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Bijections for faces of braid-type arrangements

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Abstract. We establish a general bijective framework for encoding faces of some classical hyperplane arrangements. Precisely, we consider hyperplane arrangements in $\mathbb{R}^n$ whose hyperplanes are all of the form $\{x_i - x_j = s\}$ for some $i, j \in [n]$ and $s \in \mathbb{Z}$. Such an arrangement $\mathcal{A}$ is *strongly transitive* if it satisfies the following condition: if $\{x_i - x_j = s\} \notin \mathcal{A}$ and $\{x_j - x_k = t\} \notin \mathcal{A}$ for some $i, j, k \in [n]$ and $s, t \in \mathbb{N}$, then $\{x_i - x_k = s + t\} \notin \mathcal{A}$. For any strongly transitive arrangement $\mathcal{A}$, we establish a bijection between the faces of $\mathcal{A}$ and some set of decorated plane trees.

Keywords: Hyperplane arrangements, Coxeter arrangements, regions, faces, counting

1 Introduction

In this extended abstract we establish bijective results about the faces of some classical families of hyperplane arrangements. Specifically, we consider real hyperplane arrangements made of a finite number of hyperplanes of the form

$$\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i - x_j = s\}, \tag{1.1}$$

with $i, j \in \{1, \ldots, n\}$ and $s \in \mathbb{Z}$. We call them *braid-type arrangements*. From now on, we make an abuse of notation and denote by $\{x_i - x_j = s\}$ the hyperplane in (1.1).

Given a set of integers $S \subseteq \mathbb{Z}$ we define, for every dimension $n > 0$, the braid-type arrangement $\mathcal{A}_S^n \subset \mathbb{R}^n$ as follows:

$$\mathcal{A}_S^n := \bigcup_{\substack{1 \le i < j \le n \\ s \in S}} \{x_i - x_j = s\}.$$

Classical examples include the *braid*, *Catalan*, *Shi*, *semiorder*, and *Linial* arrangements represented in Figure 1, which correspond to $S = \{0\}$, $\{-1, 0, 1\}$, $\{0, 1\}$, $\{-1, 1\}$, and $\{1\}$ respectively.

There is an extensive literature on counting regions of braid-type arrangements, starting with the work of Shi [13]. Seminal counting results were established by Stanley [15, 14], Postnikov and Stanley [12], and Athanasiadis [3]. Since then, the subject has become

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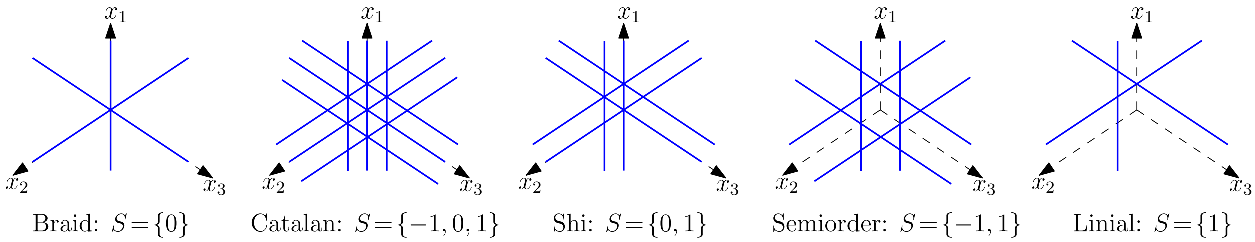


Figure 1: The braid, Catalan, Shi, semiorder, and Linial arrangements in dimension $n = 3$ (seen from the direction $(1, 1, 1)$).

quite popular among combinatorialists, and many beautiful counting formulas and bijections were discovered for various families of arrangements; see for instance [1, 4, 2, 5, 7, 8, 9, 11].

By contrast, there is a paucity of results about lower dimensional faces in braid-type arrangements. Notable exceptions are enumerative formulas by Athanasiadis [3] and bijections by Levear [10] about the Catalan and Shi arrangements.

In the present abstract, we develop a general bijective framework for encoding the faces of a large class of braid-type arrangements. Our framework applies to all *strongly transitive* arrangements, which are the braid-type arrangements $\mathcal{A}$ satisfying the following condition: if $\{x_i - x_j = s\} \notin \mathcal{A}$ and $\{x_j - x_k = t\} \notin \mathcal{A}$ for some distinct indices i, j, k and $s, t \in \mathbb{N}$, then $\{x_i - x_k = s + t\} \notin \mathcal{A}$. For any such arrangement we give an explicit bijection between the set of faces of $\mathcal{A}$ and a set of trees. Our bijection coincides with the one established by Levear [10] for the case of the Catalan and Shi arrangements, up to minor changes in notation. Our bijective results for faces of braid-type arrangements are an extension of a general bijective framework for regions of (transitive) arrangements that we established in [6]. We actually use the results from [6] to establish those in the present abstract.

The abstract is organized as follows. In Section 2 we set our notation, and recall some necessary background from [6]. In Section 3 we state our main result, which gives a bijection for faces of strongly transitive arrangements. In Section 4 we apply our bijection to a few classes of strongly transitive arrangements. In particular we discuss a multivariate generalization of the Catalan arrangement, a family of arrangements interpolating between the Catalan and Shi arrangements, and symmetric arrangements such as the semiorder arrangements. The proof of our main result is sketched in Section 5.

