Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #27, 12 pp.

Proceedings of the 37[th] *Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)*

Generic pipe dreams, lower-upper varieties, and Schwartz–MacPherson classes

Allen Knutson*[1] and Paul Zinn-Justin†[2]

[1] *Cornell University*
[2] *The University of Melbourne*

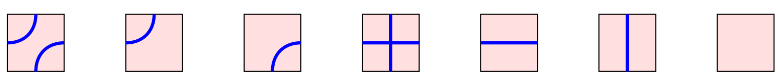
Abstract. We recall the *lower-upper varieties* E_w from [Knutson '05] and give a formula for their equivariant cohomology classes, as a sum over *generic pipe dreams*. We recover as limits the classic and bumpless pipe dream formulæ for double Schubert polynomials. As a byproduct, we obtain a formula for the degree of the nth commuting variety as a sum of powers of 2.

Generic pipe dreams also appear in the Segre–Schwartz–MacPherson analogue of the AJS/Billey formula, and when computing the Chern–Schwartz–MacPherson class of the orbit $B_-wB_+ \subseteq Mat_{k\times n}$ or of a double Bruhat cell $B_-uB_+ \cap B_+vB_-$.

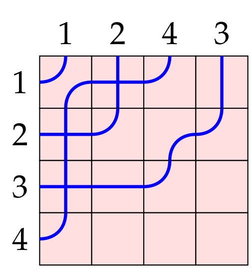
Keywords: Schubert calculus, pipe dreams, Schubert polynomials, commuting variety

1 Generic pipe dreams

Define a **generic pipe dream tile** as any of the following:



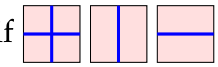
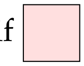
Each pipe will carry a distinct label, generally from $[n] := \{1, \ldots, n\}$. We assemble these tiles into $n \times n$ squares (and later, into quadrangulated discs), calling them **generic pipe dreams** or **GPDs**. For our first two applications of GPDs we insist that the pipes down the West side are numbered $1 \ldots n$ in order, that the pipes across the North side are numbered $w^{-1}(1), \ldots, w^{-1}(n)$ for $w \in \mathcal{S}_n$, and that the East and South sides are blank. The set of such GPDs is denoted $GPDs(w)$. Here is one of the 45 GPDs for $w = 1243$:



*allenk@math.cornell.edu
†pzinn@unimelb.edu.au

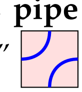
Define the **generic pipe dream polynomial** $G_w \in \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n, A, B]$ to be the sum $\sum_{\delta \in GPDs(w)} wt(\delta)$ over all GPDs δ for w, of a product of factors:

$$wt(\delta) = \prod_{i,j \in [n]} \begin{cases} A + x_i - y_j & \text{if } \blacksquare \text{ at } (i,j) \\ B - x_i + y_j & \text{if } \square \text{ at } (i,j) \\ A + B & \text{otherwise} \end{cases} \tag{1.1}$$

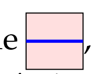
where the tiles in the first case are  and the tile in the second case is .

Let w_0 denote the longest element $\underline{n\ n-1 \ldots 3\ 2\ 1} \in \mathcal{S}_n$.

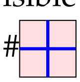
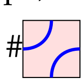
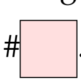
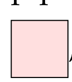
Theorem 1.1. *1. Consider the terms in G_w with the highest power of B. The only GPDs that contribute to this B-leading form are the* **classic pipe dreams***, (i) with visible pipes in only the NW triangle and (ii) no two pipes crossing twice. The result is $B^{n^2-\ell(w)}$ times the classic pipe dream formula for the double Schubert polynomial $S_w \in \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n]$, except evaluated at $S_w(A + x_1, \ldots, A + x_n, y_1, \ldots, y_n)$.*

2. Similarly, the only GPDs that contribute to the A-leading form are the **bumpless pipe dreams** *(albeit rotated 180° from their definition in [10]), with (i) no "bump tiles"*  *and (ii) no two pipes crossing twice. The result is $A^{n^2-\ell(w)}$ times the bumpless pipe dream formula for $S_{w_0ww_0}$, except evaluated at $S_{w_0ww_0}(B - x_n, \ldots, B - x_1, -y_n, \ldots, -y_1)$.*

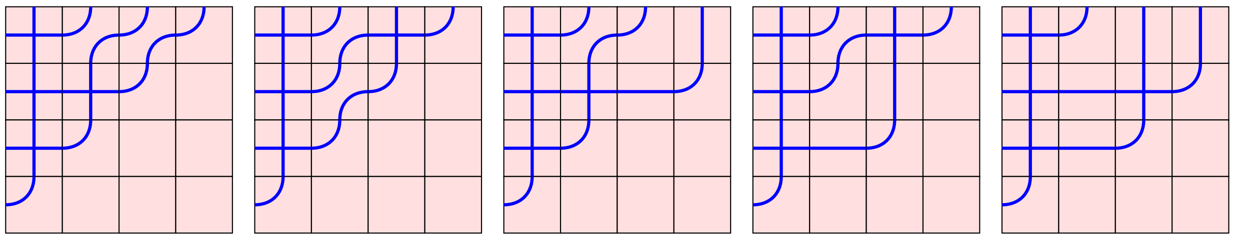
Proof. 1. If two pipes cross in tile s and then again in tile t, we can replace each crossing tile with a bump tile, swapping the two pipe colors in the range between s and t. This decreases the number of $A + x_i - y_j$ factors, allowing us to find another two factors of B.

