Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #26, 12 pp.

Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

Bijections between interlacing triangles, Schubert puzzles, and graph colorings

Christian Gaetz*[1] and Yibo Gao†[2]

[1] *Department of Mathematics, University of California, Berkeley, CA, USA.*
[2] *Beijing International Center for Mathematical Research, Peking University, Beijing, China.*

Abstract. We show that *interlacing triangular arrays*, introduced by Aggarwal–Borodin–Wheeler to study certain probability measures, can be used to compute various kinds of Schubert structure constants. We do this by establishing a splitting lemma, allowing for arrays of high rank to be decomposed into arrays of lower rank, and by constructing a bijection between interlacing triangular arrays of rank 3 and certain proper vertex colorings of the triangular grid graph that factors through generalizations of Knutson–Tao puzzles. This proves one enumerative conjecture of Aggarwal–Borodin–Wheeler and disproves another.

Keywords: interlacing triangular array, puzzle, graph coloring, Schubert calculus.

1 Introduction

The *LLT polynomials* are a 1-parameter family of symmetric polynomials introduced by Lascoux, Leclerc, and Thibon [9]. They have close connections to Macdonald polynomials and Kazhdan–Lusztig theory, among other areas of representation theory and geometry. In recent work, Aggarwal, Borodin, and Wheeler [1] studied probability measures arising from the Cauchy identity for LLT polynomials. They show that these measures asymptotically split into a continuous part, given by a product of GUE corners processes, and a discrete part, supported on *interlacing triangular arrays*. They conjectured that these arrays are equinumerous with vertex colorings of the triangular grid graph in the "rank-3" case, and with vertex colorings of another grid graph in the rank-4 case. We construct a bijection between interlacing arrays and vertex colorings, proving the first conjecture, and we disprove the second[1].

Our bijection factors through intermediate objects which are certain *edge* colorings of the triangular grid graph. We recognize these edge labelings as cryptomorphic to certain Schubert calculus *puzzles* [5]. We apply various geometric interpretations of puzzles [4, 7, 10, 11] to prove that the corresponding families of interlacing triangular arrays compute

*gaetz@berkeley.edu
†gaoyibo@bicmr.pku.edu.cn. YG was partially supported by NSFC Grants 12471309.
[1] A full version of the work summarized in this extended abstract appears in [3].

structure constants in cohomology $H^*(\mathrm{Gr}(d,n))$ and K-theory $K(\mathrm{Gr}(d,n))$ of Grassmannians, in the (localized) cohomology $H^{*\mathrm{loc}}_{\mathbb{C}^\times}(T^*\mathrm{Gr}(d,n))$ of their cotangent bundles, and for the multiplication in the cohomology of partial flag varieties of classes pulled back from smaller partial flag varieties.

We now define interlacing triangular arrays, the main objects of study.

Definition 1.1 (Aggarwal–Borodin–Wheeler [1])**.** An *interlacing triangular array* T of rank m and height n is a collection $\{T^{(i)}_{j,k} \mid 1 \le i \le m, 1 \le j \le k \le n\}$ of positive integers from $[m] = \{1, 2, \ldots, m\}$, subject to the following conditions:

(a) For each $k = 1, \ldots, n$ we have an equality of multisets:

$$\{T^{(i)}_{j,k} \mid 1 \le i \le m, 1 \le j \le k\} = \{1^k\} \cup \cdots \cup \{m^k\}.$$

(b) Let the horizontal coordinate of $T^{(i)}_{j,k}$ be $h(i,j,k) := in + j - (n+k)/2$. If $T^{(i)}_{j,k} = T^{(i')}_{j',k} = a$ for some i, j, i', j', k with $h(i,j,k) < h(i',j',k)$, then there must exist i'', j'' with $T^{(i'')}_{j'',k-1} = a$ and $h(i,j,k) < h(i'',j'',k-1) < h(i',j',k)$. This entry $T^{(i'')}_{j'',k-1}$ is said to *interlace* with $T^{(i)}_{j,k}$ and $T^{(i')}_{j',k}$.

For each $k \in [n]$, view $T^{(\bullet)}_{\bullet,k} := \{T^{(i)}_{j,k} \mid 1 \le i \le m, 1 \le j \le k\}$ as the rows of an array of m triangles, from bottom to top. Denote by $\mathcal{T}_{m,n}$ the set of interlacing triangular arrays of rank m and height n and by $\mathcal{T}_{m,n}(\lambda^{(1)}, \ldots, \lambda^{(m)})$ the subset whose top row (that is, the row $k = n$) consists of $\lambda^{(1)}, \ldots, \lambda^{(m)}$; here $\lambda^{(i)} \in [m]^n$ for $i \in [m]$. For $T \in \mathcal{T}_{m,n}$, use $T^{(i)}$ to denote the i-th triangle from left to right, and use $T^{(i)}_{\bullet,k}$ to denote its k-th row.

See Figure 1 for an interlacing trangular array of rank 3 and height 4.

