Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #28, 12 pp.
Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

Grothendieck Shenanigans: Permutons from pipe dreams via integrable probability

Alejandro H. Morales *[1], Greta Panova†[2],
Leonid Petrov‡[3], and Damir Yeliussizov§[4]

[1]*Université du Québec à Montréal, Montreal, QC, Canada.*
[2]*University of Southern California, Los Angeles, USA.*
[3]*University of Virginia, Charlottesville, USA.*
[4]*Kazakh-British Technical University, Almaty, Kazakhstan.*

Abstract. We study random permutations corresponding to pipe dreams. Our main model is motivated by the Grothendieck polynomials with parameter $\beta = 1$ arising in the K-theory of the flag variety. The probability weight of a permutation is proportional to the principal specialization (setting all variables to 1) of its Grothendieck polynomial. By mapping this random permutation to a version of TASEP (Totally Asymmetric Simple Exclusion Process), we describe the limiting permuton and fluctuations around it as the order n of the permutation grows to infinity. The fluctuations are of order $n^{\frac{1}{3}}$ and have the Tracy–Widom GUE distribution, which places this algebraic (K-theoretic) model into the Kardar–Parisi–Zhang universality class.

Inspired by Stanley's question for the maximal value of principal specializations of Schubert polynomials, we resolve the analogous question for $\beta = 1$ Grothendieck polynomials, and provide bounds for general β. This analysis uses a correspondence with the free fermion six-vertex model, and the frozen boundary of the Aztec diamond.

Keywords: Grothendieck polynomials, pipe dreams, TASEP, six vertex model

1 A story from Algebra to Probability

Algebraic Combinatorics established itself as a field that uses combinatorial methods to understand algebraic behavior in problems ranging from Group Theory to Algebraic Geometry. It started with stark exact formulas, like the celebrated hook-length formula for the dimension of irreducible modules of the symmetric group S_n; beautiful interpretations, such as the Littlewood–Richardson rule for the structure constants of representations of the general linear group GL_n; powerful and intricate bijections, such

*morales_borrero.alejandro@uqam.ca. Partially supported by NSF, FRQNT, and NSERC
†gpanova@usc.edu. Partially supported by NSF CCF:AF grant
‡lenia.petrov@gmail.com. Partially supported by NSF DMS
§yeldamir@gmail.com. Partially supported by MSHE RK (grant AP23489146)

as the Robinson–Schensted–Knuth correspondence. However, such answers only go so far, leaving room for questions like "approximately how many", "what are the typical objects", and "what is the typical behavior". These questions lead us into the realm of Asymptotic Algebraic Combinatorics, which aims to answer them with the help of tools originating outside of Combinatorics. In the present work, we employ Integrable Probability, a rapidly evolving field focused on developing and analyzing interacting particle systems and random growth models possessing a certain degree of structure or symmetry. The arising probabilistic models exhibit rich structure leading to new permutons representing "typical permutations". The connection between the algebraic model and statistical mechanics is multi-fold via a correspondence between the so-called bumpless pipe dream models for Schubert/Grothendieck and the six-vertex model. The well studied free fermion six-vertex model and the frozen boundary of the Aztec diamond are key to understanding the permutations which maximize the principal specialization of Grothendieck polynomials.

Stanley [14] asked the most basic question on the principal specializations of Schubert polynomials $\mathfrak{S}_w$ (representing cohomology classes of the flag variety): does the following limit exist

$$\lim_{n\to\infty} \frac{1}{n^2} \log_2 \max_{w\in S_n} \mathfrak{S}_w\big(\underbrace{1,\ldots,1}_{n}\big),$$

and if so, what is it and for which permutations w is this achieved. This question (including the existence of the limit) is still open. In [12], a lower bound of about 0.29 was established for *layered permutations*. An upper bound of about 0.37 comes from a remarkable connection with Alternating Sign Matrices and the six-vertex model.

The Grothendieck polynomials $\mathfrak{G}_w^\beta$, which represent K-theoretic classes of the flag variety, are a natural one-parameter generalization of the Schubert polynomials. Extending Stanley's questions, we would like to understand the asymptotic behavior of maximal principal specializations $\max_{w\in S_n} \mathfrak{G}_w^\beta(1^n)$ of Grothendieck polynomials. This question was first touched on in [11], [5].

Thanks to a combinatorial model for Grothendieck and Schubert polynomials, these questions have very natural statistical mechanics interpretations. Namely, both polynomials are partition functions of tilings into crosses and elbows of a size n triangle (staircase) shape, which result in a configuration of n "pipes". Such pipe configurations are often called *pipe dreams*. In the Schubert case, the only valid tilings are the ones where no two pipes cross more than once. This is a global (*long-range interaction*) condition, and no tools in Statistical Mechanics are known to compute the asymptotics of the energy (equivalent to Stanley's question and estimating the number of such pipe dreams). In the Grothendieck case for $\beta = 1$, all tilings are allowed, but the pipes must be resolved (reduced) to obtain a permutation. One of the key ideas leading to our analysis is that this model can be mapped to a colored stochastic six-vertex model (and further to TASEP, the Totally Asymmetric Simple Exclusion Process). The resulting interacting particle systems have only local (short-range) interactions, and are amenable to techniques from Integrable Probability.