Now, if any pipe labelled $i \in [n]$ goes through a horizontal tile , we can consider there to be an invisible pipe crossing it going North. The i-pipe heads East but eventually comes out the North side, whereas the invisible pipe heads North but comes out the East side, so the pipes must cross a second time. Apply the same argument as above.

Hence the SE triangle must be solid invisible pipes, and the NW triangle full of visible pipes, no two crossing twice.

2. The total length, in number of squares traversed, of pipe i is $i + w^{-1}(i) - 1$; summing over i we get $\binom{n+1}{2} + \binom{n+1}{2} - n = n^2$. Hence each tile used contains one visible pipe, on average. Some tiles accommodate two visible pipes, some zero, so #  + #  = # . We want to minimize the number of , as those don't admit an A term, so we must minimize the number of crosses (to $\ell(w)$, using no double crosses) and bumps (to zero). $\square$

Example 1.2. *Here are the GPDs for $w = 2431$:*



Only the first two GPDs are CPDs, and the B-leading form is related to the double Schubert polynomial $S_{2431} = (x_1 - y_1)(x_2 - y_1)(x_3 - y_1)(x_1 + x_2 - y_2 - y_3)$.

Similarly, only the last GPD is a BPD, and the A-leading form is related to $S_{4213} = (x_1 - y_1)(x_2 - y_1)(x_1 - y_2)(x_1 - y_3)$.

In the next sections we give two geometric applications of GPD polynomials.

2 Lower-upper varieties

We recall some constructions from [6]. Let B_-, B_+ denote the groups of lower and upper triangular invertible $n \times n$ matrices, respectively. Define an action of $B_- \times B_+$ on $(Mat_{n\times n})^2$ by $(b, c) \cdot (X, Y) := (bXc^{-1}, cYb^{-1})$. Slightly strengthening a result of [6], we have

Theorem 2.1. *Let $E \subseteq (Mat_{n\times n})^2$ denote the* **lower-upper scheme** *$\{(X, Y)\colon XY$ lower triangular, YX upper triangular$\}$. Then E is invariant under the above $(B_- \times B_+)$-action, and its components $(E_w\colon w \in \mathcal{S}_n)$ are describable in two ways:*

$$\begin{aligned} E_w &:= \overline{\{(X,Y) \in E\colon diag(XY) = w \cdot diag(YX) \text{ nonrepeating}\}} \\ &= \overline{(B_- \times B_+) \cdot \{(w, w^{-1}D)\colon D \text{ diagonal}\}} \end{aligned}$$

Hence the projection $(X, Y) \mapsto X$ of E_w is the **matrix Schubert variety** $\overline{X}_w := \overline{B_- w B_+}$, abbreviated below as MSV, whose $(B_- \times B_+)$-equivariant cohomology class was shown in [7] to be the double Schubert polynomial $S_w(x_1, \ldots, x_n, y_1, \ldots, y_n)$. The other projection $(X, Y) \mapsto Y$ has image $w_0 \overline{X}_{w_0 w^{-1} w_0} w_0$.

Theorem 2.2. *Let $(\mathbb{C}^\times)^2$ act on $(Mat_{n\times n})^2$ by $(s, t) \cdot (X, Y) = (sX, tY)$, commuting with the $(B_- \times B_+)$-action, and write $\mathbb{Z}[A, B]$ for $H^*_{(\mathbb{C}^\times)^2}(pt)$. The $(B_- \times B_+ \times (\mathbb{C}^\times)^2)$-equivariant cohomology class of E_w is $(A + B)^{-n}$ times the GPD polynomial G_w from Section 1. (Since every pipe must turn East to North at some point, the overall $A + B$ exponent is nonnegative.)*

Sketch of proof. These cohomology classes are shown in [8, Section 4] to be uniquely determined by certain inductive divided difference formulæ. It is not hard to show, using Yang–Baxter type arguments, that the generic pipe dream polynomials satisfy the same inductive formulæ; see [3, Proposition 6] for the K-theoretic version. □

In [8, Proposition 3] we showed that one can compute the equivariant classes of the projections of a $(\mathbb{C}^\times)^2$-invariant subvariety $Z \subseteq V \times W$ from the equivariant class of Z itself, using the A-leading and B-leading terms. Combining Theorem 2.2 with the leading-term statements of Theorem 1.1, we recover the two standard formulæ for double Schubert polynomials.

