunnel hook coverings recently introduced by the first and third authors. Then we modify our sign-reversing involutions to reduce to Sym.

Keywords: Kostka matrix, immaculate functions, sign-reversing involutions

1 Background and Introduction

The complete homogeneous functions $\{h_\lambda\}_{\lambda\vdash n}$ and Schur functions $\{s_\lambda\}_{\lambda\vdash n}$ (where $\lambda \vdash n$ means λ is a partition of n) are bases for the vector space Sym of symmetric functions. The Kostka matrix K is the transition matrix from h to s whose entries $K_{\lambda,\mu}$ count the number of semistandard Young tableaux of shape λ and content μ. The inverse Kostka matrix K^{-1} is the transition matrix from s to h whose coefficients are computed via the Jacobi–Trudi formula $s_\lambda = \det(h_{\lambda_i - i + j})$ [7]. Eğecioğlu and Remmel [3] interpret these coefficients as *special rim hook tableaux* and use this to prove the identity $KK^{-1} = I$ combinatorially, leaving the combinatorial proof of $K^{-1}K = I$ as an open problem.

Theorem 1.1 ([3]). *If $\delta_{\lambda,\mu}$ is the Kronecker delta, then $\sum_\nu K_{\lambda,\nu} K^{-1}_{\nu,\mu} = \delta_{\lambda,\mu}$.*

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Our first main result is the generalization of Theorem 1.1 to the vector space NSym of *noncommutative symmetric functions*. Here $\tilde{K}$ is the transition matrix from *complete homogeneous noncommutative symmetric functions* [5] to *immaculate functions* [2].

Theorem 1.2. *Let α, β, and γ be compositions of n. Let $\tilde{K}_{\alpha,\gamma}$ be the number of* immaculate tableaux *of shape α and content γ and let $\tilde{K}^{-1}_{\gamma,\beta}$ be the number of* tunnel hook coverings *of content γ and shape β. Then $\sum_\gamma \tilde{K}_{\alpha,\gamma}\tilde{K}^{-1}_{\gamma,\beta} = \delta_{\alpha,\beta}$.*

Sagan and Lee [8] prove the last column (indexed by (1^n)) of the identity $K^{-1}K = I$, i.e., $\sum_\nu K^{-1}_{\lambda,\nu}K_{\nu,(1^n)} = \delta_{\lambda,(1^n)}$. Loehr and Mendes [6] use bijective matrix algebra to convert the Eğecioğlu–Remmel map into a proof of the $K^{-1}K = I$ identity. Our combinatorial proof of $K^{-1}K = I$ begins with a proof of the NSym version which we then algorithmically reduce to Sym.

Theorem 1.3. *Under the same hypotheses as in Theorem 1.2, we have $\sum_\gamma \tilde{K}^{-1}_{\alpha,\gamma}\tilde{K}_{\gamma,\beta} = \delta_{\alpha,\beta}$.*

We next modify our sign-reversing involution so that it applies to pairs consisting of tunnel hook coverings and semistandard Young tableaux. This provides a solution to the problem posed by Eğecioğlu and Remmel [3] in full generality without relying on the Eğecioğlu–Remmel map.

Theorem 1.4. *If $\delta_{\lambda,\mu}$ is the Kronecker delta, then $\sum_\nu K^{-1}_{\lambda,\nu}K_{\nu,\mu} = \delta_{\lambda,\mu}$.*

In this extended abstract, Ferrers diagrams of partitions are written in English notation with (i, j) designating the cell in the ith row from the top and jth column from the left. Let $\ell(\alpha)$ denote the length of a composition α. Permutations are written in one-line notation; that is, for $\sigma \in S_n$, if $\sigma(i) = \sigma_i$ then we write $\sigma = [\sigma_1, \sigma_2, \ldots, \sigma_n]$. Transpositions, on the other hand, are represented by $s_i = (i, i+1) \in S_n$. We multiply permutations from right to left so that $\sigma\tau = [\sigma_{\tau_1}, \sigma_{\tau_2}, \ldots, \sigma_{\tau_n}]$ where $\tau = [\tau_1, \ldots, \tau_n] \in S_n$.

1.1 Immaculate functions and NSym Kostka matrices

NSym is the algebra generated by the noncommutative elements $\{H_i\}_{i\in\mathbb{Z}_{\geq 0}}$. The H_i can be thought of as the noncommutative analogues of the complete homogeneous functions h_i in Sym. Similar to Schur functions, we can define the *immaculate functions* in terms of a Jacobi–Trudi-like determinant as $\mathfrak{S}_\alpha = \mathfrak{det}(H_{\alpha_i - i + j})$, where $\alpha \vDash n$ (meaning α is a composition of n) and $\mathfrak{det}()$ is the noncommutative Laplace expansion of the determinant from row 1 through row ℓ. Immaculate functions were first defined in of [2] in terms of *noncommutative Berenstein operators* with the Jacobi–Trudi-like identity following [2, Theorem 3.27]. Similar to the Sym case, we can define the NSym Kostka matrix $\tilde{K}_{\alpha,\beta}$ for compositions α and β by $H_\beta = \sum_\alpha \tilde{K}_{\alpha,\beta}\mathfrak{S}_\alpha$.