In this work, we investigate the asymptotics of the *typical* $\beta = 1$ *Grothendieck random permutations* and characterize their limit shape which is described by a permuton. We also study fluctuations of Grothendieck random permutations around the limiting permuton. They are of order $n^{\frac{1}{3}}$ and asymptotically have the Tracy–Widom GUE distribution. This distribution was first observed in the fluctuations of the largest eigenvalue of Gaussian random matrices with unitary symmetry. By now, having Tracy–Widom fluctuations is an indication that a model is within the Kardar–Parisi–Zhang (KPZ) universality class [3], which includes a wide range of random growth models and interacting particle systems. We also derive the expected number of inversions, which are of order n^2. This is in contrast to the model of Colin Defant [4] where no pipes are resolved and the inversions are of order $n^{3/2}$.

Returning to Stanley's original question, when $\beta = 1$, we have

$$\log_2 \max_{w \in S_n} \mathfrak{G}_w^\beta(1^n) = \frac{1}{2}n^2 - O(n \log n).$$

Using the correspondence with 2-enumerated Alternating Sign Matrices (equivalently, the six-vertex model with domain wall boundary conditions and free-fermion weights; or the model of uniform domino tilings of the Aztec diamond), we show that a large family of layered permutations also achieve this asymptotic maximum which strengthens the intuition that layered also maximize Schubert polynomials asymptotically [10]. For general β, we establish certain bounds for the maximal principal specialization.

This is the extended abstract of [13], which contains all the proofs and references, background on TASEP and six-vertex model. Here we will informally describe our main results and sketch the connections.

2 Pipe dreams and permutations

We denote by S_n the set permutations of $\{1, 2, \ldots, n\}$ that we write in the one-line notation $w = w_1 w_2 \cdots w_n$ unless indicated otherwise. We also denote the image of i under w by $w(i)$, and use the notation $w_i = w(i)$ interchangeably when this does not lead to confusion. The longest permutation $n\, n-1 \ldots 2\, 1$ is denoted by $w_0 = w_0(n)$. For a permutation w of length ℓ, we denote by $R(w)$ the set of *reduced words* of w, that is, tuples $(r_1, \ldots, r_\ell)$ such that $s_{r_1} \cdots s_{r_\ell}$ is a reduced decomposition of w, where $s_i = (i, i+1)$ are the simple transpositions.

A *pipe dream* of order n is a tiling of the staircase shape (having $n-1$ boxes in the first row, $n-2$ boxes in the second row, and so on, with boxes left-justified) by tiles of two types: *bumps* and *crossings* . The n-th diagonal below the staircase is equipped with half bumps . Without any restrictions there are $2^{\binom{n}{2}}$ pipe dreams of order n.

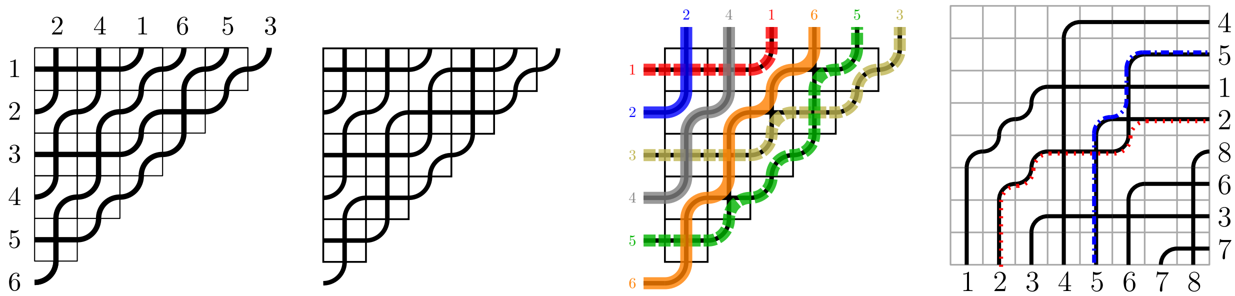


Figure 1: Left: A reduced pipe dream without any double intersections corresponding to permutation $w = 241653$, and an unreduced pipe dream D of order 6. **Center**: A reduction of the pipe dream leading to the permutation $w(D) = 241653$. **Right:** A reduction D' of a bumpless pipe dream D leading to the permutation $w(D) = w(D') = 45128637$. Dashing is added for the printed version and accessibility.

A pipe dream (a tiling of the staircase shape) forms a collection of strands (or *pipes*) labeled 1 to n from the row where they start. A pipe dream is called *reduced* if any two pipes cross through each other at most once.

Definition 2.1 (Reduction of a pipe dream)**.** Given a pipe dream D that is not necessarily reduced, the *reduction* of D is a unique reduced pipe dream D' obtained as follows: starting at the bottom left tile traverse the pipe dream upwards along columns and to the right. For each encountered crossing, replace it with a bump if the pipes have already crossed in the already traversed squares. The labeling of pipes together with a reduction is indicated by colored paths in Figure 1, center.