Corollary 2.3. *We compute the degree of E_w by setting $x_\bullet = y_\bullet = 0$ to forget the $B_\pm$-actions, and $A = B = 1$ for only the single scaling action. Hence*

$$\deg E_w = \sum_{\delta \in GPDs(w)} 2^{\#(\text{tiles with turns})-n}.$$

The proof of Theorem 2.2 above does not explain where these GPD formulæ come from (and in fact, it is not how we found them). Similarly to the case of MSVs treated in [7], one can interpret them in terms of a Gröbner degeneration. This has the added benefit that it gives us insight on "how" pipe dreams *know about their connectivity*, something which is not readily available in the study of MSVs.

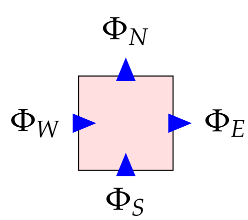
Theorem 2.4. *There is an equivariant degeneration of E_w to a union of quadratic complete intersections F_δ (possibly with some embedded components not affecting the H^*-class), one for each GPD δ, whose classes are the individual terms in the formula for $(A+B)^{-n} G_w$.*

Sketch of proof. Write $t_i = (XY)_{ii}$, so that $t_{w^{-1}(i)} = (YX)_{i,i}$ in E_w.

Introduce **flux** variables on edges of the $n \times n$ square lattice by

$$\Phi_e = \sum_{\substack{\text{squares } (i,j) \\ \text{right of } e \text{ if } e \text{ vertical} \\ \text{below } e \text{ if } e \text{ horizontal}}} X_{ij} Y_{ji}$$

In particular, note that in E_w, the fluxes on boundary edges are fixed to be zero on the South and East sides, $t_1, \dots, t_n$ on the West side, $t_{w^{-1}(1)}, \dots, t_{w^{-1}(n)}$ on the North side, which matches the connectivity of GPDs associated to E_w. Also, they satisfy the conservation equation at each square



$$\Phi_W + \Phi_S = \Phi_E + \Phi_N \tag{2.1}$$

Now consider the following degeneration of the whole scheme E: start from its defining ideal $\mathcal{I} = \langle (XY)_>, (YX)_< \rangle$, give a weight

$$[X_{ij}] = -ij \qquad [Y_{ij}] = ij \qquad i,j = 1,\dots,n$$

to each variable, and take the initial ideal $\mathrm{init}(\mathcal{I})$ with respect to the corresponding monomial order. [1] Note that this is only a *partial* Gröbner degeneration, because ties remain among monomials, which means that $\mathrm{init}(\mathcal{I})$ need not be a monomial ideal. In particular, fluxes, and (2.1), are unaffected by the degeneration.

We need to compute some of $\mathrm{init}(\mathcal{I})$. Consider the "overlap" $(XY)_{>}X - X(YX)_{<}$. An easy calculation of its initial term leads to the equation

$$X_{ij}(\Phi_S - \Phi_E) = 0 \qquad i,j = 1,\dots,n$$

for the degeneration of E, where Φ_S and Φ_E are as in (2.1) at the square (i,j).

Note that if $X_{ij} = 0$, then $\Phi_S = \Phi_N$ and $\Phi_W = \Phi_E$. We conclude that in each component of the degeneration of E, fluxes either propagate horizontal/vertically or diagonally (i.e., the equation (2.1) splits into two cases, with no more mixing of fluxes). Now starting from the bottom/left and adding one square at a time, we conclude that fluxes Φ_e can only take the values $0, t_1, \dots, t_n$. Draw a pipe (labelled i) across each edge such that $\Phi_e = t_i$, and leave the zero flux edges blank: one obtains this way a GPD.

With a bit more work, one concludes that the degeneration of E is contained inside the union of F_δ over δ GPD, where

$$F_\delta = \left\{ (X,Y) \in Mat^2_{n\times n} : \begin{array}{r} \text{At each square } (i,j),\ X_{ij} = 0 \text{ if } (i,j) \text{ is a crossing/straight pipe} \\ Y_{ji} = 0 \text{ if } (i,j) \text{ is blank} \\ \text{At each edge } e,\ \Phi_e = \begin{cases} t_i & \text{if pipe } i \text{ passes through } e \\ 0 & \text{else} \end{cases} \end{array} \right\}$$

up to lower-dimensional (necessarily embedded) components. If we degenerate E_w instead, this amounts to imposing the outgoing fluxes, i.e., that the connectivity of the GPDs be w.

Finally, one checks that F_δ is a complete intersection of dimension $n(n-1)$: each square provides one equation, but by easy linear algebra, among the flux equations, n of them are redundant. The cohomology class of F_δ can then be computed by taking the product of the weights of its equations, i.e., $wt(X_{ij}) = A + x_i - y_j$, $wt(Y_{ij}) = B - x_i + y_j$, $wt(\Phi_e) = A + B$, to be compared with (1.1).