Definition 5 ([2, Definition 3.9])**.** Let $\alpha, \beta \models n$. An *immaculate tableau* of *shape* α and *content* β is a labeling of the boxes of the diagram of shape α by positive integers in such a way that the number of boxes labeled by i is β_i, the sequence of entries in each row, from left to right, is weakly increasing, and the sequence of entries in the first column, from top to bottom, is strictly increasing. Let $IT_{\alpha,\beta}$ denote the set of immaculate tableaux of shape α and content β.

Theorem 1.6 ([2])**.** *For compositions* α, β*, we have* $\tilde{K}_{\alpha,\beta} = |IT_{\alpha,\beta}|$.

1.2 Tunnel hook coverings

A GBPR diagram [1] is a generalization of a Ferrers diagram where the cells are colored grey, blue, purple, or red. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ be an integer sequence and let $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell)$ be a weak composition. A GBPR diagram $D_{\mu/\nu}$ of shape μ/ν is constructed as follows.

1. For $1 \le i \le \ell$, place ν_i grey cells in row i.
   (a) If $\mu_i > 0$ and $\nu_i \le \mu_i$, place $\mu_i - \nu_i$ blue cells in row i situated immediately to the right of the grey cells.
   (b) If $\mu_i > 0$ and $\mu_i < \nu_i$, place $\nu_i - \mu_i$ red cells in row i situated immediately to the right of the grey cells.
   (c) If $\mu_i \le 0$, place $|\mu_i| + \nu_i$ red cells in row i situated immediately to the right of the grey cells.
2. Any cell that is not colored grey, red, or blue is purple.

Boundary cells are blue, purple, or red cells in the GBPR diagram that are vertically, horizontally, or diagonally adjacent to grey cells. A *tunnel hook* $\mathfrak{h}(j, \tau_j)$ starting in row j and ending in terminal cell $\tau_j = (p_j, q_j)$ includes all blue and red cells in row j, one purple cell in row j if there are no blue or red cells in row j, and all of the boundary cells in rows $j+1 \le i \le p_j$. The *sign* of $\mathfrak{h}(j, \tau_j)$ is $\mathrm{sgn}(\mathfrak{h}(j, \tau_j)) = (-1)^{p_j - j}$. A *tunnel hook covering* (THC) of the GBPR diagram $D_{\mu/\nu^{(0)}}$, where $\nu^{(0)} = (0, 0, \ldots)$, is constructed using the following procedure.

Procedure 7 ([1])**.** *Consider a sequence* $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \in \mathbb{Z}^\ell$ *and a partition* $\nu^{(0)} = (\nu_1, \nu_2, \ldots, \nu_\ell)$ *in which* $\nu_i = 0$ *for* $1 \le i \le \ell$*. Start with the diagram* $\mathcal{D}_{\mu/\nu^{(0)}}$ *and repeat the following steps, once for each value of r from* 1 *to* ℓ*.*

1. *Choose a tunnel hook* $\mathfrak{h}(r, \tau_r)$ *in* $D_{\mu/\nu^{(r-1)}}$*. Set* $\Delta_r = \Delta(\mathfrak{h}(r, \tau_r)) = |\mathfrak{h}(r, \tau_r)| - 2\rho_r - \psi_r$ *where* ρ_r *and* ψ_r *are the number of red cells and purple cells, respectively, in row r that are included in* $h(r, \tau_r)$*.*

2. *For each $1 \leq i \leq \ell$, let $\eta_i^{(r)}$ be the number of cells in row i of $\mathfrak{h}(r, \tau_r)$ and let $\nu^{(r)}$ be the sequence defined for $1 \leq i \leq \ell$ by $\nu_i^{(r)} = \nu_i^{(r-1)} + \eta_i^{(r)}$.*

3. *Construct the GBPR diagram $D_{\mu/\nu^{(r)}}$.*

Let $T = \big(\mathfrak{h}(1, \tau_1), \mathfrak{h}(2, \tau_2), \ldots, \mathfrak{h}(\ell, \tau_\ell)\big)$ be the resulting THC and set $\Delta(T) = (\Delta_1, \ldots, \Delta_\ell)$.

If $\operatorname{flat}(\alpha)$ is the sequence obtained by removing the zeros for an integer sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\ell)$, we call $\operatorname{flat}(\Delta(T))$ the *content* of T. For a tunnel hook covering $T = \big(\mathfrak{h}(1, \tau_1), \mathfrak{h}(2, \tau_2), \ldots, \mathfrak{h}(\ell, \tau_\ell)\big)$, define the *sign* of T to be $\operatorname{sgn}(T) = \prod_{i=1}^{\ell} \operatorname{sgn}(\mathfrak{h}(i, \tau_i))$ For integer sequences α and β, let $\mathrm{THC}_{\alpha,\beta}$ be the set of tunnel hook coverings of content α and shape β.