Definition 2.2 (Permutation from pipe dream)**.** One can associate a permutation $w(D) \in S_n$ to a pipe dream of order n as follows. If D is reduced then $w(D)^{-1}_j$ is the column where the pipe j ends up in.[1] Equivalently, the column j contains the exiting pipe of color $w(D)_j$. If D is not reduced, then $w(D)$ is the permutation associated to the reduction D' of D. Alternatively, associating transpositions s_i to each cross and reading from the bottom left to top right, we obtain a (non-necessarily reduced) word. Multiplying the transposition using the *Demazure* (0-Hecke) product rule (where $s_i^2 = s_i$) gives w.

Let $\mathrm{PD}(n)$ and $\mathrm{RPD}(n)$ be the sets of pipe dreams and reduced pipe dreams of size n. For each $w \in S_n$, let $\mathrm{PD}(w)$ and $\mathrm{RPD}(w)$ be, respectively, the sets of pipe dreams and reduced pipe dreams D such that $w(D) = w$. Note that $\#\mathrm{PD}(n) = 2^{\binom{n}{2}}$, whereas

[1]Throughout the paper, the column coordinate j increases from left to right, and the row coordinate i increases from top to bottom.

there is no simple known formula for $\#\mathrm{RPD}(n)$. Given a pipe dream D, let $\mathrm{cross}(D)$ be the set of coordinates (i, j) of the cross tiles. The *weight* of D is the monomial $\mathrm{wt}(D) := \prod_{(i,j)\in\mathrm{cross}(D)} x_i$. For example, for the unreduced pipe dream in Figure 1, left, we have $\mathrm{wt}(D) = x_1^3 x_2^2 x_3^2 x_4 x_5$. The *Grothendieck* and *Schubert* polynomials are originally defined recursively, via certain divided difference operators, but using [7] and [1] we can define them as partition functions of the pipe dream models.

Theorem 2.3. *For any $w \in S_n$, we have that the Grothendieck polynomial is given by*

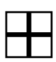
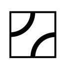
$$\mathfrak{G}_w^{\beta}(x_1, \dots, x_n) = \sum_{D\in\mathrm{PD}(w)} \beta^{\#\,\mathrm{cross}(D)-\ell(w)}\,\mathrm{wt}(D). \tag{2.1}$$

In particular, setting $\beta = 0$ forbids non-reduced pipe dreams, so the Schubert polynomial is

$$\mathfrak{S}_w(\mathbf{x}) = \sum_{D\in\mathrm{RPD}(w)} \mathrm{wt}(D). \tag{2.2}$$

3 Asymptotics of Grothendieck random permutations

Fix $p \in (0, 1)$. Equip the set of all pipe dreams with a probability measure by independently placing the tiles in each box:

 with probability p,  with probability $1 - p$. (3.1)

In particular, for $p = \frac{1}{2}$, we have the uniform measure on the set of pipe dreams.

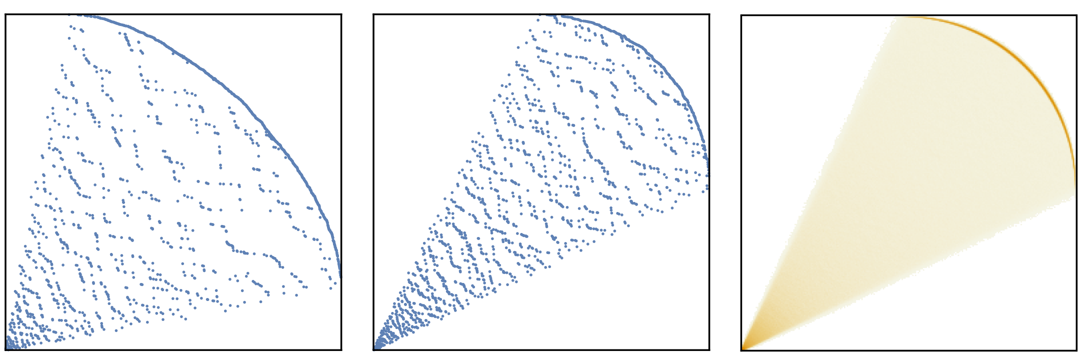


Figure 2: Left and center: A sample of a Grothendieck random permutation of order $n = 2000$ with $p = \frac{4}{5}$ and $p = \frac{1}{2}$, respectively. **Right**: An average of Grothendieck random permutations with $n = 2000$ and $p = \frac{1}{2}$ over 2000 samples. We take a sum of permutation matrices, and coarse-grain the result into 8×8 blocks. The plot is the heat map of the resulting matrix, which approximates the Grothendieck permuton.

By reducing this random pipe dream as in Definition 2.1, we obtain a random permutation $\mathbf{w} \in S_n$ which we call the *Grothendieck random permutation* (of order n and with parameter p; we suppress the dependence on n and p in the notation). The name is justified by a connection with the polynomials $\mathfrak{G}_w^{\beta=1}$. Indeed, we have for any $w \in S_n$:

$$\mathbb{P}(\mathbf{w} = w) = \sum_{D \in \mathrm{PD}(w)} p^{\mathrm{cross}(D)}(1-p)^{\mathrm{elbow}(D)} = (1-p)^{\binom{n}{2}} \mathfrak{G}_w^{\beta=1}\left(\frac{p}{1-p}, \dots, \frac{p}{1-p}\right). \quad (3.2)$$

For $w \in S_n$, define its *height function* as in Figure 3 by

$$H(x,y) := \#\left(\left\{w^{-1}(x), w^{-1}(x+1), \dots, w^{-1}(n)\right\} \cap \{y, y+1, \dots, n\}\right),$$

where $1 \leqslant x, y \leqslant n$. In terms of the pipe dream as in Figure 1, center, $H(x,y)$ is the number of pipes of color $\geqslant x$ which exit through the positions $\geqslant y$ at the top. $H(x,y)$ gives also the number of 1s in the rectangle with lower left corner at (x,y).