2 Notation and background

In this section we set our notation and recall some relevant results from [6].

For a positive integer n, we define $[n] := \{1, 2, \dots, n\}$, and we denote by $\mathfrak{S}_n$ the set of permutations of $[n]$. For any integers $m < n$ we define $[m; n] := \{k \in \mathbb{Z} \mid m \leq k \leq n\}$. For a real hyperplane arrangement $\mathcal{A}$, we denote by $\mathcal{R}(\mathcal{A})$ the set of regions of $\mathcal{A}$.

Definition 1. *Let $m, n \in \mathbb{N}$. The m-*Catalan arrangement in dimension *n is*

$$\mathcal{A}_m^n = \bigcup_{\substack{1 \leq i < j \leq n \\ s \in [-m; m]}} \{x_i - x_j = s\} \subseteq \mathbb{R}^n.$$

We will now introduce a canonical way to label the hyperplanes in braid-type arrangements.

Notation 2. *For $m, n \in \mathbb{N}$ we define*

$$\mathit{Triple}_{\mathrm{m}}^{\mathrm{n}} := \{(i, j, s) \mid i, j \in [n],\ s \in [0; m] \text{ such that } i \neq j \text{ and } (s > 0 \text{ or } i > j)\}.$$

Observe that each hyperplane of $\mathcal{A}_m^n$ is of the form $\{x_i - x_j = s\}$ for exactly one triple (i, j, s) in $\mathrm{Triple}_{\mathrm{m}}^{\mathrm{n}}$. For an arrangement $\mathcal{A} \subseteq \mathcal{A}_m^n$ we define

$$\mathrm{Triple}_{\mathrm{m}}^{\mathrm{n}}(\mathcal{A}) := \{(i, j, s) \in \mathrm{Triple}_{\mathrm{m}}^{\mathrm{n}} \mid \{x_i - x_j = s\} \in \mathcal{A}\},$$

so that

$$\mathcal{A} = \bigcup_{(i,j,s) \in \mathrm{Triple}_{\mathrm{m}}^{\mathrm{n}}(\mathcal{A})} \{x_i - x_j = s\}.$$

Definition 3. *An arrangement $\mathcal{A} \subseteq \mathcal{A}_m^n$ is* transitive *if for all distinct indices $i, j, k \in [n]$ and integers $s, t \geq 0$ such that (i, j, s) and (j, k, t) are in $\mathit{Triple}_{\mathrm{m}}^{\mathrm{n}}$ the following holds:*
if $\{x_i - x_j = s\} \notin \mathcal{A}$ and $\{x_j - x_k = t\} \notin \mathcal{A}$, then $\{x_i - x_k = s + t\} \notin \mathcal{A}$.

Definition 4. *A (m, n)-*tree *is a rooted $(m+1)$-ary tree with n nodes labeled with distinct labels in $[n]$ (the leaves have no labels). We denote by $\mathcal{T}_m^n$ the set of all (m, n)-trees (there are $\frac{n!}{mn+1}\binom{(m+1)n}{n}$ of them).*

For a tree $T \in \mathcal{T}_m^n$ we identify the nodes with their labels in $[n]$ (so that the node set of T is $[n]$). By definition, a node $j \in [n]$ of T has exactly $m + 1$ (ordered) children, which are denoted by 0-child(j), 1-child(j), …, m-child(j) respectively.

*The node $i \in [n]$ is the s-*cadet *of the node $j \in [n]$ in T if $i = s$-child(j) and t-child(j) is a leaf for all $t \in [s + 1; m]$. In this case, we write $i = s$-cadet(j), and we call $\{i, j\}$ a* cadet-edge.

Definition 5. *Let $\mathcal{A} \subseteq \mathcal{A}_m^n$ be an arrangement. We define*

$$\mathcal{T}_{\mathrm{m}}^{\mathrm{n}}(\mathcal{A}) := \{T \in \mathcal{T}_m^n \mid \forall (i, j, s) \in \mathit{Triple}_{\mathrm{m}}^{\mathrm{n}} \setminus \mathit{Triple}_{\mathrm{m}}^{\mathrm{n}}(\mathcal{A}),\ i \neq s\text{-cadet}(j)\}.$$

It was established in [6] that, for any transitive arrangement $\mathcal{A}$, the regions of $\mathcal{A}$ are in bijection with the trees in $\mathcal{T}_{\mathrm{m}}^{\mathrm{n}}(\mathcal{A})$. In order to describe the bijection we need to introduce a total order on the vertices of each tree in $\mathcal{T}_m^n$.