Theorem 1.8 ([1]). *For compositions α, β of a positive integer n, we have*

$$\tilde{K}^{-1}_{\alpha,\beta} = \sum_{T \in \mathrm{THC}_{\alpha,\beta}} \operatorname{sgn}(T)$$

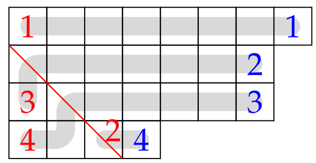
An important ingredient in the proof of the above theorem is the association of a unique permutation to each tunnel hook covering of a diagram. First, the d^{th} *diagonal* $\mathcal{L}_d$ of a GBPR diagram of shape α is the set of all cells of the form $(d + k, 1 + k)$ for some $k \in \mathbb{Z}^{\geq 0}$; that is, $\mathcal{L}_d = \{(d + k, 1 + k) \mid k \in \mathbb{Z}^{\geq 0}\}$. For a tunnel hook covering $T = \big(\mathfrak{h}(1, \tau_1), \mathfrak{h}(2, \tau_2), \ldots, \mathfrak{h}(\ell, \tau_\ell)\big)$, its *permutation* $\sigma \in S_\ell$, denoted $\operatorname{perm}(T)$, is given by setting $\sigma(i) = j$ if $\tau_i \in \mathcal{L}_j$ for each $i \in [\ell]$. See Figure 1 for an example.

Theorem 1.9 ([1]). *For each composition α, the map $T \mapsto \operatorname{perm}(T)$ is a bijection from tunnel hook coverings of shape α to permutations of $\ell(\alpha)$ such that $\operatorname{sgn}(T) = \operatorname{sgn}(\operatorname{perm}(T))$.*

We use the permutation of a tunnel hook covering to construct the sign-reversing involutions necessary to prove $K\tilde{K}^{-1} = \tilde{K}^{-1}K = I$ combinatorially. We need the following two lemmas. Recall that for compositions α and β of n, we say α *dominates* β, denoted $\alpha \trianglerighteq \beta$, if $\alpha_1 + \cdots + \alpha_i \geq \beta_1 + \cdots + \beta_i$ for all $i \in [n]$, where we pad the ends of compositions with zeros as necessary to make the comparisons.

Lemma 10 (Dominance Lemma for IT [2]). *For $\alpha, \beta \vDash n$, if $\alpha \triangleleft \beta$, then $IT_{\alpha,\beta} = \emptyset$. Moreover, if $\alpha = \beta$, then $|IT_{\alpha,\beta}| = 1$.*

Lemma 11 (Dominance Lemma for THC). *For $\alpha, \beta \vDash n$, if $\alpha \triangleleft \beta$, then $\mathrm{THC}_{\alpha,\beta} = \emptyset$ Moreover, if $\alpha = \beta$, then $\mathrm{THC}_{\alpha,\beta}$ has 1 element and it is of positive sign.*

$T =$ 

$\mathrm{perm}(T) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}$

Figure 1: The permutation (in 2-line notation) associated to a tunnel hook covering.

2 A combinatorial proof of Theorem 1.1

In this section, we provide a combinatorial proof that $\tilde{K}\tilde{K}^{-1} = I$. For compositions α, β of n, we have

$$(\tilde{K}\tilde{K}^{-1})_{\alpha\beta} = \sum_{\gamma \models n} \tilde{K}_{\alpha,\gamma}\tilde{K}^{-1}_{\gamma,\beta} = \sum_{(S,T)\in A_{\alpha,\beta}} \mathrm{sgn}(T)$$

where $A_{\alpha,\beta}$ is the set of pairs (S, T) such that S is an immaculate tableau of shape α, T is a tunnel hook covering of shape β, and the content of S is identical to the content of T. To prove the matrix identity $\tilde{K}\tilde{K}^{-1} = I$ combinatorially, we generalize the approach from the Sym case [3]. Specifically, we prove the following.

1. If $\alpha \neq \beta$, there is a sign-reversing involution $\phi_{\alpha,\beta}$ on $A_{\alpha,\beta}$.
2. If $\alpha = \beta$, $A_{\alpha,\alpha}$ contains exactly one pair (S, T), and $\mathrm{sgn}(T) = 1$.

We start with the latter.

Lemma 1. *For any composition α, $A_{\alpha,\alpha}$ contains exactly one pair (S, T), and for this pair (S, T), we have* $\mathrm{perm}(T) = id$.

For $\alpha \neq \beta$, we construct the sign-reversing involution recursively.

Algorithm 2. Suppose α and β are distinct compositions of n and let $(S, T) \in A_{\alpha,\beta}$ of say mutual content $\gamma = (\gamma_1, \ldots, \gamma_\ell)$. We construct another pair $\phi_{\alpha,\beta}(S, T) = (U, V) \in A_{\alpha,\beta}$. Let S^1 denote the first row of S and set $\sigma = \mathrm{perm}(T)$.