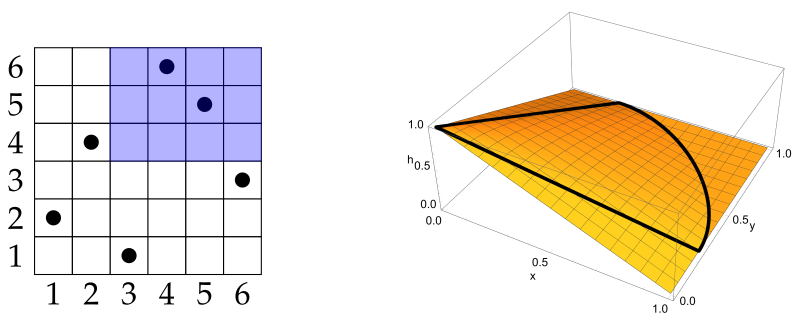


Figure 3: Left: Permutation matrix of $w = (2,4,1,6,5,3)$ coming from the pipe dream in Figure 1, center (dots indicate 1's). The highlighted rectangle has $H(4,3) = 2$ entries. Right: the limit shape h of the rescaled H.

Let now $\mathbf{w} \in S_n$ be the Grothendieck random permutation with the fixed parameter $p \in (0,1)$. We show that the random $H(x,y)$ satisfies the law of large numbers, and characterize its asymptotic fluctuations illustrated in Figure 2.

Theorem 3.1. 1. *There exists a limiting height function* h° *such that*

$$\lim_{n\to\infty} n^{-1} H(\lfloor n\mathsf{x}\rfloor, \lfloor n\mathsf{y}\rfloor) = \mathsf{h}^\circ(\mathsf{x},\mathsf{y}), \qquad (\mathsf{x},\mathsf{y}) \in [0,1]^2,$$

where the convergence is in probability. The function $\mathsf{h}^\circ(\mathsf{x},\mathsf{y})$ *is explicit, see [13]. It is continuous and depends only on* p. *The graph of* h° *is given in Figure 3, right.*

2. *The fluctuations of* $H(x,y)$ *around* h° *belong to the KPZ universality class:*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{H(\lfloor n\mathsf{x}\rfloor, \lfloor n\mathsf{y}\rfloor) - n\mathsf{h}^\circ(\mathsf{x},\mathsf{y})}{\mathsf{v}(\mathsf{x},\mathsf{y})\, n^{1/3}} \le r\right) = F_2(r), \qquad r \in \mathbb{R},$$

where F_2 is the cumulative distribution function of the Tracy–Widom GUE distribution, and $\mathsf{v}(\mathsf{x},\mathsf{y})$ is given in [13].

The law of large numbers in the first part of Theorem 3.1 means that the Grothendieck random permutations $\mathbf{w} \in S_n$ converge to a deterministic *permuton*. Recall that permutons are Borel probability measures μ on $[0,1]^2$ with uniform marginals, that is, $\mu([0,1] \times [a,b]) = \mu([a,b] \times [0,1]) = b - a$. Our limiting *Grothendieck permuton* is completely determined by the limiting height function via $\mu([\mathsf{x},1] \times [\mathsf{y},1]) = \mathsf{h}^\circ(\mathsf{x},\mathsf{y})$. Its singular part (a positive portion of its mass) is concentrated along the curve

$$\mathcal{E}_p := \left\{ (\mathsf{x},\mathsf{y}) \colon (\mathsf{y}-\mathsf{x})^2/p + (\mathsf{y}+\mathsf{x}-1)^2/(1-p) = 1,\ 1-p \leqslant \mathsf{x} \leqslant 1 \right\} \subset [0,1]^2.$$

The total mass supported on this curve is equal to γ_p defined below in (3.3).

As a corollary of the permuton convergence of Theorem 3.1, one can also obtain laws of large numbers for arbitrary pattern counts in Grothendieck random permutations. The simplest example is the number of inversions:

Proposition 3.2. *Let $\mathbf{w} = \mathbf{w}(n) \in S_n$ be the Grothendieck random permutations with a fixed parameter $p \in (0,1)$. We have*

$$\lim_{n\to\infty} \frac{\operatorname{inv}(\mathbf{w}(n))}{\binom{n}{2}} = \gamma_p := 1 - \sqrt{\frac{1-p}{p}} \arccos \sqrt{1-p}. \tag{3.3}$$

In the original model we have $\gamma_{\frac{1}{2}} = 1 - \frac{\pi}{4}$. The fact that the scaled number of inversions converges to γ_p, the singular part of the Grothendieck permuton, is surprising. Besides exact computations we do not have a conceptual explanation for this phenomenon.