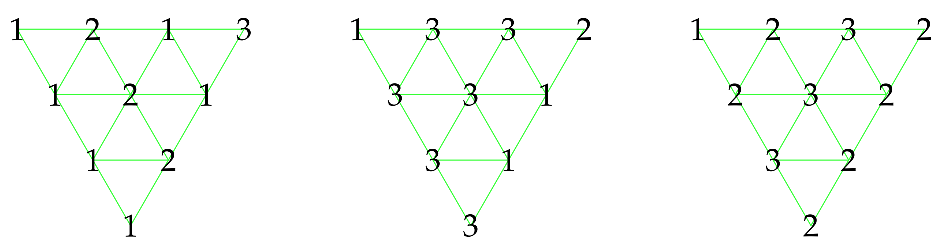


Figure 1: An interlacing triangular array T of rank 3 and height 4. Throughout the paper, we draw interlacing triangular arrays in green (■), 1/2/3-puzzles in blue (■), 0/1/10-puzzles in cyan (■), and vertex-colored graphs in red (■).

1.1 Interlacing triangular arrays and graph colorings

The following conjecture was our original motivation for this work.

Conjecture 1.2 (Conjecture A.3 of Aggarwal–Borodin–Wheeler [1]). *For $n \geq 1$, we have*

$$|\mathcal{T}_{3,n}| = \frac{1}{4}\,|\{\text{proper vertex 4-colorings of } \Delta_n\}|\,,$$

where Δ_n denotes the equilateral triangular grid graph with n edges on each side.

Our first main theorem resolves and significantly refines and extends Conjecture 1.2.

Theorem 1.3. *Fix $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)} \in [3]^n$. Then the following sets of objects are in bijection:*

(1) Interlacing triangular arrays $\mathcal{T}_{3,n}(\boldsymbol{\lambda})$ of rank 3 with top row $\boldsymbol{\lambda}$;

(2) 1/2/3-puzzles $\mathcal{P}_n(\boldsymbol{\lambda})$ with boundary conditions $\boldsymbol{\lambda}$;

(3) 0/1/10-puzzles $\widetilde{\mathcal{P}}_n(\boldsymbol{\xi})$ with boundary conditions $\boldsymbol{\xi} = \mathsf{str}(\boldsymbol{\lambda})$;

(4) Proper vertex 4-colorings $\mathcal{C}_n(\boldsymbol{\kappa})$ of Δ_n with boundary colors $\boldsymbol{\kappa} = \mathsf{col}(\boldsymbol{\lambda})$.

The functions $\mathsf{str}, \mathsf{col}$, and top are straightforward conversions between the different kinds of indexing data. The colorings $\bigsqcup_{\boldsymbol{\kappa}=\mathsf{col}(\lambda)} \mathcal{C}_n(\boldsymbol{\kappa})$ from Theorem 1.3(4) are exactly those proper vertex 4-colorings of Δ_n in which a fixed base vertex is colored with the first color. The number of these is one fourth the total number of proper vertex 4-colorings, so Theorem 1.3 implies Conjecture 1.2.

The particular bijections underlying Theorem 1.3 allow us to give geometric interpretations for certain sets of interlacing triangular arrays (see Section 1.2).

Aggarwal, Borodin, and Wheeler also conjectured a connection between interlacing triangular arrays of rank 4 and graph colorings. Let $\boxtimes_n$ be the graph obtained from the $n \times n$ square grid graph $\square_n$ by adding the two diagonal edges of each face of $\square_n$.

Conjecture 1.4 (Conjecture A.5 of Aggarwal–Borodin–Wheeler [1]). *For $n \geq 1$, we have*

$$|\mathcal{T}_{4,n}| = \frac{1}{5}\,|\{\text{proper vertex 5-colorings of } \boxtimes_n\}|\,. \tag{1.1}$$

In Theorem 1.5 we give a bijection between interlacing triangular arrays of rank 4 and certain edge labelings of the square grid graph $\square_n$, which are analogous to the 1/2/3-puzzles of Theorem 1.3(2).

Theorem 1.5. *Fix $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda^{(4)} \in [4]^n$. Then the followings are in bijection:*

(1) Interlacing triangular arrays $\mathcal{T}_{4,n}(\boldsymbol{\lambda})$ of rank 4 with top row $\boldsymbol{\lambda}$;

(2) Edge labelings $\mathcal{D}_n(\boldsymbol{\lambda})$ of the square grid graph $\square_n$ having boundary conditions $\boldsymbol{\lambda}$ and satisfying the conditions of Section 3.2.

Unlike in the rank-3 case, there is no straightforward way to biject the objects in Theorem 1.5 with proper *vertex* colorings. In particular, by enumerating $\mathcal{D}_4$ we show that Conjecture 1.4 is false[2], as

$$|\mathcal{T}_{4,4}| = 191232 \neq 187008 = \frac{1}{5}\,|\{\text{proper vertex 5-colorings of } \boxtimes_4\}|\,.$$

1.2 Geometric interpretations of interlacing triangular arrays

The 0/1/10-puzzles appearing in Theorem 1.3(3) are known to have various geometric interpretations when certain puzzle pieces are forbidden and when the boundary conditions are appropriate. The number of such puzzles with boundary conditions $\boldsymbol{\xi} = (\xi^{(1)}, \xi^{(2)}, \xi^{(3)})$ computes the coefficient of the basis element indexed by $\xi^{(3)}$ in the product of basis elements indexed by $\xi^{(1)}$ and $\xi^{(2)}$ in cohomology $H^*(\mathrm{Gr}(d,n))$ and K-theory $K(\mathrm{Gr}(d,n))$ of Grassmannians, in the (appropriately localized) cohomology $H^{*\mathrm{loc}}_{\mathbb{C}^\times}(T^*\,\mathrm{Gr}(d,n))$ of their cotangent bundles, and for the multiplication in the cohomology of the 2-step flag variety [6, 7, 10, 11].