Because we already know equality of cohomology classes according to Theorem 2.2, we conclude that the degeneration of E_w can differ from the union of corresponding F_δ by at most lower-dimensional embedded components. □

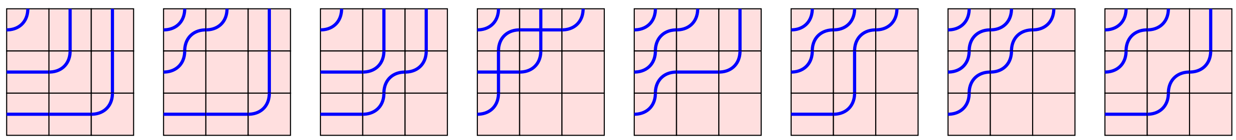
2.1 Degree of the commuting variety

The lower-upper scheme was invented in [6] to study the **commuting scheme** $C := \{(X,Y) \in (Mat_{n\times n})^2 \colon XY = YX\}$. Specifically, the first author showed that C has a de-

[1] If we view $(Mat_{n\times n})^2$ as $T^*Mat_{n\times n}$, then this degeneration preserves the symplectic form.

generation to the lower-upper variety E_1 union (possibly) some embedded components; ergo, $\deg C = \deg E_1$. With Corollary 2.3, we can compute that as a 2-enumeration:

Example 2.5. *When* $n = 3$, $\deg C = 1 + 2 + 2 + 2 + 4 + 4 + 8 + 8 = 31$*:*



This formula is quite computationally effective, allowing us to go up to

$$\deg C_{n=16} = 8\,152\,788\,880\,952\,641\,347\,488\,179\,079\,698\,833\,772\,730\,621\,821\,001\,288\,826\,319\,965\,501\,665$$

See also [3, Theorem 4] for an independent proof of this formula for the (multi)degree of the commuting variety.

3 Schwartz–MacPherson classes of Kazhdan–Lusztig varieties and of some unions thereof

3.1 Segre–Schwartz–MacPherson classes on G/B_+

Let $X \subseteq Y$ be a locally closed subscheme in a smooth ambient variety, over $\mathbb{C}$. To this one associates a **Chern–Schwartz–MacPherson class** $\mathrm{csm}(X)$ in $H_*^{BM}(Y) \cong H^{\dim_{\mathbb{R}} Y - *}(Y)$, where H_*^{BM} denotes the Borel–Moore homology. These classes are characterized by three properties:

1. If $X = X_1 \sqcup X_2$ is a disjoint union, then $\mathrm{csm}(X) = \mathrm{csm}(X_1) + \mathrm{csm}(X_2)$.
2. If $f : Y \to Z$ makes X a bundle over $f(X)$ with fiber F, then $f_*(\mathrm{csm}(X)) = \mathrm{csm}(f(X))\, \chi_c(F)$, where χ_c is the compactly supported Euler characteristic.
3. If $X = Y$ is proper, then $\mathrm{csm}(X) = c(TY)$, the total Chern class.

These CSM classes behave well under pushforward; for good Poincaré-duality properties, one defines the **Segre–Schwartz–MacPherson class** $\mathrm{ssm}(X \subseteq Y) := \mathrm{csm}(X)/c(TY) \in H^*(Y)$. Here are two naturality properties these SSM classes possess:

Lemma 3.1. 1. *Under the isomorphism* $H^*(Y) \cong H^*(Y \times V)$*, for* V *a vector space, we have* $\mathrm{ssm}(X) \mapsto \mathrm{ssm}(X \times V)$*.*

2. *[13] Let* $X_1, X_2 \subseteq Y$ *be locally closed submanifolds, whose closures* $\overline{X}_1, \overline{X}_2$ *we stratify by manifolds (having* X_1, X_2 *as strata). If each stratum in* $\overline{X}_1$ *is transverse to each in* $\overline{X}_2$*, then* $\mathrm{ssm}(X_1 \cap X_2 \subseteq Y)$ *is the product* $\mathrm{ssm}(X_1 \subseteq Y)\ \mathrm{ssm}(X_2 \subseteq Y)$*.*

All of the foregoing generalizes nicely to equivariant cohomology, i.e. if T acts on Y preserving X then $\mathrm{csm}(X)$ can be defined in $H^*_T(Y)$. We adapt [4, Lemma 2] to compute point restrictions of SSM classes on the flag variety G/B_+, where G is a semisimple algebraic group and B_+ is its Borel subgroup, giving an alternate proof of [14, Theorem 1.1].

Let W be the Weyl group of G.