Definition 6. *For every vertex v of in a tree T, we consider the sequence $v_0, v_1, \ldots, v_k = v$ on the path of T from the root v_0 to the vertex v, and define $path_T(v) := (s_1, \ldots, s_k)$, where $v_i = s_i\text{-}child(v_{i-1})$ for all $i \in [k]$. We also define the* drift *of v as $drift_T(v) := \sum_{i=1}^{k} s_i$.*

Definition 7. *Let T be a tree in $\mathcal{T}_m^n$. We define a total order $\preceq_T$ on the vertices of T as follows. Let v, w be distinct vertices of T. Let $path_T(v) = (s_1, \ldots, s_k)$ and $path_T(w) = (t_1, \ldots, t_\ell)$. Then $v \prec_T w$ if either*
- *$drift_T(v) < drift_T(w)$, or*
- *$drift_T(v) = drift_T(w)$ and there exists $j \leq k$ such that $(s_1, \ldots, s_j) = (t_1, \ldots, t_j)$ and ($j = k$ or $s_{j+1} > t_{j+1}$).*

Definition 8. *Let $\mathcal{A} \subseteq \mathcal{A}_m^n$ be an arrangement. For every tree $T \in \mathcal{T}_m^n(\mathcal{A})$ we define the polyhedron*

$$\phi_{\mathcal{A}}(T) := \Bigg(\bigcap_{\substack{(i,j,s) \in Triple_m^n(\mathcal{A}) \\ i \prec_T s\text{-}child(j)}} \{x_i - x_j < s\} \Bigg) \cap \Bigg(\bigcap_{\substack{(i,j,s) \in Triple_m^n(\mathcal{A}) \\ i \succeq_T s\text{-}child(j)}} \{x_i - x_j > s\} \Bigg).$$

The following result was proved in [6, Theorem 8.8].

Theorem 9 ([6]). *If an arrangement $\mathcal{A} \subseteq \mathcal{A}_m^n$ is transitive, then $\Phi_{\mathcal{A}}$ is a bijection between the set $\mathcal{T}_m^n(\mathcal{A})$ of trees and the set $\mathcal{R}(\mathcal{A})$ of regions of $\mathcal{A}$.*

3 Main results

In this section we state our main result, which is a bijection between the set of faces of any "strongly transitive" arrangement and some set of marked trees.

Definition 10. *An arrangement $\mathcal{A} \subseteq \mathcal{A}_m^n$ is* strongly transitive *if for all distinct indices $i, j, k \in [n]$ and integers $s, t \geq 0$ the following holds:*
if $\{x_i - x_j = s\} \notin \mathcal{A}$ and $\{x_j - x_k = t\} \notin \mathcal{A}$, then $\{x_i - x_k = s + t\} \notin \mathcal{A}$.

The only difference between transitive and strongly transitive arrangements is that the conclusion $\{x_i - x_k = s + t\} \notin \mathcal{A}$ needs to hold even in the cases ($s = 0$ and $i < j$) or ($t = 0$ and $j < k$). In fact, an arrangement $\mathcal{A} \subseteq \mathcal{A}_m^n$ is strongly transitive if and only if $\pi(\mathcal{A})$ is transitive for every permutation $\pi \in \mathfrak{S}_n$.

Example 11. The (extended) Catalan, Shi and semiorder arrangements are strongly transitive. The Linial arrangement is transitive, but not strongly transitive. Any transitive arrangement containing the braid arrangement is strongly transitive.

Let us state which arrangements of the form $\mathcal{A}_S^n$ are strongly transitive (the proof is omitted):

Lemma 12. *Let $S \subseteq \mathbb{Z}$ be a set of integers. The following are equivalent:*
(i) $\mathcal{A}_S^n$ *is strongly transitive for every integer* $n > 0$,
(ii) $\mathcal{A}_S^n$ *is strongly transitive for at least one integer* $n \geq 3$,
(iii) for all integers $s, t \notin S$, *if* $st \geq 0$ *then* $s + t \notin S$ *and if* $st \leq 0$ *then* $s - t, t - s \notin S$.

We will now define the trees in bijection with the faces of a strongly transitive arrangement.

Definition 13. *A* marked (m, n)-tree *is a pair* (T, μ), *where* $T \in \mathcal{T}_m^n$ *is a* (m, n)*-tree and* μ *is a set of cadet-edges of* T *such that if an edge* $e \in \mu$ *is of the form* $e = \{j, 0\text{-}cadet(j)\}$ *then* $j < 0\text{-}cadet(j)$. *We refer to the edges in* μ *as the* marked edges.

We denote by $\overline{\mathcal{T}}_m^n$ *the set of marked* (m, n)*-trees.*

Note that the marked edges of a marked tree in $\overline{\mathcal{T}}_m^n$ form a collection of vertex-disjoint paths. Let $(T, \mu) \in \overline{\mathcal{T}}_m^n$ be a marked (m, n)-tree. For nodes $i, j \in [n]$, we write $i \overset{\mu}{\sim} j$ if $i = j$ or $i \neq j$ and all the edges on the path of T between i and j are marked. This is an equivalence relation, and we call its equivalence classes the *blocks* of (T, μ).