We use the specific bijections underlying Theorem 1.3 to show that interlacing triangular arrays with forbidden patterns and specified top row likewise compute these coefficients. One advantage of interlacing triangular arrays is that they allow for an interpretation of coefficients in the expansion of an $(m-1)$-fold product, without the need to iteratively apply a rule for products of two elements.

For ξ a 0, 1-string with content 0^d1^{n-d}, let G_ξ denote the class of the structure sheaf of the Schubert variety $X_\xi \subset \mathrm{Gr}(d,n)$ inside $K(\mathrm{Gr}(d,n))$. These classes can be represented by the *(Grassmannian) Grothendieck polynomials*. The $\{G_\xi\}$ form a basis for $K(\mathrm{Gr}(d,n))$. In Theorem 1.6 we show that the structure constants for multiplication in the basis $\{G_\xi\}$ are equal (up to signs) to the number of certain interlacing triangular arrays. Even the positivity of these structure constants (up to predictable signs) is not obvious, and is due originally to Buch [2].

Given $\boldsymbol{\xi} = (\xi^{(1)}, \dots, \xi^{(m)})$ of 0, 1-strings, define $|\boldsymbol{\xi}| = \sum_i |\xi^{(i)}|$, where $|\xi^{(i)}|$ is the number of inversions of $\xi^{(i)}$. For ξ of content 0^d1^{n-d}, denote by $\xi^\perp$ the reversed string.

Theorem 1.6. *Let $\xi^{(1)}, \dots, \xi^{(m)}$ have content 0^d1^{n-d}. Let coefficients $g_{\boldsymbol{\xi}} = g_{\xi^{(1)},\dots,\xi^{(m)}}$ be determined by*

$$\prod_{i=1}^{m-1} G_{\xi^{(i)}} = \sum_{\xi^{(m)}} g_{\boldsymbol{\xi}} G_{(\xi^{(m)})^\perp}.$$

[2] Leonid Petrov has independently observed the failure of Conjecture 1.4 (personal communication).

Then $(-1)^{d(n-d)-|\xi|}g_{\boldsymbol{\xi}}$ *is the number of interlacing triangular arrays* T *from* $\mathcal{T}_{m,n}(\mathsf{top}(\boldsymbol{\xi}))$ *such that, for* $i = 2, \ldots, m-1$, $T^{(i)}$ *avoids*

(1.2)

Dual to $\{G_\xi\}$ is the basis $\{G^*_\xi\}$ of ideal sheaves: functions on Schubert varieties vanishing on smaller Schubert varieties.

Theorem 1.7. *Let* $\xi^{(1)}, \ldots, \xi^{(m)}$ *have content* $0^d 1^{n-d}$. *Let coefficients* $g^*_{\boldsymbol{\xi}} = g^*_{\xi^{(1)},\ldots,\xi^{(m)}}$ *be determined by*

$$\prod_{i=1}^{m-1} G^*_{\xi^{(i)}} = \sum_{\xi^{(m)}} g^*_{\boldsymbol{\xi}} G^*_{(\xi^{(m)})^\perp}.$$

Then $(-1)^{d(n-d)-|\xi|}g^*_{\boldsymbol{\xi}}$ *is the number of interlacing triangular arrays* T *from* $\mathcal{T}_{m,n}(\mathsf{top}(\boldsymbol{\xi}))$ *such that, for* $i = 2, \ldots, m-1$, $T^{(i)}$ *avoids*

(1.3)

Interlacing triangular arrays avoiding *both* the patterns (1.2) and (1.3) compute structure constants in the ordinary cohomology of the Grassmannian. We can generalize this case to products of certain classes in the cohomology of arbitrary partial flag varieties.

For $\boldsymbol{d} = (0 = d_0 \leq d_1 \leq \cdots \leq d_m = n)$, let $\mathrm{Fl}(\boldsymbol{d}; n)$ denote the partial flag variety of flags of subspaces of $\mathbb{C}^n$ with dimension vector $\boldsymbol{d}$. Let $S_n^{\boldsymbol{d}}$ denote the set of permutations whose descents are contained in $\boldsymbol{d}$. Then $H^*(\mathrm{Fl}(\boldsymbol{d}; n))$ has a basis $\{\sigma_w\}_{w \in S_n^{\boldsymbol{d}}}$ consisting of the classes of the Schubert varieties in $\mathrm{Fl}(\boldsymbol{d}; n)$. In particular, the class $\sigma_{w_0^{\boldsymbol{d}}}$ of the longest element of $S_n^{\boldsymbol{d}}$ is the class of a point. For $w \in S_n^{\boldsymbol{d}}$, write $w^{\vee_{\boldsymbol{d}}} := w_0 w w_0(\boldsymbol{d}) \in S_n^{\boldsymbol{d}}$. For $w^{(1)}, \ldots, w^{(m)} \in S_n^{\boldsymbol{d}}$, let coefficients $c_{\boldsymbol{w}} = c_{w^{(1)},\ldots,w^{(m)}}$ be determined by

$$\prod_{i=1}^{m-1} \sigma_{w^{(i)}} = \sum_{w^{(m)}} c_{\boldsymbol{w}} \sigma_{(w^{(m)})^{\vee_{\boldsymbol{d}}}},$$