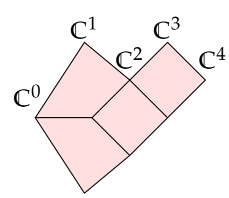
Lemma 3.2. *Let $X^v_\circ := B_+vB_+/B_+$, $X^\circ_w := B_-wB_+/B_+$ be the Bruhat and opposite Bruhat cells in G/B_+. Then the bases $(\mathrm{csm}(X^v_\circ))_{v\in W}$, $(\mathrm{ssm}(X^\circ_w))_{w\in W}$ are dual bases under the Poincaré pairing $\langle\alpha,\beta\rangle := \int \alpha\beta$ on $H^*_T(G/B_+)$ and localizations thereof.*

Proof. Let $\pi\colon G/B_+ \to pt$ be the map to a point, so π_* is integration over G/B_+. Then we compute the Poincaré pairings:

$$\begin{aligned}
\int_{G/B_+} \mathrm{csm}(X^v_\circ)\,\mathrm{ssm}(X^\circ_w) &= \int_{G/B_+} \mathrm{csm}(X^v_\circ \cap X^\circ_w) && \text{by Lemma 3.1(2)}\\
&= \chi_c(X^v_\circ \cap X^\circ_w) && \text{by property (2) of CSM classes}\\
&= \chi_c((X^v_\circ \cap X^\circ_w)^T) && \text{a property of } \chi_c\\
&= \chi_c(\{vB_+/B_+\} \cap \{wB_+/B_+\}) = \delta_{vw} \quad \square
\end{aligned}$$

We will make use of a similar pair of cell decompositions of the *degenerate Bott–Samelson variety* from [11]. For Q a word (not necessarily reduced) in the simple reflections of G's Weyl group, one associates a **Bott–Samelson manifold** BS^Q and B_+-equivariant map $BS^Q \to G/B_+$. BS^Q can be defined in terms of the **heap** of Q, which in type A can be drawn as the dual quadrangulation of the wiring diagram of Q, e.g.,

$$Q = r_3 r_1 r_2 r_1$$



where vector subspaces of $\mathbb{C}^n$ sit at every vertex of the diagram, satisfy left-to-right inclusion, and the vertices at the top of the picture are fixed to form the standard flag $\mathbb{C}^0 \subset \mathbb{C}^1 \subset \cdots \subset \mathbb{C}^n$. The map to the complete flag variety G/B_+ is reading off the bottom flag.

This BS^Q has a nice B_+-invariant cell decomposition $BS^Q = \coprod_{R\subseteq Q} BS^R_\circ$ indexed by the 2^Q "subwords" of Q, which also label the T-fixed points $(BS^R)^T$; the cell $BS^R_\circ$ corresponds to imposing equality (resp. inequality) of top and bottom subspaces of a square corresponding to a letter of R (resp. of $Q\setminus R$).

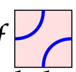
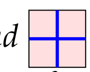
There is a fascinating toric degeneration $BS^Q \rightsquigarrow TV^Q$ in which each stratum $BS^R := \overline{BS^R_\circ}$ degenerates to a toric subvariety TV^R (i.e., it stays irreducible). This toric variety

TV^Q is modeled on a combinatorial cube $\square^Q$ [5] with vertices indexed by 2^Q, and the TV^R correspond to the faces of $\square^Q$ containing the bottom vertex (the empty subword). As such TV^Q has a second cell decomposition $TV^Q = \coprod_{R\subseteq Q} TV^\circ_R$ transverse to the first, whose closures correspond to the faces containing the top vertex $R = Q$. These two cell decompositions then enjoy the same dual-basis property as in Lemma 3.2.

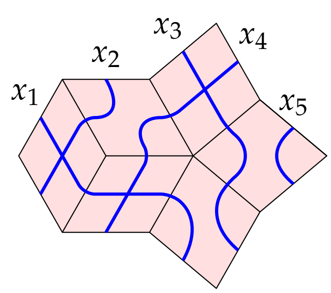
This toric degeneration has the uncommon property of being smooth, and consequently, there is a T_c-equivariant diffeomorphism $TV^Q \to BS^Q$ where T_c is the maximal compact subgroup of the complex torus T. One can carry the algebraic submanifolds $\overline{TV^\circ_R} \subseteq TV^Q$ across this diffeomorphism to give *non*algebraic submanifolds of BS^Q; see [1, Chapter 18]. We involve this variety TV^Q so as to use the same calculation as in Lemma 3.2.

Theorem 3.3 (see also [14])**.** *Let Q be a word in the simple reflections of W, with product $v \in W$. Then the restriction $\mathrm{ssm}(X^\circ_w)|_v$ of the class $\mathrm{ssm}(X^\circ_w)$ to the point vB_+/B_+ is given by the sum over all subwords $R \subseteq Q$ with product w, of*

$$\prod_{i\in I} \frac{\beta_i}{\beta_i+1} \prod_{i\notin I} \frac{1}{\beta_i+1} \qquad \beta_i = \left(\prod_{j<i} r_{Q_j}\right)\cdot \alpha_{Q_i} \qquad R = \prod_{i\in I} Q_i \tag{3.1}$$

In type A, we can index these terms using pipe dreams made of  *and*  *in the heap of Q, where the simple roots are parameterised as $\alpha_{r_i} = x_i - x_{i+1}$, and the roots β_i can be read off the diagram as the differences of labels propagating on parallel sides of squares,*

e.g., if $Q = r_3r_4r_2r_1r_2r_3$, $R = r_3\_\_r_1r_2\_$,



$$\beta_1 = x_3 - x_4$$
$$\beta_2 = x_3 - x_5$$
$$\beta_3 = x_2 - x_4$$
$$\beta_4 = x_1 - x_4$$
$$\beta_5 = x_1 - x_2$$
$$\beta_6 = x_1 - x_5$$

Proof. We compose $\{Q\} \hookrightarrow TV^Q \xrightarrow{\sim} BS^Q \to G/B_+$. The image of the point Q in G/B_+ is vB_+/B_+ by assumption. We first compute the pushforward map of the composite $TV^Q \to G/B_+$ in one pair of bases, then transpose it to compute pullback in the dual bases.