1. Pick q such that q appears in S^1 and $\sigma(q) \geq \sigma(k)$ for all $k \in S^1$.
2. Suppose $\sigma(q) = 1$. Then q is the only entry in S^1 and therefore $q = 1$.
   (a) Remove S^1 from S and decrease all other entries in S by 1 to make an immaculate tableau $\overline{S}$ with content $\overline{\gamma} = (\gamma_2, \ldots, \gamma_\ell)$ and shape $\overline{\alpha} = (\alpha_2, \ldots, \alpha_\ell)$. Remove the tunnel hook starting in row 1 of T to make a tunnel hook covering $\overline{T}$ of content $\overline{\gamma}$ and shape $\overline{\beta} = (\beta_2, \ldots, \beta_\ell)$.
   (b) Apply the map $\phi_{\overline{\alpha},\overline{\beta}}$ on $(\overline{S}, \overline{T})$ to produce $(\overline{U}, \overline{V})$.
   (c) Increase all entries of $\overline{U}$ by 1 and append S^1 to make U. Append the first row of T to the top of $\overline{V}$ to make V.

3. Suppose $\sigma(q) > 1$.
   (a) Let V be the unique tunnel hook covering of shape β with $\operatorname{perm}(V) = s_{\sigma(q)-1}\sigma$.
   (b) Let U be obtained by turning a q in S^1 into $p = \sigma^{-1}(\sigma(q)-1)$ and then sorting the first row weakly increasingly.

See Figure 2 for an example.

Theorem 2.3. *Let $\alpha, \beta \vDash n$ have length ℓ. Given a pair $(S,T) \in A_{\alpha,\beta}$, Algorithm 2 is a sign-reversing involution. Moreover, let $\phi_{\alpha,\beta}(S,T) = (U,V)$. If p and q are chosen as in the description of Algorithm 2, we have*

$$\Delta_k(V) = \begin{cases} \Delta_p(T)+1 & k = p \\ \Delta_q(T)-1 & k = q \\ \Delta_k(T) & k \notin \{p,q\}, \end{cases}$$

for all $1 \leq k \leq \ell$.

Proof (Sketch). Let $(S,T) \in A_{\alpha,\beta}$, let S^1 be the first row of S, and set $\sigma = \operatorname{perm}(T)$. First verify the validity of the recursive step by assuming $\sigma(q) = 1$ and proving $q = 1$ and every entry in S^1 is 1. Then removing this first row and applying the algorithm to the resulting pair produces the desired result.

Next verify the nonrecursive step by supposing $\sigma(q) > 1$ and noting that $s_{\sigma(q)-1}\sigma$ is indeed a permutation with opposite sign as σ. Since the size Δ_k(T) of the tunnel hook starting in row k of T is $\beta_k - k + \sigma_k$ for each k, if $p = \sigma^{-1}(\sigma(q)-1)$, then

$$\Delta_q(V) = \beta_q - q + \operatorname{perm}(V)_q = \beta_q - q + \operatorname{perm}(T)_q - 1 = \Delta_q(T) - 1,$$

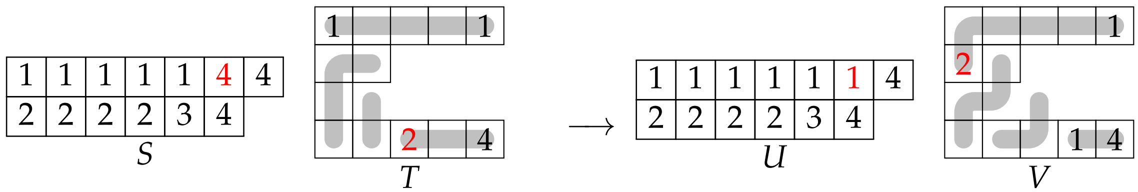
$$\Delta_p(V) = \beta_p - p + \operatorname{perm}(V)_p = \beta_p - p + \operatorname{perm}(T)_p + 1 = \Delta_p(T) + 1.$$

Since no other entries in the permutation are altered by applying the transposition $s_{\sigma(q)-1}$, we have $\Delta_k(V) = \Delta_k(T)$ for all $k \neq p, q$, as desired.