For a finite alphabet with a total order $\Sigma = \{q_1 < \cdots < q_k\}$, we say that λ has *type* $q^\alpha = q_1^{\alpha_1} \cdots q_k^{\alpha_k}$ if λ contains α_i copies of q_i, for $i = 1, \ldots, k$. For such a string, we associate a permutation $w(\lambda)_\Sigma = w \in S_n$ such that $w(d_{i-1}+1) < \cdots < w(d_i)$ are the positions of q_i's in λ, where $\alpha = \alpha(\boldsymbol{d})$ is defined by $\alpha_1 + \cdots + \alpha_i = d_i$ for $i = 1, \ldots, k$.

Theorem 1.8. *Let* $\boldsymbol{d} = (0 = d_0 \leq d_1 \leq \cdots \leq d_m = n)$. *For* $i \in [m]$ *let* $\Sigma_i = \{m - i < m < m - i + 1\}$, *let* $\lambda^{(i)}$ *be a string of type* $(m-i)^{d_{m-i}} m^{d_{m-i+1}-d_{m-i}} (m-i+1)^{n-d_{m-i+1}}$, *and let* $w^{(i)} = w(\lambda^{(i)})_{\Sigma_i} \in S_n^{\boldsymbol{d}}$. *Then* $c_{\boldsymbol{w}}$ *is the number of interlacing triangular arrays* T *from* $\mathcal{T}_{m,n}(\boldsymbol{\lambda})$ *such that, for* $i = 2, \ldots, m-1$, $T^{(i)}$ *avoids the patterns from (1.2) and (1.3).*

The Schubert classes $\sigma_{w^{(i)}}$ appearing in Theorem 1.8 are the pullbacks of Schubert classes under the projection from $\mathrm{Fl}(\boldsymbol{d};n)$ to certain 2-step flag varieties.

Finally, let $H^{*\mathrm{loc}}_{\mathbb{C}^\times}(T^*\,\mathrm{Gr}(d,n))$ denote the equivariant cohomology of the cotangent bundle of $\mathrm{Gr}(d,n)$ with respect to the $\mathbb{C}^\times$-action scaling the cotangent spaces, localized as in [7, Section 2.2]. For ξ of content 0^d1^{n-d}, let $S_\xi \in H^{*\mathrm{loc}}_{\mathbb{C}^\times}(T^*\,\mathrm{Gr}(d,n))$ denote the *Segre–Schwartz–MacPherson (SSM) class* of the corresponding Schubert variety, using the conventions of [7, Sections 2.4 and 5.2]. In our last main theorem, we show that interlacing triangular arrays, with *no* forbidden patterns, compute structure constants for the $\{S_\xi\}$.

Theorem 1.9. *Let $\xi^{(1)},\ldots,\xi^{(m)}$ have content 0^d1^{n-d}. Let coefficients $s_{\boldsymbol{\xi}} = s_{\xi^{(1)},\ldots,\xi^{(m)}}$ be determined by*

$$\prod_{i=1}^{m-1} S_{\xi^{(i)}} = \sum_{\xi^{(m)}} s_{\boldsymbol{\xi}} S_{(\xi^{(m)})^\perp}.$$

Then $(-1)^{d(n-d)-|\xi|}s_{\boldsymbol{\xi}}$ is the cardinality of $\mathcal{T}_{m,n}(\mathsf{top}(\boldsymbol{\xi}))$.

1.3 Examples of the geometric interpretations

Example 1.10. Let $m = 4$ and $n = 4$. Consider the 0, 1-strings $\xi^{(1)} = \xi^{(2)} = \xi^{(3)} = 0101$ with length 1. Correspondingly, $\lambda^{(1)} = \mathsf{top}(\xi^{(1)}) = 3434$, and analogously, $\lambda^{(2)} = 2323$ and $\lambda^{(3)} = 1212$. They also correspond to the permutation 1324 and the partition with one box. As an example of Theorem 1.8, in $H^*(\mathrm{Gr}(2,4))$, $\sigma_{1324}^3 = 2\sigma_{2413}$, whose 0, 1-string is 1010 and the coefficient 2 counts the interlacing triangular arrays in Figure 2.