3.1 Proofs via colored vertex models and TASEP

The *colored stochastic six-vertex model* on $\mathbb{Z}^2$ has the following vertex weights

$$w_p(a,b;c,d) = w_p\Big({}_b \overset{c}{\underset{a}{+}} {}_d \Big),$$

where $a,b,c,d \in \{0,1,\ldots,n\}$. Here 0 represents the absence of a pipe, and positive numbers indicate the pipes' colors. We view (a,b) and (c,d) as incoming and outgoing pipes, respectively. The weights are defined as follows: $w_p(a,a;a,a) = 1$, $w_p(b,a;b,a) = p$, $w_p(b,a;a,b) = 1-p$, $w_p(a,b;a,b) = 0$, and $w_p(a,b;b,a) = 1$, where $0 \leqslant a < b \leqslant n$ (weights of all other configurations are 0). The weights *conserve* the pipes, meaning $w_p(a,b;c,d) = 0$ unless $\{a,b\} = \{c,d\}$ as sets, and are *stochastic*: $w_p(a,b;c,d) \geqslant 0$ and $\sum_{c,d=0}^n w_p(a,b;c,d) = 1$ for all a,b. Assign coordinates $(i,j) \in \mathbb{Z}^2_{\geqslant 1}$ to the boxes of the staircase shape, with i increasing downward and j increasing to the right. The staircase shape is then $\delta_n := \{(i,j) : i+j \leqslant n\}$.

Place a stochastic vertex with the weight w_p at each box $(i,j) \in \delta_n$. Let the initial condition along the left boundary of δ_n be the *rainbow* one, with colors 1 to n from top to bottom. Then we can sample a random configuration of pipes as in Figure 1, center, by running a discrete time Markov chain with time $\tau = j - i$, where $-(n-1) \leqslant \tau \leqslant n-1$. At each step $\tau \to \tau+1$, the configuration with $j - i \leqslant \tau$ is already sampled, which determines the incoming colors at all boxes $(i,j) \in \delta_n$ with $j - i = \tau + 1$. The next step $\tau \to \tau+1$ consists in an independent update of the outgoing colors at all boxes $(i,j) \in \delta_n$ with $j - i = \tau + 1$, using the stochastic vertex weights $w_p(a,b;\cdot,\cdot)$, $1 \leqslant a, b \leqslant n$. Here we view each $w_p(a,b;\cdot,\cdot)$ as a probability distribution on possible outputs, (a,b) or (b,a). Reading off the outgoing colors at the top boundary of δ_n, we arrive at a random permutation $\mathbf{w} \in S_n$.

Proposition 3.3. *The random permutation $\mathbf{w} \in S_n$ obtained from the colored stochastic six-vertex model as described above has the same distribution as the Grothendieck random permutation defined in Section 3.*

The *permutation height function* $H(x,y)$ is also the height function for that model:

$$H(x,y) := \#\{\text{pipes of colors} \geqslant x \text{ which exit through positions } j \geqslant y \text{ at the top}\} \quad (3.4)$$

Consider an *uncolored* (*color-blind*) stochastic vertex model with all pipes of the same color (1) or no pipes (0) and the following weights $w_p^{\bullet}(0,0;0,0) = 1$, $w_p^{\bullet}(1,1;1,1) = 1$ $w_p^{\bullet}(1,0;1,0) = p$, $w_p^{\bullet}(1,0;0,1) = 1-p$, $w_p^{\bullet}(0,1;0,1) = 0$, $w_p^{\bullet}(0,1;1,0) = 1$.

Proposition 3.4. *Fix $1 \leqslant x \leqslant n$. In the colored stochastic vertex model, erase all pipes of colors $< x$, and identify all the remaining pipes for colors $\geqslant x$. The resulting random configuration of uncolored pipes evolves according to the color-blind stochastic vertex model $w_p^{\bullet}$.*

Definition 3.5. Let $k \geqslant 1$. Let $\xi(t) := (\xi_1(t) > \ldots > \xi_k(t)) \subset \mathbb{Z}$ be the k-particle discrete time TASEP (Totally Asymmetric Simple Exclusion Process) having parallel updates and geometrically distributed jumps. In detail, $\xi(t)$, $t \in \mathbb{Z}_{\geqslant 0}$, is a discrete time Markov chain on particle configurations in $\mathbb{Z}$ which at each time step $t \to t+1$ evolves as follows:

$$\xi_i(t+1) = \xi_i(t) + \min\left(G_i(t+1), \xi_{i-1}(t) - \xi_i(t) - 1\right), \qquad 1 \leqslant i \leqslant k, \quad (3.5)$$

where $G_i(t+1)$ are independent geometric random variables with parameter $p \in (0,1)$, i.e. $\mathbb{P}\left(G_i(t+1) = m\right) = (1-p)\,p^m, \qquad m \in \mathbb{Z}_{\geqslant 0}$. The update (3.5) occurs in parallel for all particles $1 \leqslant i \leqslant k$, that is, the new positions $\xi_i(t+1)$ depend only on the configuration $\xi(t)$ at the previous time step, and new independent random variables. By agreement, we have $\xi_0(t) \equiv +\infty$, so that the first particle $\xi_1(t)$ performs an independent random walk with geometrically distributed jumps.