Definition 14. *Let* $\mathcal{A} \subseteq \mathcal{A}_m^n$ *be an arrangement. A block* $B \subseteq [n]$ *of a marked tree* $(T, \mu) \in \overline{\mathcal{T}}_m^n$ *is called* $\mathcal{A}$-connected *if the graph* G *with vertex set* B *and edge set*

$$E = \left\{ \{i, j\} \mid i, j \in B \text{ such that } \{x_i - x_j = drift_{\mathrm{T}}(i) - drift_{\mathrm{T}}(j)\} \in \mathcal{A} \right\}$$

is connected. We say that the marked tree (T, μ) *is* $\mathcal{A}$-connected *if every block of* (T, μ) *is* $\mathcal{A}$*-connected.*

We say that a marked tree $(T, \mu) \in \overline{\mathcal{T}}_{\mathrm{m}}^{\mathrm{n}}$ *satisfies the* $\mathcal{A}$-cadet condition *if every non-marked cadet-edge* $e = \{i, j\}$ *of* (T, μ), *with* $i = s\text{-}cadet(j)$, *satisfies (*$s = 0$ *and* $i < j$*) or (there exists* $i' \overset{\mu}{\sim} i$ *and* $j' \overset{\mu}{\sim} j$ *such that the hyperplane* $\{x_{i'} - x_{j'} = drift_{\mathrm{T}}(i') - drift_{\mathrm{T}}(j')\}$ *is in* $\mathcal{A}$*).*

We define $\overline{\mathcal{T}}_{\mathrm{m}}^{\mathrm{n}}(\mathcal{A})$ *as the set of marked trees in* $\overline{\mathcal{T}}_{\mathrm{m}}^{\mathrm{n}}$ *which are* $\mathcal{A}$*-connected and satisfy the* $\mathcal{A}$*-cadet condition.*

Definition 15. *Let* $\mathcal{A} \subseteq \mathcal{A}_m^n$. *We associate to each marked tree* (T, μ) *in* $\overline{\mathcal{T}}_{\mathrm{m}}^{\mathrm{n}}(\mathcal{A})$ *a polyhedron*

$$\begin{aligned} \overline{\Phi}_{\mathcal{A}}(T, \mu) \;:=\; & \Bigg(\bigcap_{\substack{\{i,j\} \in \mu \\ i = s\text{-}child(j)}} \{x_i - x_j = s\} \Bigg) \\ & \cap \Bigg(\bigcap_{\substack{(i,j,s) \in Triple_{\mathrm{m}}^{\mathrm{n}}(\mathcal{A}) \\ i \overset{\mu}{\nsim} j,\; i \prec_{\mathrm{T}} s\text{-}child(j)}} \{x_i - x_j < s\} \Bigg) \cap \Bigg(\bigcap_{\substack{(i,j,s) \in Triple_{\mathrm{m}}^{\mathrm{n}}(\mathcal{A}) \\ i \overset{\mu}{\nsim} j,\; i \succeq_{\mathrm{T}} s\text{-}child(j)}} \{x_i - x_j > s\} \Bigg). \end{aligned}$$

We are now ready to state our main result, which is illustrated in Figure 2.