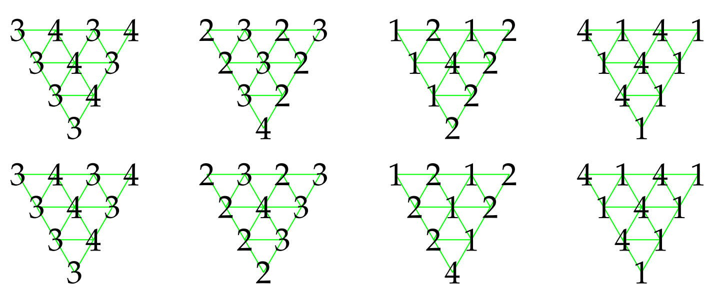


Figure 2: Interlacing triangular arrays with top row $\lambda^{(1)} = 3434, \lambda^{(2)} = 2323, \lambda^{(3)} = 1212, \lambda^{(4)} = 4141$.

There are more interlacing triangular arrays whose top row starts with $\lambda^{(1)} = 3434$, $\lambda^{(2)} = 2323$, $\lambda^{(3)} = 1212$. Besides the ones in Figure 2, all the others have top row $\lambda^{(4)} = 4411$ shown in Figure 3. The first five of them contain patterns from (1.2) and the last one contains patterns from (1.3), highlighted in the figure.

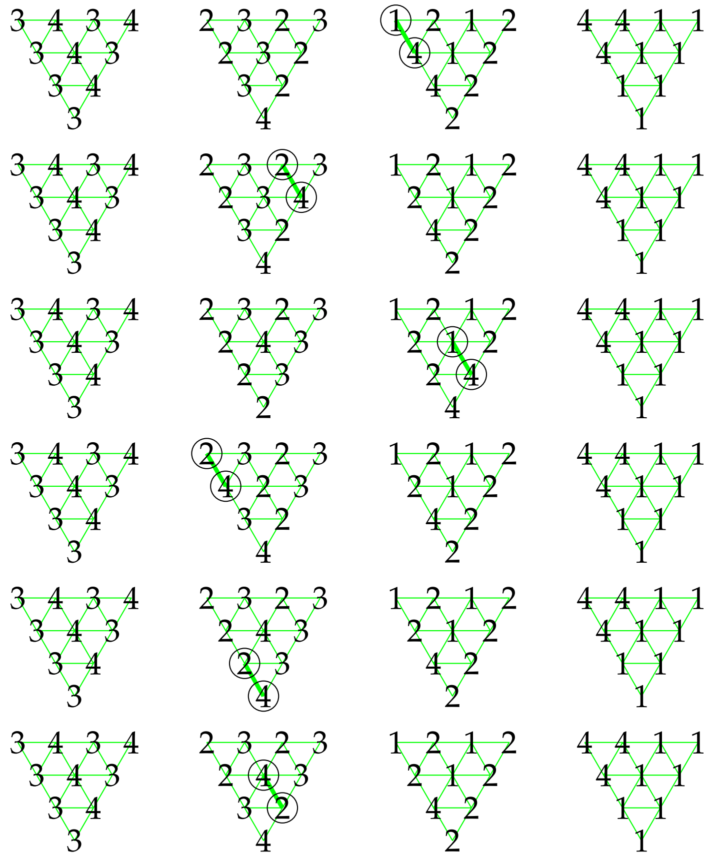


Figure 3: Interlacing triangular arrays with top row $\lambda^{(1)} = 3434, \lambda^{(2)} = 2323, \lambda^{(3)} = 1212, \lambda^{(4)} = 4411$.

Now, Theorem 1.6, Theorem 1.7 and Theorem 1.9 imply that

$$G_{0101}^3 = 2G_{1010} - G_{1100}, \quad (G_{0101}^*)^3 = 2G_{1010}^* - 5G_{1100}^*, \quad S_{0101}^3 = 2S_{1010} - 6S_{1100}.$$

1.4 Outline

In Section 2, we prove Theorem 2.4, establishing bijections between rank-3 interlacing triangular arrays $\mathcal{T}_{3,n}$ and 1/2/3-puzzles $\mathcal{P}_n$. In Section 3 we in turn prove Theorem 3.2, giving bijections between 1/2/3-puzzles $\mathcal{P}_n$ and proper vertex colorings $\mathcal{C}_n$ of Δ_n. In Theorem 3.4 we also give bijections between rank-4 interlacing triangular arrays $\mathcal{T}_{4,n}$ and certain edge labelings $\mathcal{D}_n$ of $\square_n$. Finally, in Section 4 we sketch some key ideas that go into the proofs of the geometric interpretations in Theorems 1.6 to 1.9. These include a correspondence between 1/2/3-puzzles and 0/1/10-puzzles for which we are grateful to Allen Knutson and a *splitting lemma* on certain interlacing triangular arrays allowing us to reduce to the case $m = 3$.

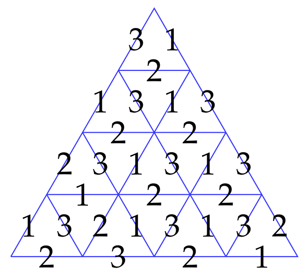


Figure 4: A 1/2/3-puzzle P with boundary conditions $(1213, 1332, 1232)$.

2 From interlacing triangular arrays to puzzles

The goal of this section is to establish a bijection $\mathscr{T}$ between interlacing triangular arrays of rank 3 and certain edge labelings of Δ_n. We call these labelings 1/2/3-*puzzles* since they in turn are in bijection with the 0/1/10-puzzles of Knutson–Tao [5] as generalized by Knutson–Zinn-Justin [7, Section 4].