For U, observe that there are at least two distinct numbers in S^1 and that there is an occurrence of q in S^1 that is not the first element of S^1. Hence, the entries in the leftmost column of U must strictly increase since they will equal those in the leftmost column of S. Therefore U is an immaculate tableau. □

We use Algorithm 2 to provide a new combinatorial proof of the classical Kostka matrix identity $KK^{-1} = I$. From [1], for each $\lambda, \mu \vdash n$ we have

$$K^{-1}_{\lambda,\mu} = \sum_{\substack{\alpha \vDash n \\ \operatorname{dec}(\alpha)=\lambda}} \sum_{T \in \mathrm{THC}_{\alpha,\mu}} \operatorname{sgn}(T), \tag{2.1}$$



perm$(T) = [1,4,3,2]$ perm$(V) = [2,4,3,1]$

Figure 2: An example of Algorithm 2. Here $q = 4$.

where $\mathrm{dec}(\alpha)$ is the partition obtained by writing the composition α in weakly decreasing order. Let $B_{\lambda,\mu}$ be the set of pairs (S,T) where S is a semistandard Young tableau of shape λ and T is a tunnel hook covering of shape μ and content α such that $\mathrm{dec}(\alpha)$ is the content of S. Then for $\lambda \neq \mu$, we define a map $\chi_{\lambda,\mu} : B_{\lambda,\mu} \to B_{\lambda,\mu}$ exactly the same as in Algorithm 2, except we replace Step (3b) by the procedure described in [3, p. 71] to produce a semistandard Young tableau of the desired type.

Theorem 2.4. *For all $\lambda, \mu \vdash n$ such that $\lambda \neq \mu$, $\chi_{\lambda,\mu}$ is a sign-reversing involution.*

Therefore, our combinatorial proof of $\tilde{K}\tilde{K}^{-1} = I$ reduces to a new combinatorial proof of $KK^{-1} = I$ in the Sym case, different from the one found by Eğecioğlu and Remmel [3].

3 A combinatorial proof of Theorem 1.3

The goal of this section is to prove that $(\tilde{K}^{-1}\tilde{K})_{\alpha,\beta} = \delta_{\alpha,\beta}$ for all compositions α, β of n. The combinatorial interpretations of these matrices imply that

$$(\tilde{K}^{-1}\tilde{K})_{\alpha,\beta} = \sum_{\gamma \vDash n} \tilde{K}^{-1}_{\alpha,\gamma}\tilde{K}_{\gamma,\beta} = \sum_{(T,S)\in C_{\alpha,\beta}} \mathrm{sgn}(T), \tag{3.1}$$

where $C_{\alpha,\beta}$ is the set of pairs (T,S) such that T is a THC with content α, S is a immaculate tableau of content β, and $sh(S) = sh(T)$. We prove the following.

1. If $\alpha \neq \beta$, there is a sign-reversing involution $\psi_{\alpha,\beta}$ on $C_{\alpha,\beta}$.
2. If $\alpha = \beta$, then $C_{\alpha,\beta}$ contains exactly one pair (T,S). Moreover, $\mathrm{sgn}(T) = 1$.

We start with the latter point. This is the same as Lemma 1 *mutatis mutandis*.

Lemma 1. *For any composition α, $C_{\alpha,\alpha}$ contains exactly one pair (T,S) and for this pair, we have* perm$(T) = id$.

Assume $\alpha \neq \beta$. We describe a sign-reversing involution $\psi_{\alpha,\beta}$ on $D_{\alpha,\beta}$ recursively. In the following, let $\sigma = \mathrm{perm}(T)$ be the permutation associated to T (Theorem 1.9), embedded into S_n by setting $\sigma_i = i$ for $\ell(\alpha) < i \leq n$.

Algorithm 2. Let $\alpha, \beta \vDash n$ where $n > 1$ and $\alpha \neq \beta$, let $(T, S) \in C_{\alpha,\beta}$ with $\alpha \neq \beta$, and set $\sigma = \mathrm{perm}(T)$. We construct another pair $\psi_{\alpha,\beta}(T, S) = (V, U) \in C_{\alpha,\beta}$ as follows.

1. Let m be the maximum element of S. Let $\mathcal{R} = \{\sigma_r | m \text{ appears in row } r \text{ of } S\}$. Select r such that σ_r is the minimal element of $\mathcal{R}$.

2. Suppose $\sigma_r = r$ and S has only m's in row r. Note that this means row r is the last row of S and m only appears in row r of S; otherwise the row containing m above row r would have a lower σ value.

   (a) Let $\overline{S}$ be S with its final row removed. Define $\overline{\sigma} \in S_{n-1}$ by setting

   $$\overline{\sigma}_i = \begin{cases} \sigma_i & i < r \\ i & i \geq r. \end{cases}$$

   Let $\overline{T}$ be the tunnel hook covering whose shape is the shape of T minus the final row with associated permutation $\overline{\sigma}$.

   (b) Set $(\overline{V}, \overline{U}) = \psi_{\overline{\alpha},\overline{\beta}}(\overline{T}, \overline{S})$ where $\overline{\alpha} = (\alpha_1, \ldots, \alpha_{\ell(\alpha)-1})$ and $\overline{\beta} = (\beta_1, \ldots, \beta_{\ell(\beta)-1})$. Let $\overline{\tau} \in S_{n-1}$ be the permutation associated to $\overline{V}$.