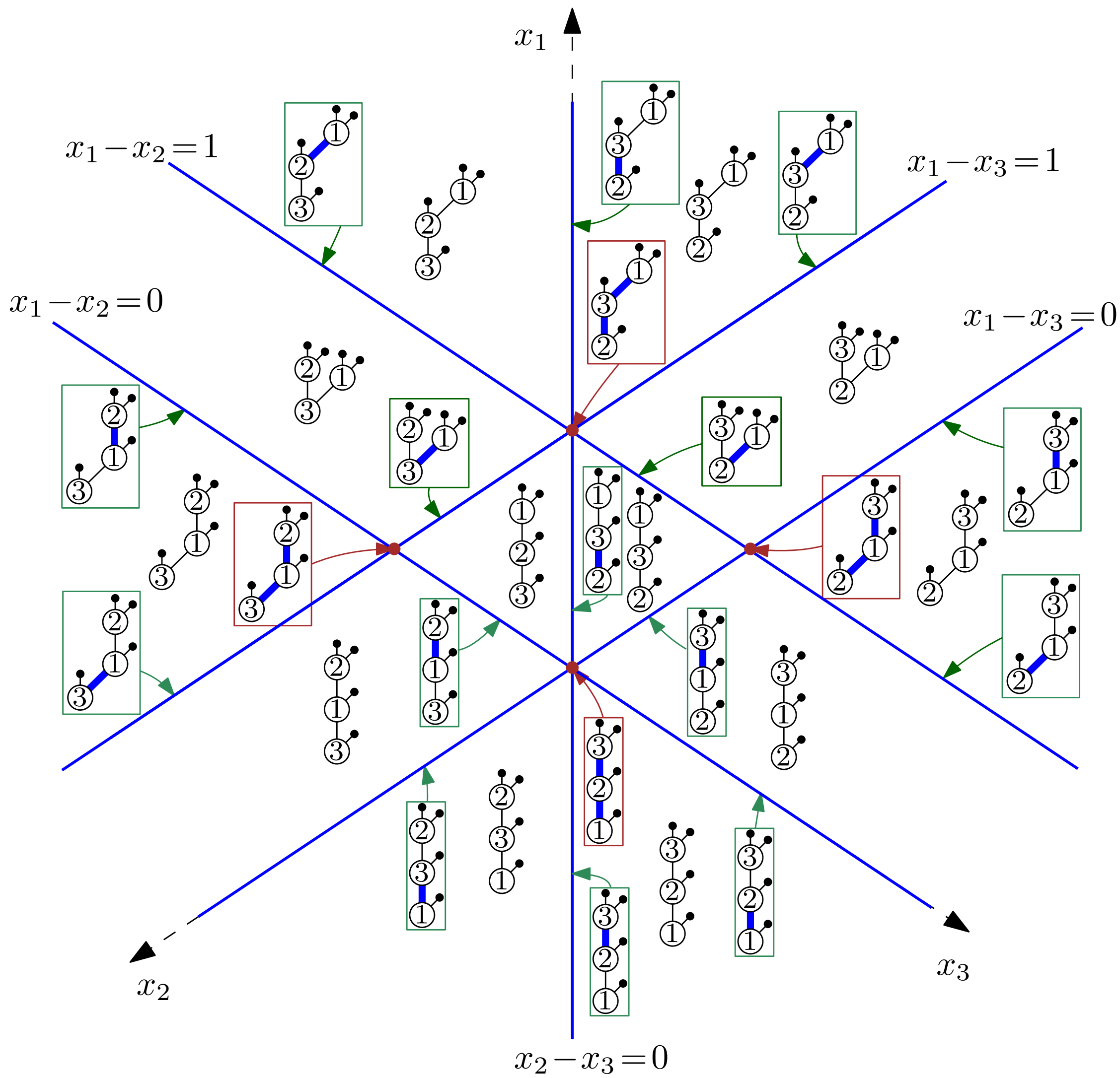


Figure 2: The bijection $\overline{\Phi}_{\mathcal{A}}$ between the faces of an arrangement $\mathcal{A}$ and the set $\overline{\mathcal{T}}^{\mathrm{n}}_{\mathrm{m}}(\mathcal{A})$.

Theorem 16. *If an arrangement $\mathcal{A} \subseteq \mathcal{A}^n_m$ is strongly transitive, then $\overline{\Phi}_{\mathcal{A}}$ is a bijection between the set $\overline{\mathcal{T}}^{\mathrm{n}}_{\mathrm{m}}(\mathcal{A})$ of marked trees and the set $\mathcal{F}(\mathcal{A})$ of faces of $\mathcal{A}$. The number of marked edges of (T, μ) is equal to the codimension of the corresponding face $\overline{\Phi}_{\mathcal{A}}(T, \mu)$.*

The special cases of Theorem 16 corresponding to $\mathcal{A} = \mathcal{A}^n_m$ (the m-Catalan arrangement) or $\mathcal{A} = \mathcal{A}^n_{[-m+1;m]}$ (the m-Shi arrangement) give bijections which are the same as that of Levear [10] (up to small differences of presentation). We discuss these special cases, and some other examples, in the next section.

4 Applications

In this section we apply Theorem 16 to several families of braid-type arrangements.

4.1 From the Catalan arrangement to the Shi arrangement, and back

Theorem 16 readily implies the following results of Levear [10] about the m-Catalan and m-Shi arrangements.

Corollary 17 ([10]). *Let m, n be positive integers. The faces (of codimension k) of the m-Catalan arrangement $\mathcal{A}_m^n$ are in bijection with the marked trees in $\overline{\mathcal{T}}_{\mathrm{m}}^{\mathrm{n}}$ (having k marked edges).*

Corollary 18 ([10]). *Let m, n be positive integers. The faces (of codimension k) of the m-Shi arrangement $\mathcal{S}hi_m^n = \mathcal{A}_{[-m+1;m]}^n$ are in bijection with the set $\overline{\mathcal{S}}_{\mathrm{m}}^{\mathrm{n}}$ of marked trees (T, μ) in $\overline{\mathcal{T}}_{\mathrm{m}}^{\mathrm{n}}$ (having k marked edges) such that if $i = m$-child(j) for some nodes $i, j \in [n]$ of T, then $i < j$.*

It is clear that $\mathcal{A}_m^n$ and $\mathcal{S}hi_m^n$ are strongly transitive. We can therefore apply Theorem 16, and we only need to check that $\overline{\mathcal{T}}_{\mathrm{m}}^{\mathrm{n}}(\mathcal{A}_m^n) = \overline{\mathcal{T}}_{\mathrm{m}}^{\mathrm{n}}$ and $\overline{\mathcal{T}}_{\mathrm{m}}^{\mathrm{n}}(\mathcal{S}hi_m^n) = \overline{\mathcal{S}}_{\mathrm{m}}^{\mathrm{n}}$. These proofs are rather straightforward. We will actually prove a more general result that interpolates between the case of the Catalan arrangement and the case of the Shi arrangement.

Observe that, for all $m, n > 0$,, any arrangement $\mathcal{A}$ such that $\mathcal{A}_{m-1}^n \subseteq \mathcal{A} \subseteq \mathcal{A}_m^n$ is strongly transitive. We now describe a family of arrangements $\mathcal{A}$ which interpolates between $\mathcal{A}_{m-1}^n$ and $\mathcal{A}_m^n$ and for which the set $\overline{\mathcal{T}}_{\mathrm{m}}^{\mathrm{n}}(\mathcal{A})$ admits a simple description.

Let

$$R_n := \{(i,j) \in [n]^2 \mid i \neq j\} \text{ and } R_n^+ := \{(i,j) \in [n]^2 \mid i < j\}.$$

For a subset $I \subseteq R_n$ we define the arrangement

$$\mathcal{B}_{m,I}^n = \mathcal{A}_{m-1}^n \cup \left(\bigcup_{(i,j) \in I} \{x_i - x_j = m\} \right).$$

Note that $\mathcal{A}_{m-1}^n = \mathcal{B}_{m,\emptyset}^n$, $\mathcal{A}_m^n = \mathcal{B}_{m,R_n}^n$, and $\mathcal{S}hi_m^n = \mathcal{B}_{m,R_n^+}^n$.