Definition 2.1. We denote by Δ_n the *triangular grid graph* with side length n; see Figure 4. We view Δ_n as embedded in the plane as pictured, allowing us to distinguish between the Δ-oriented and ∇-oriented faces. We take the lower left corner as a distinguished base point, and view Δ_{n-1} as a subgraph of Δ_n, sharing the base point.

A 1/2/3-*puzzle* is a labeling of the edges of Δ_n with labels 1, 2, and 3 so that each face has distinct edge labels. We write $\mathcal{P}_n$ for the set of these puzzles.

The *boundary conditions* $\boldsymbol{\mu} = (\mu^{(1)}, \mu^{(2)}, \mu^{(3)})$ of a puzzle $P \in \mathcal{P}_n$ are the labelings of the three sides of the triangle Δ_n, read clockwise starting from the base point. We write $\mathcal{P}_n(\boldsymbol{\mu})$ for the set of puzzles from $\mathcal{P}_n$ with boundary conditions $\boldsymbol{\mu}$.

Definition 2.2. Given a 1/2/3-puzzle $P \in \mathcal{P}_n$, we produce a collection

$$\mathscr{T}(P) = \{T^{(i)}_{j,k} \mid 1 \leq i \leq 3, 1 \leq j \leq k \leq n\}$$

of integers from $\{1, 2, 3\}$ as follows. For each $k = 1, \ldots, n$, consider the copy of Δ_k inside Δ_n justified into the lower left corner of Δ_n. The k-th row of $\mathscr{T}(P)$ is obtained by reading the labels of P clockwise around Δ_k, starting from the lower left vertex.

Example 2.3. The 1/2/3-puzzle P from Figure 4 is sent by $\mathscr{T}$ to the array T from Figure 1.

Theorem 2.4. *For any $P \in \mathcal{P}_n$, the array $\mathscr{T}(P)$ is an interlacing triangular array of rank 3 and height n. Furthermore, map $\mathscr{T}$ is a bijection $\mathcal{P}_n \to \mathcal{T}_{3,n}$ restricting, for each boundary condition $\boldsymbol{\lambda}$, to a bijection $\mathcal{P}_n(\boldsymbol{\lambda}) \to \mathcal{T}_{3,n}(\boldsymbol{\lambda})$.*

3 From puzzles to graph colorings

In this section, we relate interlacing triangular arrays and 123-puzzles to graph colorings. We prove Conjecture A.3 of [1] and disprove Conjecture A.5. We also formulate a new conjecture (Conjecture 3.5) on the enumeration of $\mathcal{T}_{4,n}$.

3.1 $\mathcal{T}_{3,n}$ and the triangular grid

Recall from Section 2 that Δ_n denotes the triangular grid graph with side length n, a graph on $\binom{n+2}{2}$ vertices. Let $\mathcal{C}_n$ denote the set of proper colorings of the vertices of Δ_n by $\{0,1,2,3\}$ such that the base point is colored 0. We write $\mathcal{C}_n(\boldsymbol{\kappa})$ for the set of colorings from $\mathcal{C}_n$ with fixed coloring $\boldsymbol{\kappa}$ of the boundary vertices. We have proven in Theorem 2.4 that the map $\mathscr{T}$ is a bijection $\mathscr{T} : \mathcal{P}_n \to \mathcal{T}_{3,n}$. We now define a map $\mathscr{P} : \mathcal{C}_n \to \mathcal{P}_n$, proven in Theorem 3.2 to be a bijection.

Definition 3.1. Given a proper vertex coloring $C \in \mathcal{C}_n$, define an edge labeling $\mathscr{P}(C)$ as follows: for an edge e of Δ_n incident to vertices v and v', set $\mathscr{P}(C)(e) = i$ if $\{c(v), c(v')\} = \{0, i\}$ or $\{0,1,2,3\} \setminus \{0, i\}$. See Figure 5 for an example. By Theorem 3.2 below, $\mathscr{P}$ is invertible on 1/2/3-puzzles and so determines boundary colors $\mathsf{col}(\boldsymbol{\lambda})$ given boundary conditions $\boldsymbol{\lambda}$ of a puzzle.

Theorem 3.2. *The map $\mathscr{P}$ is a bijection $\mathcal{C}_n \to \mathcal{P}_n$ sending $\mathcal{C}(\mathsf{col}(\boldsymbol{\lambda}))$ to $\mathcal{P}_n(\boldsymbol{\lambda})$.*

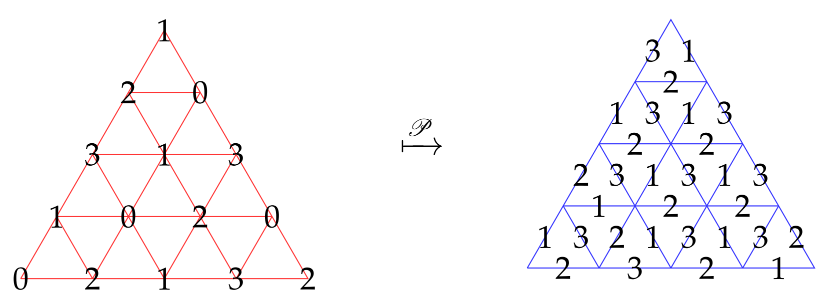


Figure 5: The bijection $\mathscr{P}$ from Theorem 3.2.