Push forward $\mathrm{csm}(TV^R_\circ) \mapsto \mathrm{csm}(BS^R_\circ)$ along the middle map, then consider the B_+-equivariant map $BS^R_\circ \to G/B_+$. Its image is some union $\cup X^w_\circ$ of B_+-orbits, and the map must be a bundle over each target orbit. (In fact a trivial bundle, as each $X^v_\circ$ is a *free* orbit under the subgroup $B_+ \cap (v[B_-, B_-]v^{-1})$). So far we know $\mathrm{csm}(BS^R_\circ) \mapsto \sum_w \chi_c(F_w)\,\mathrm{csm}(X^w_\circ)$, where F_w is the fiber over wB_+/B_+. Now use

$$\chi_c(F_w) = \chi_c((F_w)^T) = \chi_c(F_w \cap (BS^R_\circ)^T) = \chi_c(F_w \cap \{R\}) = [R \in F_w] = \left[\prod R = w\right]$$

where [assertion] = 1 if true, 0 if false. In all, $\text{csm}(TV_\circ^R) \mapsto \text{csm}(X_\circ^{\prod R})$.

Transposing, $\text{ssm}(X_w^\circ) \mapsto \sum\{\text{ssm}(TV_R^\circ) \colon R \subseteq Q,\ \prod R = w\}$. Consequently,

$$\text{ssm}(X_w^\circ)|_v = \sum \left\{ \text{ssm}(TV_R^\circ)|_Q \colon R \subseteq Q,\ \prod R = w \right\}$$

Note that $TV_R^\circ = \bigcap_{r \in R} TV_r \cap \bigcap_{r \in Q \setminus R} (TV^Q \setminus TV_r)$, and this intersection is transverse in the sense of Lemma 3.1(2). To calculate the factor $\text{ssm}(TV_R^\circ)|_Q$, we work in the open set $TV_\circ^Q \cong \mathbb{C}^Q$, whose intersection with TV_r corresponds to the subset $\{\vec{v} \in \mathbb{C}^Q \colon v_r = 0\}$. (3.1) then follows from Lemma 3.1. □

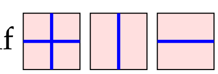
It is a standard fact that the pullback of a *Schubert* class $[\overline{B_- w P_+}/P_+] \in H^*(G/P_+)$ from a partial flag manifold, along the projection $G/B_+ \twoheadrightarrow G/P_+$ ($P_+ \supseteq B_+$), is again a Schubert class $[\overline{B_- w B_+}/B_+] \in H^*(G/B_+)$, where w is the unique smallest element in its coset wW_P. The situation is more complicated for CSM and SSM classes. The argument used in Theorem 3.3 lets one show $\pi_*(\text{csm}(X_\circ^{v'})) = \text{csm}(X_\circ^v)$ for any $v' \in vW_P$, so transposing, $\pi^*(\text{ssm}(X_v^\circ)) = \sum_{f \in W_P} \text{ssm}(X_{vf}^\circ)$. Thus to compute the point restriction of a G/P_+ SSM class we can sum the pipe dreams over all vW_P. This gives us combinatorial formulaæ for SSM classes in G/P_+, which are spelled out in [9, Lemma 2.4 and Section 5].

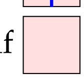
Here we focus on a different direction, which will allow us to reconnect to GPDs.

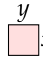
3.2 Open Kazhdan–Lusztig and matrix Schubert varieties

Given a word Q for $v \in S_n$, define a modified weight for a GPD δ on the heap of Q:

$$\widetilde{wt}(\delta) = \prod_{\substack{y \\ \square\, x}} \begin{cases} x - y & \text{if } (\text{tiles 1}) \\ x - y + 1 & \text{if } (\text{tile 2}) \\ 1 & \text{otherwise} \end{cases} \qquad (3.2)$$







It differs from the original weight (1.1) by signs and the specialization $A = 0$, $B = -1$. [2]

Given a set of GDPs, define the modified GPD polynomial to be the sum of its modified weights. (If locations of endpoints and connectivity of pipes are fixed, then GPD and modified GPD polynomials only differ by an overall sign and the specialization above.)