We say that $I \subseteq R_n$ is an *ideal* if the following holds for all $(i,j), (i',j') \in R_n$:

if (i,j) is in I and $i' \leq i$ and $j' \geq j$, then (i',j') is in I.

Note that $\emptyset$, R_n and R_n^+ are ideals.

Theorem 19. *Let m, n be positive integers. For any ideal $I \subseteq R_n$, the faces (of codimension k) of the arrangement $\mathcal{A} = \mathcal{B}_{m,I}^n$ are in bijection, via the bijection $\overline{\Phi}_{\mathcal{A}}$, with the set of marked trees $(T, \mu) \in \overline{\mathcal{T}}_{\mathrm{m}}^{\mathrm{n}}$ (with k marked edges) such that if $i = m$-child(j) for some nodes $i, j \in [n]$ of T, then (i,j) is in I.*

Note that Theorem 19 generalizes both Corollary 17 (which corresponds to $I = R_n$) and Corollary 18 (which corresponds to $I = R_n^+$).

Proof. We apply Theorem 16 to the strongly transitive arrangement $\mathcal{A} = \mathcal{B}^n_{m,I}$, and need to check that $\overline{\mathcal{T}}^{\mathrm{n}}_{\mathrm{m}}(\mathcal{A})$ is the set $\mathcal{S}$ of marked trees (T, μ) in $\overline{\mathcal{T}}^{\mathrm{n}}_{\mathrm{m}}$ such that if $i = m$-child(j) for some nodes $i, j \in [n]$ of T, then (i, j) is in I. Before starting this proof, let us observe that for any marked tree $(T, \mu) \in \overline{\mathcal{T}}^{\mathrm{n}}_{\mathrm{m}}$ and any nodes i, j such that $i = s$-cadet(j) the hyperplane $\{x_i - x_j = \mathrm{drift}_{\mathrm{T}}(i) - \mathrm{drift}_{\mathrm{T}}(j)\} = \{x_i - x_j = s\}$ is in $\mathcal{A}$ unless $s = m$ and $(i, j) \notin I$.

Now we will determine under which conditions a marked tree $(T, \mu) \in \overline{\mathcal{T}}^{\mathrm{n}}_{\mathrm{m}}$ is $\mathcal{A}$-connected. Let B be a block of (T, μ), and let G be the graph with vertex set B and edge set

$$E = \{\{i, j\} \mid i, j \in B \text{ such that } \{x_i - x_j = \mathrm{drift}_{\mathrm{T}}(i) - \mathrm{drift}_{\mathrm{T}}(j)\} \in \mathcal{A}\}.$$

Recall that B is of the form $B = \{i_1, \ldots, i_\ell\}$, where for all $k \in [\ell - 1]$, $i_{k+1} = s_k$-cadet(i_k) for some $s_k \leq m$. By the above observation, the edge $\{i_k, i_{k+1}\}$ is in E whenever $s_k < m$. Hence the graph G is connected if and only if for all $k \in [\ell - 1]$ such that $i_{k+1} = m$-cadet(i_k) there exist $k' \leq k$ and $k'' \geq k + 1$ such that $\mathrm{drift}_{\mathrm{T}}(i_{k'}) = \mathrm{drift}_{\mathrm{T}}(i_k)$ and $\mathrm{drift}_{\mathrm{T}}(i_{k''}) = \mathrm{drift}_{\mathrm{T}}(i_{k+1})$ (so that $\mathrm{drift}_{\mathrm{T}}(i_{k''}) - \mathrm{drift}_{\mathrm{T}}(i_{k'}) = m$) and $(i_{k''}, i_{k'}) \in I$ (so that $\{i_{k''}, i_{k'}\} \in E$). Moreover, $\mathrm{drift}_{\mathrm{T}}(i_{k'}) = \mathrm{drift}_{\mathrm{T}}(i_k)$ and $k' \leq k$ imply $i_{k'} \leq i_k$ (since $(T, \mu) \in \overline{\mathcal{T}}^{\mathrm{n}}_{\mathrm{m}}$) and similarly $\mathrm{drift}_{\mathrm{T}}(i_{k''}) = \mathrm{drift}_{\mathrm{T}}(i_{k+1})$ and $k'' \geq k + 1$ imply $i_{k''} \geq i_{k+1}$. Hence, there exists k' and k'' satisfying the above conditions if and only if $(i_{k+1}, i_k) \in I$ (because I is an ideal). This shows that a marked tree $(T, \mu) \in \overline{\mathcal{T}}^{\mathrm{n}}_{\mathrm{m}}$ is $\mathcal{A}$-connected if and only if (i, j) is in I for every marked edge $\{i, j\}$ such that $i = m$-child(j).