3.2 $\mathcal{T}_{4,n}$ and the square grid

Let $\square_n$ be the square grid graph with side length n. For $k < n$, we view $\square_k$ as a southwest-justified subgraph of $\square_n$, and we fix the southwest corner as the basepoint. Let $\mathcal{D}_n$ be the set of edge labelings of $\square_n$ with four labels satisfying the following:

- for each face on the main SW-NE diagonal, all four edges are labeled differently;

- for each of the off-diagonal faces, the four edges are labeled with exactly two distinct labels so that either one label is assigned to the west and south boundaries (with the other label assigned to the north and east boundaries), or one label is assigned to the north and south boundaries (with the other label assigned to the west and east boundaries).

We now define a map $\mathscr{D}'$ from $\mathcal{D}_n$ to certain triangular arrays of integers.

Definition 3.3. Given an edge labeling $D \in \mathcal{D}_n$, let $\mathscr{D}'(D)$ be the triangular array of integers whose k-th row is obtained by reading the D-labels around the boundary of $\square_k$ in the clockwise direction, starting from the base point. See Figure 6 for an example.

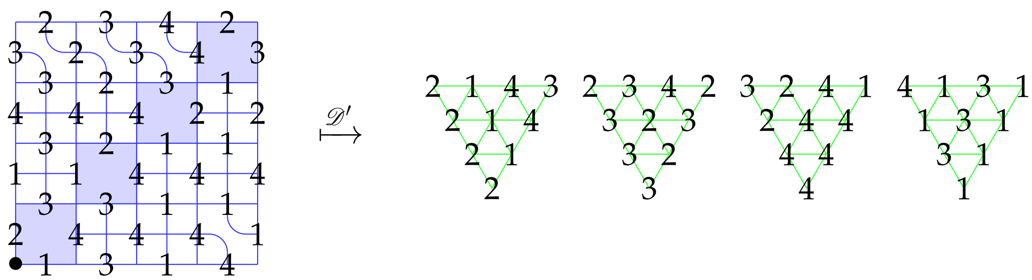


Figure 6: The bijection $\mathscr{D}'$ from Theorem 3.4.

Theorem 3.4. *The map $\mathscr{D}'$ is a bijection $\mathcal{D}_n \to \mathcal{T}_{4,n}$.*

By enumerating the labelings $\mathcal{D}_n$, we have computed for $n = 0, 1, 2, 3$ that $|\mathcal{T}_{4,n}| =$ 1, 24, 1344, 191232. This last value disagrees with $\frac{1}{5}|\{\text{proper vertex 5-colorings of } \boxtimes_n\}|$ from [1, Conjecture A.5] which for $n = 0, 1, 2, 3$ is equal[3] to 1, 24, 1344, 187008, disproving the conjecture of Aggarwal–Borodin–Wheeler. However we make a new conjecture for $|\mathcal{T}_{4,n}|$ that has been checked up to $n = 7$. This replaces vertex colorings of $\boxtimes_n$ with edge labelings of $\square_n$ and is a direct extension of the equinumerosity of $\mathcal{P}_n$ and $\mathcal{T}_{3,n}$.

Conjecture 3.5. *$|\mathcal{T}_{4,n}|$ equals the number of edge labelings of $\square_n$ with four labels such that the four sides of each face have distinct labels.*

4 Ingredients in the geometric interpretations

We end by giving a couple of combinatorial ingredients used in our proofs of Theorems 1.6 to 1.9. The full details are available in [3, Section 5].

[3] See the OEIS entry A068294.

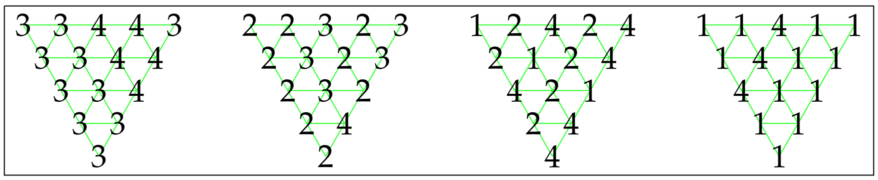
4.1 The splitting lemma

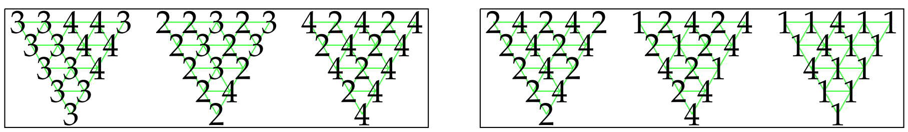
Lemma 4.1. *Fix $m \geq 3$ and $\boldsymbol{d}$ and strings $\lambda^{(i)}$ of type $(m-i)^{d_{m-i}} m^{d_{m-i+1}-d_{m-i}} (m-i+1)^{n-d_{m-i+1}}$. Then there is a bijection* split *from $\mathcal{T}_{m,n}(\lambda^{(1)}, \ldots, \lambda^{(m)})$ to*

$$\bigsqcup_{\mu} \mathcal{T}_{3,n}(\lambda^{(1)}, \lambda^{(2)}, \mu) \times \mathcal{T}_{m-1,n}(\mu^{\dagger}, \lambda^{(3)}, \ldots, \lambda^{(m)}), \tag{4.1}$$

where μ runs over type $(m-2)^{n-d_{m-2}} m^{d_{m-2}}$ and $\mu^{\dagger}$ reverses and swaps the letters of μ.