Start the TASEP from the *densely packed* (*step*) initial configuration $\{1, 2, \ldots, k\}$, that is, $\xi_i(0) = k + 1 - i$, $1 \leqslant i \leqslant k$. Fix $n \geqslant k$, and introduce a *moving exit boundary* of staircase shape The *exit time* of a particle $\xi_i(t)$ is then $T_{\text{exit}}(i) := \min\{t\colon \xi_i(t) \geqslant n + 1 - t\}$.

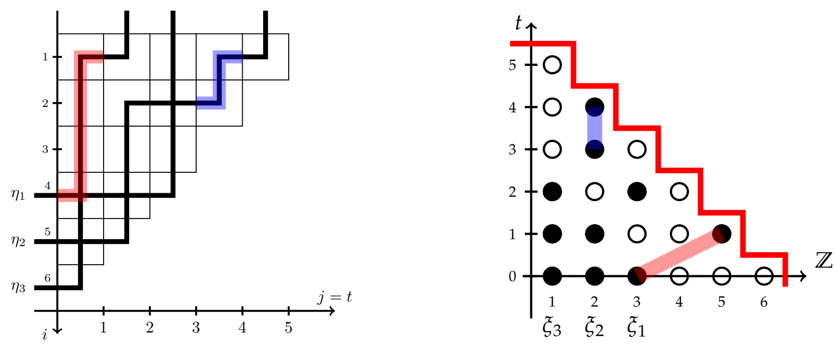


Figure 4: Left: The evolution of the uncolored pipes $\eta(t)$. Here $n = 6$ and $x = 4$, so $k = 3$. At time $t = 2$, we have $\eta_2(2) = 2$, $\eta_3(2) = 4$, and the pipe η_1 has exited before $t = 2$. We have $H(4,3) = 2$. **Right**: The evolution of the process $\xi(t)$, in bijection with the pipe configuration on the left (that is, $\xi_m(t) = n + 1 - t - \eta_m(t)$). In detail, a move of the particle ξ_ℓ by $r \geqslant 0$ steps corresponds to the pipe moving $r + 1$ steps up.

Theorem 3.6. *For any* $1 \leqslant x, y \leqslant n$ *and* $0 \leqslant h \leqslant n - x + 1$, *we have*

$$\mathbb{P}_{\mathbf{w}}\big(H(x,y) \leqslant h\big) = \mathbb{P}_{\text{TASEP}}\big(\xi_{n-x+1-h}(y-1) \geqslant n - y + 2\big).$$

Here $\mathbb{P}_{\mathbf{w}}$ *corresponds to the Grothendieck random permutation of order* n, *and* $\mathbb{P}_{\text{TASEP}}$ *is the probability distribution of the TASEP with* $k = n - x + 1$ *particles and moving exit boundary.*

To prove Theorem 3.1 we use the integrability of TASEP [9] with the given moving boundary (via asymptotics of Schur functions) to analyze the asymptotics of $H(x, y)$.

4 Asymptotics of Grothendieck principal specializations

We are interested in the *principal specializations* of the Grothendieck polynomials:

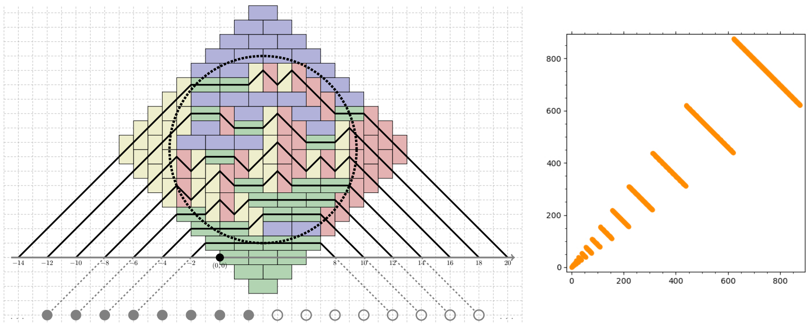
$$\Upsilon_w(\beta) := \mathfrak{G}_w^\beta(\underbrace{1, 1, \ldots, 1}_{n}), \qquad w \in S_n. \tag{4.1}$$

In particular, $\Upsilon_w(0)$ is the principal specialization of the Schubert polynomial studied in [12] and [14]. For $\beta = 1$, these quantities are the (unnormalized) probability weights of the Grothendieck random permutation $\mathbf{w}$ with $p = \frac{1}{2}$ (see (3.2)). Set

$$v_n(\beta) := \sum_{w \in S_n} \Upsilon_w(\beta) \quad \text{and} \quad u_n(\beta) := \max_{w \in S_n} \Upsilon_w(\beta).$$

From Theorem 2.3, we have $v_n(1) = 2^{\binom{n}{2}}$ and $u_n(1) = 2^{\binom{n}{2} - o(n^2)}$ as $n \to \infty$. The first equality is exact, while the second one is asymptotic. This asymptotic behavior follows from the cardinality $n! \sim e^{n \log n + O(n)} \ll 2^{\binom{n}{2}}$ of the set of all $n!$ permutations.