A similar reasoning shows that (T, μ) satisfies the $\mathcal{A}$-cadet condition if and only if (i, j) is in I for every non-marked edge $\{i, j\}$ such that $i = m$-child(j). $\square$

4.2 Multi-Catalan arrangements

Let n be a positive integer. Given a n-tuple of integers $\mathbf{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$, we define the ***m**-Catalan arrangement* as

$$\mathcal{A}_{\mathbf{m}} := \bigcup_{\substack{1 \leq i < j \leq n \\ s \in [-m_i; m_j]}} \{x_i - x_j = s\}.$$

Given a marked tree $(T, \mu) \in \overline{\mathcal{T}}^{\mathrm{n}}_{\mathrm{m}}$, we define the ***m**-reach* of a node $j \in [n]$ of T as

$$r_{\mathbf{m}}(j) := \max(m_k + \mathrm{drift}_{\mathrm{T}}(k) - \mathrm{drift}_{\mathrm{T}}(j) \mid k \overset{\mu}{\sim} j \text{ and } k \text{ ancestor of } j).$$

It is easy to see that $\mathcal{A}_{\mathbf{m}}$ is strongly transitive for all $\mathbf{m} \in \mathbb{N}^n$, and that applying Theorem 16 gives the following bijection.

Proposition 20. *Let n be a positive integer, let $\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{N}^n$, and let $m = \max(m_i \mid i \in [n])$. The faces of the $\boldsymbol{m}$-Catalan arrangement $\mathcal{A}_{\boldsymbol{m}}$ are in bijection with the set of marked trees $(T, \mu) \in \overline{\mathcal{T}}^{\mathrm{n}}_{\mathrm{m}}$ such that for every node $j \in [n]$, the vertex s-child(j) is a leaf for all $s > r_{\boldsymbol{m}}(j)$.*

4.3 Generating function for symmetric transitive arrangements

In this section we focus on *symmetric* braid-type arrangements, that is, braid-type arrangements $\mathcal{A}$ such that $\pi(\mathcal{A}) = \mathcal{A}$ for all $\pi \in \mathfrak{S}_n$. It is easy to see that the symmetric braid-type arrangements are the arrangements of the form $\mathcal{A}^n_S$ for a set $S \subseteq \mathbb{Z}$ such that $S = -S$. From Lemma 12, one gets that the strongly transitive symmetric arrangements in dimension $n \geq 3$ are precisely the arrangements of the form $\mathcal{A}^n_S$, where the set $S \subseteq \mathbb{Z}$ satisfies

$$S = -S \text{ and } \forall s, t \in \mathbb{N} \setminus S, \quad s + t \notin S. \tag{4.1}$$

Given a finite set S, we define the *face generating function* of the arrangements $\mathcal{A}^n_S$ as

$$F_S(x,t) := \sum_{n=0}^{\infty} \sum_{k=0}^{n} c_{n,k} t^k \frac{x^n}{n!},$$

where $c_{n,k}$ is the number of faces of codimension k of $\mathcal{A}^n_S$. Applying Theorem 16 gives the following result for any finite set $S \subseteq \mathbb{Z}$ satisfying (4.1):

$$F_S(x,t) = \sum_{(T,\mu) \in \overline{\mathcal{T}}(S)} t^{|\mu|} \frac{x^{|T|}}{|T|!},$$

where $\overline{\mathcal{T}}(S) := \bigcup_{n=0}^{\infty} \overline{\mathcal{T}}^{\mathrm{n}}_{\mathrm{m}}(\mathcal{A}^n_S)$, $|T|$ is the number of nodes of the tree T, and $|\mu|$ is the number of marked edges. Using this expression one can establish the following result:

Theorem 21. *Let $S \subset \mathbb{Z}$ be a finite set satisfying (4.1). The face generating function $F_S(x,t)$ is characterized by a finite equation, which is computable from S. This equation takes the form $P(F_S(x,t), e^{xt}, e^x, t) = 0$ for some (non-zero) polynomial P with coefficients in $\mathbb{Q}$.*

We omit the (non-trivial) proof of Theorem 21. Let us write down two cases explicitly: the case of m-Catalan arrangements for which a generating function equation was established by Levear [10], and the case of the semiorder arrangement (which is new).

Proposition 22 ([10]). *The generating function $G \equiv F_{[-m;m]}(x,t)$ counting the faces of the m-Catalan arrangements is characterized by the following equation:*

$$G = 1 + \frac{1}{t}\Omega(e^{xt} - 1, F), \quad \textit{where} \quad \Omega(X,Y) = \frac{XY^{m+1}}{1 - XY\frac{1-Y^m}{1-Y}}.$$

Proposition 23. *The generating function $H \equiv F_{[-m;m]\setminus\{0\}}(x,t)$ counting the faces of the m-extended semiorder arrangements is characterized by the following equation:*

$$H = 1 + (1 - e^{-x})H^{m+1} + \frac{1}{t}\tilde{\Omega}(e^{xt} - 1, H), \quad \textit{where} \quad \tilde{\Omega}(X,Y) = X^2 \frac{Y^{m+2} - Y^{2m+2}}{1 - Y - XY + XY^{m+1}}.$$

5 Proof of Theorem 16 (sketch)

In this section we sketch the proof Theorem 16. Roughly speaking, the proof consists in seeing the faces of an arrangement $\mathcal{A} \subseteq \mathcal{A}_m^n$ as the regions of the restrictions of $\mathcal{A}$ to each subspace in the intersection lattice, and applying Theorem 9 to these arrangements.