   (c) Define U by reattaching the row of m's to the end of $\overline{U}$. If r' is the row of m's in U, then define $\tau \in S_n$ by setting, for all $i \in [n]$,

   $$\tau_k = \begin{cases} \overline{\tau}_i & i < r' \\ i & i \geq r'. \end{cases}$$

3. Suppose either $\sigma_r = r$ and the final row is not all m's or $\sigma_r \neq r$.

   (a) Set $U(c) = S(c)$ for all cells c except for the last cell of row r. Append a cell with value m to the end of row $p = \sigma^{-1}(q + 1)$, where $q = \sigma_r$.

   (b) Let V be the THC of shape $sh(U)$ with $\mathrm{perm}(V) = s_q\sigma$.

See Example 6 for an example of this algorithm. The properties of the output of our involution are described in the following Theorem.

Theorem 3.3. *Assume that Algorithm 2 applied to $(T, S) \in C_{\alpha,\beta}$ produces a pair $(V, U) \in C_{\alpha,\beta}$. Let $\sigma = \mathrm{perm}(T)$ and $\tau = \mathrm{perm}(V)$. Then*

1. $\beta_i - i + \tau_i = \alpha_i - i + \sigma_i$ *for all* $1 \leq i \leq n$, *where* $\beta = sh(U)$ *and* $\alpha = sh(S)$,

2. $\mathrm{sgn}(\tau) = -\mathrm{sgn}(\sigma)$, *and*

3. $U(c) = S(c)$ *for all cells c in the diagram, except for the last cell in row r and in row p.*

It is not hard to see that U is an immaculate tableau. To prove Theorem 3.3, we first show that $q < n$ in Step (3) of Algorithm 2 so that $\tau \in S_n$. Then, we prove Property (1) of Theorem 3.3. Properties (2) and (3), however, are immediate from Step (3) of the algorithm. We need the following Lemmas for our proof.

Lemma 4. *Let $\alpha, \beta \vDash n$, $(T,S) \in C_{\alpha,\beta}$ and r be as in Algorithm 2. If $\mathrm{perm}(T)_r = n$, then $sh(S) = (1^n)$ and $\beta = (1^n)$. If $(V,U) = \psi_{\alpha,\beta}(T,S)$ and $\mathrm{perm}(T)_r \neq n$, then $sh(U)$ is a composition of n.*

Now, we are ready to complete the proof of Theorem 3.3.

Proof of Theorem 3.3. Property (1) of Theorem 3.3. We need to prove that $\beta_i - i + \tau_i = \alpha_i - i + \sigma_i$ for all $1 \leq i \leq n$. The algorithm either begins with Step (3) or applies Step (2) (possibly several times) before applying Step (3). First assume the algorithm begins with Step (3). By construction, $\beta_i - i + \tau_i = \alpha_i - i + \sigma_i$ for $i \neq r, p$. We also have $\beta_r - r + \tau_r = \alpha_r - 1 - r + \sigma_r + 1 = \alpha_r - r + \sigma_r$ and $\beta_p - p + \tau_p = \alpha_p + 1 - p + \sigma_p - 1 = \alpha_p - p + \sigma_p$.

Next assume the algorithm begins with k applications of Step (2), where $1 \leq k < \ell(\alpha)$. This means apply Steps (2a) and (2b) a total of k times, then apply Step (3) once, then apply step (2c) a total of k times. Since $\beta_i = \alpha_i$ and $\tau_i = \sigma_i$ for all $i \neq p, r$, we have $\beta_i - i + \tau_i = \alpha_i - i + \sigma_i \geq 0$ for $i \neq p, r$. Since rows r and p change via Step 3, as do the corresponding permutation entries, we again have $\beta_r - r + \tau_r = \alpha_r - 1 - r + \sigma_r + 1 = \alpha_r - r + \sigma_r$ and $\beta_p - p + \tau_p = \alpha_p + 1 - p + \sigma_p - 1 = \alpha_p - p + \sigma_p$. □

Proposition 5. *Algorithm 2 is an involution. That is, if (V,U) is the image of (T,S) under Algorithm 2, then (T,S) is the image of (V,U).*

Proof. Algorithm 2 begins with either Step (2) or Step (3). We show that under both cases, the algorithm is an involution.

Case 1: *Algorithm 2 begins with Step (3).* Recall that r is chosen so that $\sigma(r) = q$ where q is the smallest element of $\mathcal{R}$. Step (3b) places m into row p where $\sigma(p) = q+1$. Furthermore, note that since $\tau = s_q\sigma$ we have $\tau(p) = q$. Let $\mathcal{R}' = \{\tau_r | m \text{ appears in row r of } U\}$. If m appears exactly once in row r and m does not appear in row p of S, then $\mathcal{R} = \mathcal{R}'$. If m appears more than once in row r of S and does not appear in row p of S, then $\mathcal{R}' = \mathcal{R} \cup \{q+1\}$. If m appears exactly once in row r and at least once in row p then $\mathcal{R}' = \mathcal{R} \setminus \{j+1\}$. In all of the previous cases, q is the smallest entry in $\bar{\mathcal{R}}$. Algorithm 2 applies s_q to τ yielding $s_q\tau = s_q^2\sigma = \sigma$. Also, the algorithm moves m from row p in U to row r yielding our original pair (T,S).