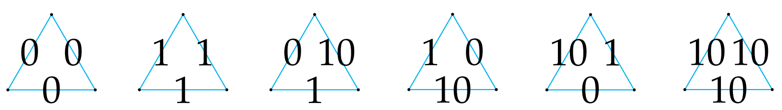
Example 4.2. Let $T \in \mathcal{T}_{4,5}$ be as below; then $\mathsf{split}(T) \in \mathcal{T}_{3,5} \times \mathcal{T}_{3,5}$ is shown on bottom.





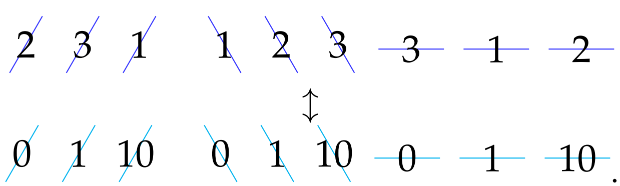
4.2 Puzzle conversion

A 0/1/10-*puzzle* is a labeling of the edges of Δ_n with labels 0, 1, and 10 so that each Δ-oriented face is labeled in one of the following ways



and so that each ∇-oriented face is labeled by a 180° rotation of one of these[4]. We write $\widetilde{\mathcal{P}}_n(\boldsymbol{\xi})$ for the set of 0/1/10-puzzles on Δ_n with boundary conditions $\boldsymbol{\xi}$.

We are grateful to Allen Knutson for sharing with us the following correspondence between 1/2/3-puzzles and 0/1/10-puzzles. Given $\boldsymbol{\lambda} = (\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)})$, let $\mathsf{str}(\boldsymbol{\lambda})$ be the 0, 1-strings $(\xi^{(1)}, \xi^{(2)}, \xi^{(3)})$ obtained by applying the transformation below:



Proposition 4.3. *For any boundary condition $\boldsymbol{\lambda}$, the transformation of edge labels shown above determines a bijection $\mathcal{P}_n(\boldsymbol{\lambda}) \to \widetilde{\mathcal{P}}_n(\mathsf{str}(\boldsymbol{\lambda}))$, where* top *denotes the inverse to* str.

[4] See [8, Section 5] for the relationship between these puzzles and others which have appeared in the literature.

Acknowledgements

We are grateful to Richard Stanley and Alexei Borodin for introducing us to this problem. We also thank Allen Knutson and Paul Zinn-Justin for their insightful comments.

References

[1] A. Aggarwal, A. Borodin, and M. Wheeler. "Coloured corner processes from asymptotics of LLT polynomials". *Adv. Math.* **451** (2024), Paper No. 109781, 85 pp. DOI.

[2] A. S. Buch. "A Littlewood-Richardson rule for the K-theory of Grassmannians". *Acta Math.* **189**.1 (2002), pp. 37–78. DOI.

[3] C. Gaetz and Y. Gao. "Interlacing triangles, Schubert puzzles, and graph colorings" (2024). arXiv:2408.07863.

[4] I. Halacheva, A. Knutson, and P. Zinn-Justin. "Restricting Schubert classes to symplectic Grassmannians using self-dual puzzles". *Sém. Lothar. Combin.* **82B** (2020), Art. 83, 12 pp.

[5] A. Knutson and T. Tao. "Puzzles and (equivariant) cohomology of Grassmannians". *Duke Math. J.* **119**.2 (2003), pp. 221–260. DOI.

[6] A. Knutson, T. Tao, and C. Woodward. "The honeycomb model of $\mathrm{GL}_n(\mathbb{C})$ tensor products. II. Puzzles determine facets of the Littlewood-Richardson cone". *J. Amer. Math. Soc.* **17**.1 (2004), pp. 19–48. DOI.

[7] A. Knutson and P. Zinn-Justin. "Schubert puzzles and integrability II: multiplying motivic Segre classes". 2021. arXiv:2102.00563.

[8] A. Knutson and P. Zinn-Justin. "Schubert puzzles and integrability III: separated descents". 2023. arXiv:2306.13855.

[9] A. Lascoux, B. Leclerc, and J.-Y. Thibon. "Ribbon tableaux, Hall-Littlewood functions, quantum affine algebras, and unipotent varieties". *J. Math. Phys.* **38**.2 (1997), pp. 1041–1068. DOI.

[10] R. Vakil. "A geometric Littlewood-Richardson rule". *Ann. of Math. (2)* **164**.2 (2006). Appendix A written with A. Knutson, pp. 371–421. DOI.

[11] M. Wheeler and P. Zinn-Justin. "Littlewood-Richardson coefficients for Grothendieck polynomials from integrability". *J. Reine Angew. Math.* **757** (2019), pp. 159–195. DOI.