An equivalent formulation of Theorem 3.3 is:

Corollary 3.4. *The SSM class of the* **open Kazhdan–Lusztig variety** $X_w^\circ \cap X_\circ^v$ *inside the cell* $X_\circ^v$ *is computed by the same formula* (3.1) *as in Theorem 3.3. In type* A*, and taking the word* Q

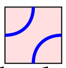
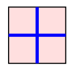
[2] We could have kept B unspecialized, which would correspond to the natural homogenization of CSM classes in $H^{\dim Y}(Y)[B] \cong H^{\dim Y}_{\mathbb{C}^\times}(Y) \cong H^{\dim Y}_{\mathbb{C}^\times}(T^*Y)$, where B is interpreted as equivariant parameter for the scaling of the fiber of T^*Y.

of v to be reduced, the CSM class of $X^\circ_w \cap X^v_\circ$ inside $X^v_\circ$ is given by the modified GPD polynomial for the set of pipe dreams on the heap of Q in Theorem 3.3.

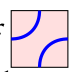
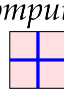
Proof. Let $U_v := vB_-B_+/B_+$ be the big cell in G/B_+ centered at the point vB_+/B_+. The Kazhdan–Lusztig lemma is a T-equivariant isomorphism of pairs

$$(X^\circ_w \cap U_v \ \subseteq\ U_v) \cong (X^\circ_w \cap X^v_\circ \ \subseteq\ X^v_\circ) \times X^\circ_v$$

We then apply Lemma 3.1(1).

To compute the CSM class we must multiply by the total Chern class $c(TX^v_\circ)$, which is nothing but $\prod_{i=1}^{|Q|}(1+\beta_i)$. This removes the denominator in the expression of (3.1), so that now a  contributes 1, whereas a  contributes $\beta_i = x - y$ where x and y are labels attached to the two sides of the square. This matches the weights of (3.2) (noting that these GPDs have no blanks). □

One well-studied family of such varieties arises in the theory of cluster algebras:

Proposition 3.5. *Let $(B_+uB_+) \cap (B_-vB_-) \subseteq GL_n$ be w_0 times the* **double Bruhat cell**. *Its $(T^n \times T^n)$-equivariant CSM class can be computed as the modified GPD polynomial for pipe dreams on a $n \times n$ square, made of*  *and* , *with boundary $1 \ldots 2n$ down the West then along the South side, $u^{-1}(1) \ldots u^{-1}(n)$ across the North, and $n + v^{-1}(1) \ldots n + v^{-1}(n)$ down the East.*

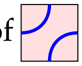
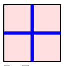
Proof. Let $u \oplus v \in \mathcal{S}_{2n}$ have one-line notation $\underline{u(1) \ldots u(n)\ n+v(1) \ldots n+v(n)}$. Its Fulton essential set is $ess(u \oplus v) = \begin{bmatrix} ess(u) & \\ & ess(v) \end{bmatrix}$. The pullback of $X^\circ_{u \oplus v}$ along the Fulton isomorphism [2] $Mat_{n\times n} \xrightarrow{\sim} X_\circ^{\underline{n+1 \ldots 2n\ 1 \ldots n}} \subseteq GL(2n)/B_+$, $M \mapsto \begin{bmatrix} M & I_n \\ I_n & 0 \end{bmatrix}$ is the double Bruhat cell. The cut-the-deck permutation $\underline{n+1 \ldots 2n\ 1 \ldots n}$ is 321-avoiding, hence is "fully commutative" i.e. has only one reduced word up to commuting moves. The unique resulting heap is the $n \times n$ square. Now apply Corollary 3.4. □

Our final application is the following:

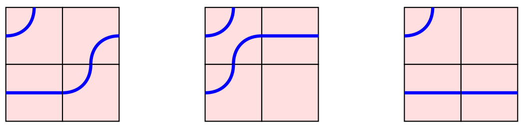
Theorem 3.6. *Let w be a* **partial permutation**, *viewed as a matrix in $Mat_{k\times n}$. Then the CSM class of B_-wB_+ is the modified GPD polynomial of GPDs on a $k \times n$ rectangle, such that pipes come in from the West side, and the i^{th} pipe on the West side (counted from bottom to top) emerges on the North side at location j if $w_{ij} = 1$, anywhere on the East side otherwise.*

Proof. Consider $v = \underline{n+1 \ldots n+k\, 1 \ldots n}$. The setup is similar to that of Proposition 3.5, and it is clear that via the Fulton isomorphism,

$$B_-wB_+ = \coprod_{\substack{w' \in S_{n+k} \\ w'|_{k\times n} = w}} X^\circ_{w'} \cap X^v_o$$

Therefore, $\mathrm{csm}(B_-wB_+) = \sum_{w'|_{k\times n}=w} \mathrm{csm}(X^\circ_{w'} \cap X^v_o)$, with each of the summands being described according to Corollary 3.4 by pipe dreams in a $k \times n$ rectangle made of  and  such that the pipes coming from the West corresponding to rows of 1s of w come out North as prescribed by w, whereas the other pipes coming from the West or South come out North or East as prescribed by w'. Summing over w' removes this last condition. Now *erase* all the pipes coming from the South, noting that the only information that's lost is when two such paths bump into or cross each other; but according to (3.2), the weight of a blank is the sum of the weight of a bump and of that of a cross. □