Case 2: *Algorithm 2 begins with Step (2).* Let m be the largest entry in S and assume there is a $k < \ell(\alpha)$ such that for each $1 \leq i \leq k$, all entries in row $\ell(\alpha) - i$ equal $m - i$ and $\sigma(\ell(\alpha) - i) = \ell(\alpha) - i$. Algorithm 2 first applies Steps (2a) and (2b) a total of k times to produce a pair $(\overline{\sigma}, \overline{S})$, where we abuse notation and use just one overline to represent repeated iterations of Steps (2a) and (2b). Next, Step (3) is applied, resulting in the pair $(\overline{V}, \overline{U})$. Finally, Step (2c) is applied a total of k times, resulting in the pair (V, U). Applying Algorithm 2 to the pair (V, U) first reduces the pair to $(\overline{V}, \overline{U})$ by repeated application of Steps (2a) and (2b). The proof of the case beginning with Step (3) now applies to this pair, so that the result of Step (3) is $(\overline{T}, \overline{S})$. Repeated application of Step (2c) then produces the initial pair (T, S), as desired. □

$$(S,T) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 6 \\ \hline 2 & 3 & 5 & \multicolumn{1}{c}{} \\ \hline 4 & 4 & 6 & 6 \\ \hline \end{array} \quad \rightarrow \quad (U,V) = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 6 \\ \hline 2 & 3 & 5 & 6 \\ \hline 4 & 4 & 6 & \multicolumn{1}{c}{} \\ \cline{1-3} \end{array}$$

Figure 3: An example of Algorithm 2

Example 6. For $\alpha = (2, 2, 1, 2, 1, 3)$ and $\beta = (6, 3, 2)$, consider the pair $(S, T) \in C_{\alpha,\beta}$ in Figure 3, where we superimpose the two objects, both of shape $(4, 3, 4)$. The content of T is $(6, 3, 2)$, corresponding to the permutation $\sigma = [3, 2, 1, 4, \cdots]$. Rows 1 and 3 contain the maximal element $m = 6$. Since $\sigma_1 = 3 > 1 = \sigma_3$, we set $r = 3$ in Algorithm 2.

Since $\sigma_r = 1 \neq n$, we move from Step (1) directly to Step (3) in the algorithm. Since $\sigma_3 = 1 =: q$, we have $\tau = [3, 1, 2, 4, \cdots]$. We move the cell in row $r = 3$ to row $p = \sigma^{-1}(1 + 1) = 2$ to create U of shape $(4, 4, 3)$, and then V define to be the unique tunnel hook covering with shape $(4, 4, 3)$ and content $(6, 3, 2)$.

4 Reduction to the Sym Case

In this section, we provide a combinatorial proof that $K^{-1}K = I$ by utilizing our algorithm for showing $\tilde{K}^{-1}\tilde{K} = I$ from Section 3. In particular, we extend Algorithm 2 to pairs of the form (T, S), where T is a THC of content α and S is a semistandard Young tableau (rather than an immaculate tableau) of shape $\mathrm{sh}(T)$.

For $\lambda, \mu \vdash n$, let $E_{\lambda,\mu}$ be the set of pairs (T, S) such that T is tunnel hook covering of content α with $\mathrm{dec}(\alpha) = \lambda$ and S is a semistandard Young tableau of shape $sh(T)$ and content μ. Note that S is partition shaped. Then writing $K^{-1}_{\lambda,\mu}$ as a signed sum of tunnel hook coverings as in Equation (2.1), we have

$$(K^{-1}K)_{\lambda,\mu} = \sum_{(T,S) \in E_{\lambda,\mu}} \mathrm{sgn}(T). \tag{4.1}$$

We prove Theorem 1.4 by exhibiting a sign-reversing involution on $E_{\lambda,\mu}$ when $\lambda \neq \mu$.

Define $F_{\lambda,\mu}$ to be the set of pairs (T, S) as above, except we assume $\lambda \neq \mu$ and replace the condition that "S is a semistandard Young tableau" with "S is an immaculate tableau." Note that the shape of S does not need to be a partition.

For an element $(T,S) \in F_{\lambda,\mu}$, say that c is a *bad cell* of S if the position above c is either not a cell in S or is a cell in S and contains a number that is larger than or equal to the number in the cell c. We have the following two observations.

1. If $(T,S) \in E_{\lambda,\mu}$, then applying Algorithm 2 to (T,S) produces an element of $F_{\lambda,\mu}$.

2. Let $(T,S) \in F_{\lambda,\mu}$. Then S has a bad cell if and only if $(T,S) \notin E_{\lambda,\mu}$.

In order to describe our involution on $E_{\lambda,\mu}$, it is useful to describe an involution on $F_{\lambda,\mu} \setminus E_{\lambda,\mu}$. This map takes heavy inspiration from the sign-reversing involution that Gasharov used in a Schur positivity proof for certain chromatic symmetric functions [4].