For $u \in S_k$ and $w \in S_m$, we denote $u \times w := (u(1), \dots, u(k), w(1)+k, \dots, w(m)+k) \in S_{k+m}$. For a composition $b = (b_\ell, \dots, b_1)$ of $n = b_1 + \cdots + b_\ell$, the *layered permutation* $w(b) \in S_n$ is defined as $w(b) := w_0(b_\ell) \times \cdots \times w_0(b_1)$, where $w_0(k) = (k, k-1, \dots, 1)$ is the full reversal permutation in the one-line notation. Denote by $L_n \subset S_n$ the subset of layered permutations of size n, and let $u'_n(1) := \max_{w \in L_n} \Upsilon_w(1)$.



n	b_1	$f(n)$
$\vdots$	$\vdots$	$\vdots$
311	91	0.490735
439	128	0.493404
621	182	0.495329
877	256	0.496684

Figure 5: Left: Domino tilings with Schroder paths, frozen boundary indicated as the inscribed circle. **Right**: Permutation matrix of a layered permutation $w(b) \in S_{877}$, where the composition is $b = (256, 182, 128, 91, 64, 46, 32, 23, 16, 12, 8, 6, 4, 3, 2, 2, 2, 1, 1)$. Note that $b_i/b_{i+1} \approx 1/\sqrt{2}$. Table of exact values for $3 \leqslant k \leqslant 19$ of layered permutations $w(b)$ with $b_i/b_{i+1} \approx 1/\sqrt{2}$. The third column is $f(n) := \frac{1}{n^2}\log_2 \Upsilon_{w(b)}(1)$ for $n = \sum_i b_i$.

Regarding Stanley's problem on permutations achieving the maximal Schubert specialization $u_n(0)$ (the case $\beta = 0$), the Merzon–Smirnov conjecture [10] states that *the maximum is attained on layered permutations*. In [12], $\frac{1}{n^2}\max_{w \in L_n} \log_2 \mathfrak{S}_w(1^n) \approx 0.29$ was computed. As some evidence that indeed layered permutations are good candidates, we prove that $\Upsilon_w(\beta)$ at $\beta = 1$ attain their asymptotic maximum in L_n:

Theorem 4.1. *There are sequences of layered permutations $w(b^{(n)}) \in S_n$ so that*
$$\lim_{n\to\infty} \frac{1}{n^2} \log_2 \Upsilon_{w(b^{(n)})}(1) = \frac{1}{2}.$$

Explicit constructions of such sequences of layered permutations can be obtained with geometric $b_i \sim (1-\alpha)\alpha^{i-1} n$ for any $\alpha \in [1/\sqrt{2}, 1)$. See Figure 5 for an illustration. Note, however that we do not know in the limit what compositions b of size n yield the global maximum of $\Upsilon_{w(b)}$ over all layered permutations. For Schubert polynomials, the analogous question was settled in [12] with an explicit limiting composition. Our Theorem 4.1 establishes that an asymptotic analog of the Merzon–Smirnov conjecture holds for the $\beta = 1$ Grothendieck polynomials. Bounds on the quantities $v_n(\beta)$ and $u_n(\beta)$ for general values of β can also be obtained (see [13, Section 6.5]).

Sketch of proof of Theorem 4.1. Let $u \in S_k, w \in S_n$. We have $\Upsilon_{u \times w}(\beta) = \Upsilon_u(\beta) \cdot \Upsilon_{\mathrm{id}_k \times w}(\beta)$, where $u \times w$ is the block permutation. Let s_k be the little Schröder numbers, which count paths from $(0,0)$ to $(2k,0)$ of steps $(1,1),(1,-1),(2,0)$ and which do not have a horizontal step on the x-axis. We have the following identity:

Theorem 4.2 ([11, Theorem 5.9])**.** *For nonnegative integers k and n, we have*

$$\Upsilon_{w_0(k;n)}(1) = \mathfrak{G}^{\beta=1}_{w_0(k;n)}(1,\dots,1) = 2^{-\binom{k}{2}} \det[s_{n-2+i+j}]_{i,j=1}^{k}. \tag{4.2}$$

Lemma 4.3. *Let $k \in (n/\sqrt{2}, n]$. We have*

$$\log_2 \Upsilon_{w_0(k;n-k)}(1) = \frac{n^2}{2} - \frac{k^2}{2} - O(n), \qquad n \to \infty. \tag{4.3}$$

Proof of Lemma 4.3. By Lindstrom–Gessel–Viennot, the determinant (4.2) counts k-tuples of nonintersecting Schöder paths staying above height 0 and starting from $(-2i,0)$ and ending at $(2(n-k+i-1),0)$, respectively, where $i = 1,\dots,k$. Denote this space of configurations by $\mathcal{M}_{k,n-k}$.

A well-known correspondence (e.g., see [6]) between domino tilings of the Aztec diamond and nonintersecting paths allows to interpret

$$\frac{\det[s_{n-k-2+i+j}]_{i,j=1}^{k}}{2^{\binom{n}{2}}} \tag{4.4}$$

as a probability of a certain event in the uniformly random domino tiling of the Aztec diamond D_{n-1} of order $n-1$. The ratio $|\mathcal{M}_{k,n-k}|/|\mathcal{S}_n|$ given by (4.4) is equal to the probability that the $n-k$ paths in $\mathcal{S}_n \setminus \mathcal{M}_{k,n-k}$ are in their *lowest possible configuration*.