Example 3.7. *If* $w = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, *there are three GPDs:*



Renaming the variables (i.e., equivariant parameters) x_i, $i = 1, \ldots, k$ *for rows,* y_j, $j = 1, \ldots, n$ *for columns, one finds*

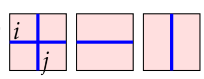
$$\begin{aligned}\mathrm{csm}(B_-wB_+) = {} & \left(x_1 + x_2 - y_1 - y_2\right) + \left(x_2^2 + x_1x_2 + y_2^2 + y_1y_2 - x_2y_1 - 2x_2y_2 - x_1y_2\right) \\ & + \left(x_2y_1y_2 + x_1y_1y_2 + x_1x_2^2 - y_1y_2^2 - x_2^2y_2 + x_2y_2^2 - x_1x_2y_1 - x_1x_2y_2\right)\end{aligned}$$

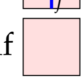
In particular if $k = n$ and w is a full permutation, we recover the class of GPDs defined in Section 1, and the polynomials G_w up to the sign $(-1)^{\dim(B_-wB_+)}$.

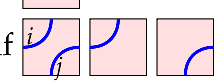
4 Extensions to K-theory

Our results admit (in some cases conjectural) K-theoretic versions. For instance, one can obtain using similar arguments as in Section 3 the **motivic Chern class** of an open Kazhdan–Lusztig variety; all the theorems of Section 3.2 thus generalize, using the same GPDs, provided their weights are replaced with their K-theoretic analogues

$$\widetilde{wt}_K(\delta) = \prod_{\square_x^y} \begin{cases} t^{[\,|\,\text{ or } i>j]}(1 - y/x) & \text{if} \\ 1 - t\,y/x & \text{if} \\ (1-t)(y/x)^{[\,\lrcorner\,\text{ or } i<j]} & \text{if} \end{cases}$$







This allows to recover various formulæ as special cases, e.g., those of [12].

Conjecturally, K-classes of lower-upper varieties should be given by K-theoretic GPD polynomials as well; see in particular [3, Theorem 3] for such a conjectural formula for the K-class of the commuting variety.

References

[1] D. Anderson and W. Fulton. *Equivariant cohomology in algebraic geometry*. Vol. 210. Cambridge Stud. Adv. Math. Cambridge University Press, Cambridge, 2024, xv+446 pp. DOI.

[2] W. Fulton. "Flags, Schubert polynomials, degeneracy loci, and determinantal formulas". *Duke Math. J.* **65**.3 (1992), pp. 381–420. DOI.

[3] A. Garbali and P. Zinn-Justin. "Shuffle algebras, lattice paths and the commuting scheme". *Hypergeometry, integrability and Lie theory*. Vol. 780. Contemp. Math. Amer. Math. Soc., [Providence], RI, [2022] ©2022, pp. 29–68. DOI.

[4] R. Goldin and A. Knutson. "Schubert structure operators and $K_T(G/B)$". *Sém. Lothar. Combin.* **82B** (2020), Art. 90, 12 pp.

[5] M. Harada and J. J. Yang. "Newton-Okounkov bodies of Bott-Samelson varieties and Grossberg-Karshon twisted cubes". *Michigan Math. J.* **65**.2 (2016), pp. 413–440. DOI.

[6] A. Knutson. "Some schemes related to the commuting variety". *J. Algebraic Geom.* **14**.2 (2005), pp. 283–294. DOI.

[7] A. Knutson and E. Miller. "Gröbner geometry of Schubert polynomials". *Ann. of Math. (2)* **161**.3 (2005), pp. 1245–1318. DOI.

[8] A. Knutson and P. Zinn-Justin. "The Brauer loop scheme and orbital varieties". *J. Geom. Phys.* **78** (2014), pp. 80–110. DOI.

[9] A. Knutson and P. Zinn-Justin. "Schubert puzzles and integrability II: multiplying motivic Segre classes". 2021. arXiv:2102.00563.

[10] T. Lam, S. J. Lee, and M. Shimozono. "Back stable Schubert calculus". *Compos. Math.* **157**.5 (2021), pp. 883–962. DOI.

[11] A. Parameswaran and K. Paramasamy. "Toric Degeneration of Bott–Samelson–Demazure–Hansen varieties". 2016. arXiv:1604.01998.

[12] P. Rudnicki and A. Weber. "Characteristic classes of Borel orbits of square-zero upper-triangular matrices". *J. Algebra* **598** (2022), pp. 351–384. DOI.

[13] J. Schürmann. "Chern classes and transversality for singular spaces". *Singularities in geometry, topology, foliations and dynamics*. Trends Math. Birkhäuser/Springer, Cham, 2017, pp. 207–231. DOI.

[14] C. Su. "Restriction formula for stable basis of the Springer resolution". *Selecta Math. (N.S.)* **23**.1 (2017), pp. 497–518. DOI.