Definition 1. For $\lambda, \mu \vdash n$ such that $\lambda \neq \mu$, define a map $b_{\lambda,\mu} : F_{\lambda,\mu} \setminus E_{\lambda,\mu} \to F_{\lambda,\mu} \setminus E_{\lambda,\mu}$ algorithmically as follows.

1. Let $(V,U) \in F_{\lambda,\mu} \setminus E_{\lambda,\mu}$ where V has content β with $\mathrm{dec}(\beta) = \lambda$.

2. Let i be the leftmost column of U containing a bad cell and let t be the largest value such that row t contains a bad cell in column i. Swap the cells of U in the sets $A = \{(t,j) \in U \mid j > i\}$, and $B = \{(t-1,j) \in U \mid j \geq i\}$ to create U'.

3. Let V' be the THC of shape $\mathrm{sh}(U')$ such that $\mathrm{perm}(V') = \mathrm{perm}(V)s_{t-1}$.

4. Set $b_{\lambda,\mu}(V,U) = (V',U')$.

Lemma 2. *For all $\lambda, \mu \vdash n$ with $\lambda \neq \mu$, $b_{\lambda,\mu}$ is a sign-reversing involution on $F_{\lambda,\mu} \setminus E_{\lambda,\mu}$.*

Algorithm 3. Let $\lambda, \mu \vdash n$ such that $\lambda \neq \mu$. Let $(T,S) \in E_{\lambda,\mu}$.

1. Apply Algorithm 2 to (T,S) to get (V,U). If $(V,U) \in E_{\lambda,\mu}$, then output (V,U).

2. Otherwise, let $(V',U') = b_{\lambda,\mu}(V,U)$ and then go back to Step (1).

3. Repeat steps (1) and (2) until $(V,U) \in E_{\lambda,\mu}$.

Theorem 4.4. *For $\lambda, \mu \vdash n$ with $\lambda \neq \mu$, Algorithm 3 is a sign-reversing involution on $E_{\lambda,\mu}$.*

Proof. Turn $F_{\lambda,\mu}$ into a graph by connecting (V,U) to (V',U') if they are related by either

1. (V',U') is output of (V,U) under Algorithm 2, or

2. $(V,U) \in F_{\lambda,\mu} \setminus E_{\lambda,\mu}$ and $b_{\lambda,\mu}(V,U) = (V',U')$.

Since both Algorithm 2 and $b_{\lambda,\mu}$ are involutions, every vertex of $F_{\lambda,\mu}$ has degree 1 or 2. Hence $F_{\lambda,\mu}$ is a disjoint union of paths or cycles (which are all finite since $F_{\lambda,\mu}$ is finite).

Every element of $E_{\lambda,\mu}$ must have degree one while every element of $F_{\lambda,\mu} \setminus E_{\lambda,\mu}$ must have degree two since only Algorithm 2 can produce an element of $E_{\lambda,\mu}$. Therefore every connected component involving an element of $E_{\lambda,\mu}$ must be a path with an odd number of vertices whose endpoints are in $E_{\lambda,\mu}$. Thus, Algorithm 3 is an involution, and sign-reversing since every path has an odd number of vertices. $\square$

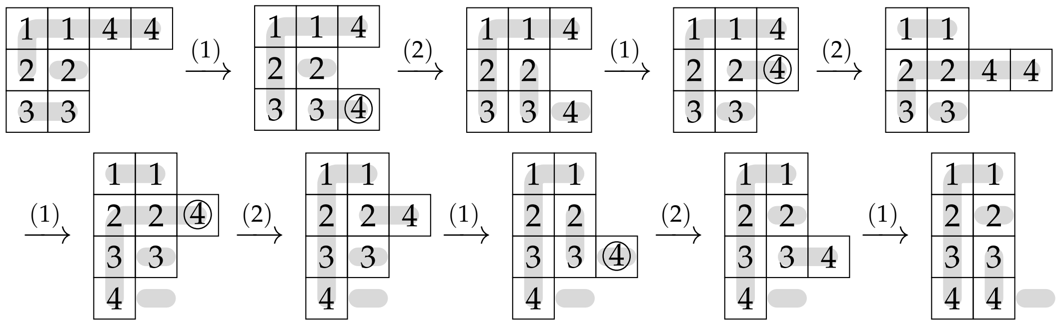


Figure 4: An example of Algorithm 3

Example 5. Figure 4 depicts Algorithm 3 applied to a pair $(S,T) \in E_{\lambda,\mu}$ for $\lambda = (4,2,1)$ and $\mu = (2,2,2,2)$. The (1) and (2) above the arrows indicates which step is performed for the preceding pair. The circled number indicates the bad cell for (2).

Although our map differs from the Loehr–Mendes bijection [6], we conjecture that when our map is restricted to standard Young tableaux, our map is equivalent to the Sym version of Sagan and Lee [8].

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