The model of uniformly random domino tilings of the Aztec diamond develops an arctic circle [8]. In particular, the configuration outside of the circle inscribed in the Aztec diamond (illustrated in Figure 5) is frozen (nonrandom). Qualitatively, by [2, Proposition 13], this means that with probability $1 - e^{-O(n)}$, the $n-k$ paths in $\mathcal{S}_n \setminus \mathcal{M}_{k,n-k}$ are indeed in their lowest possible configuration. Note that here we rely on the assumption $k > n/\sqrt{2}$, which guarantees that the top of the $n-k$ paths in $\mathcal{S}_n \setminus \mathcal{M}_{k,n-k}$ does not reach the arctic circle. Therefore, we get desired asymptotics (4.3). $\square$

Let $w_0(k;n) := \mathrm{id}_k \times w_0(n) = (1,\dots,k,k+n,k+n-1,\dots,k+1) \in S_{k+n}$. Using Lemma 4.3 we can estimate that if $k > n/\sqrt{2}$ then $\log_2 \mathfrak{G}_{w_0(k;n)}(1^n) = \frac{n^2}{2} - \frac{k^2}{2} - O(n)$. Via the multiplicativity property $\mathfrak{G}_{u\times v}(1^n) = \mathfrak{G}_u(1^{|u|}) \cdot \mathfrak{G}_{\mathrm{id}_{|u|}\times v}(1^n)$ we can then estimate the values for a layered permutation recursively as

$$\log_2 \mathfrak{G}_{w(\dots,b_2,b_1)}(1^n) = \log_2 \mathfrak{G}_{w(\dots,b_3,b_2)}(1^{n-b_1}) + \frac{n^2}{2} - \frac{k_1^2}{2} - O(n),$$

where $k_i := n - b_1 - \cdots - b_i$ and $k_i/k_{i-1} \in (1/\sqrt{2}, 1)$ and establish Theorem 4.1.

Acknowledgements

We are very grateful to Jehanne Dousse for her valuable insights in this project and help with experiments. We thank Igor Pak and AIM for bringing us together as part of a SQuaRE. We thank Dave Anderson, Philippe Biane, Jacopo Borga, Filippo Colomo, Colin Defant, Philippe DiFrancesco, Rick Kenyon, Matthew Nicoletti, Khaydar Nurligareev, Mikhail Tikhonov, and Anna Weigandt for helpful discussions. We thank IPAM for the hospitality as three of the four authors were part of a long program.

References

[1] N. Bergeron and S. Billey. "RC-graphs and Schubert polynomials". *Experiment. Math.* **2**.4 (1993), pp. 257–269. Link.

[2] H. Cohn, N. Elkies, and J. Propp. "Local statistics for random domino tilings of the Aztec diamond". *Duke Math. J.* **85**.1 (1996), pp. 117–166. DOI.

[3] I. Corwin. "The Kardar-Parisi-Zhang equation and universality class". *Random Matrices Theory Appl.* **1**.1 (2012), 1130001, 76 pp. DOI.

[4] C. Defant. "Random Subwords and Pipe Dreams". 2024. arXiv:2408.05182.

[5] H. Dennin. "Pattern bounds for principal specializations of β-Grothendieck Polynomials". 2022. arXiv:2206.10017.

[6] S.-P. Eu and T.-S. Fu. "A simple proof of the Aztec diamond theorem". *Electron. J. Combin.* **12** (2005), Research Paper 18, 8 pp. DOI.

[7] S. Fomin and A. N. Kirillov. "The Yang-Baxter equation, symmetric functions, and Schubert polynomials". *Discrete Math.* **153**.1-3 (1996), pp. 123–143. DOI.

[8] W. Jockusch, J. Propp, and P. Shor. "Random domino tilings and the arctic circle theorem". 1998. arXiv:math/9801068.

[9] K. Johansson. "Shape fluctuations and random matrices". *Comm. Math. Phys.* **209**.2 (2000), pp. 437–476. DOI.

[10] G. Merzon and E. Smirnov. "Determinantal identities for flagged Schur and Schubert polynomials". *Eur. J. Math.* **2**.1 (2016), pp. 227–245. DOI.

[11] A. H. Morales, I. Pak, and G. Panova. "Hook Formulas for Skew Shapes IV. Increasing Tableaux and Factorial Grothendieck Polynomials". *J. Math. Sci.* **261**.5 (2022), pp. 630–657. DOI.

[12] A. H. Morales, I. Pak, and G. Panova. "Asymptotics of principal evaluations of Schubert polynomials for layered permutations". *Proc. Amer. Math. Soc.* **147**.4 (2019), pp. 1377–1389. DOI.

[13] A. H. Morales, G. Panova, L. Petrov, and D. Yeliussizov. "Grothendieck Shenanigans: Permutons from pipe dreams via integrable probability". 2024. arXiv:2407.21653.

[14] R. Stanley. "Some Schubert shenanigans". 2017. arXiv:1704.00851.