Recall that the *intersection lattice* of $\mathcal{A}$ is

$$\mathcal{L}(\mathcal{A}) := \left\{ \bigcap_{i=1}^{k} H_i \;\middle|\; k \geq 0, H_1, \ldots, H_k \in \mathcal{A} \text{ such that } \bigcap_{i=1}^{k} H_i \neq \emptyset \right\}.$$

For an affine space $L \in \mathcal{L}(A)$ we want to identify the restriction $\mathcal{A}_L$ of $\mathcal{A}$ to L with an arrangement $\widetilde{\mathcal{A}}_L \subseteq \mathcal{A}_{mn}^d$, where $d = \dim(L)$.

If L is contained in a hyperplane of the form $\{x_i - x_j = s\}$, we write $i \overset{L}{\sim} j$. This is an equivalence relation, and we denote by $\text{blocks}(L)$ the set of equivalence classes. Let $\{B_1, \ldots, B_d\} = \text{blocks}(L)$. For all $i \in [n]$ we define $\delta_L(i) = \max(x_i - x_j \mid j \overset{L}{\sim} i)$ for any point $(x_1, \ldots, x_n)$ in L. Finally, we define the arrangement $\widetilde{\mathcal{A}}_L \subseteq \mathbb{R}^d$ as follows:

$$\widetilde{\mathcal{A}}_L := \bigcup_{\substack{k,\ell \in [d] \\ k \neq \ell}} \; \bigcup_{\substack{i \in B_k,\, j \in B_\ell,\, s \in \mathbb{Z} \\ \{x_i - x_j = s\} \in \mathcal{A}}} \{x_k - x_\ell = s - \delta_L(i) + \delta_L(j)\}.$$

It is easy to check that the arrangements $\mathcal{A}_L$ and $\widetilde{\mathcal{A}}_L$ are isomorphic, and that $\widetilde{\mathcal{A}}_L \subseteq \mathcal{A}_{mn}^d$, where $d = \dim(L)$. A key observation (whose proof we omit) is the following.

Lemma 24. *If $\mathcal{A} \subseteq \mathcal{A}_\infty^n$ is strongly transitive, then for any affine space L in $\mathcal{L}(\mathcal{A})$, the arrangement $\widetilde{\mathcal{A}}_L$ is strongly transitive.*

The proof of Theorem 16 consists in establishing the commutative diagram of bijections represented in Figure 3, where $\mathcal{F}(\mathcal{A})$ is the set of faces of $\mathcal{A}$ and

$$\widetilde{\mathcal{T}}(\mathcal{A}) := \{(L, \widetilde{T}) \mid L \in \mathcal{L}(\mathcal{A}),\ \widetilde{T} \in \mathcal{T}_{mn}^{\dim(L)}(\widetilde{\mathcal{A}}_L)\},$$
$$\widetilde{\mathcal{F}}(\mathcal{A}) := \{(L, \widetilde{R}) \mid L \in \mathcal{L}(\mathcal{A}),\ \widetilde{R} \in \mathcal{R}(\widetilde{\mathcal{A}}_L)\}.$$

$$\begin{array}{ccc}
(T,\mu) \in \overline{\mathcal{T}}^{\mathrm{n}}_{\mathrm{m}}(\mathcal{A}) & \xrightarrow{\quad \overline{\Phi}_{\mathcal{A}} \quad} & F \in \mathcal{F}(\mathcal{A}) \\
\text{bijection } \Gamma \big\downarrow & & \text{bijection } \Theta \big\uparrow \\
\left(L, \widetilde{T}\right) \in \widetilde{\mathcal{T}}(\mathcal{A}) & \xrightarrow[\text{bijection } \widetilde{\Phi}_{\mathcal{A}}]{} & \left(L, \widetilde{R}\right) \in \widetilde{\mathcal{F}}(\mathcal{A})
\end{array}$$

Figure 3: Commutative diagram representing the proof of Theorem 16.

We start by defining $\widetilde{\Phi}_{\mathcal{A}} : \widetilde{\mathcal{T}}(\mathcal{A}) \to \widetilde{\mathcal{F}}(\mathcal{A})$ as the map which associates to each pair $(L, \widetilde{T}) \in \widetilde{\mathcal{T}}(\mathcal{A})$ the pair $(L, \Phi_{\widetilde{\mathcal{A}}_L}(T)) \in \widetilde{\mathcal{F}}(\mathcal{A})$. By combining Lemma 24 with Theorem 9 from [6], we deduce that $\widetilde{\Phi}_{\mathcal{A}}$ is a bijection between the sets $\widetilde{\mathcal{T}}(\mathcal{A})$ and $\widetilde{\mathcal{F}}(\mathcal{A})$. It is also clear that the sets $\widetilde{\mathcal{F}}(\mathcal{A})$ and $\mathcal{F}(\mathcal{A})$ are in bijection. It remains to describe the bijection Γ between $\overline{\mathcal{T}}^{\mathrm{n}}_{\mathrm{m}}(\mathcal{A})$ and $\widetilde{\mathcal{T}}(\mathcal{A})$. We first need to encode the subspaces in $\mathcal{L}(\mathcal{A})$.

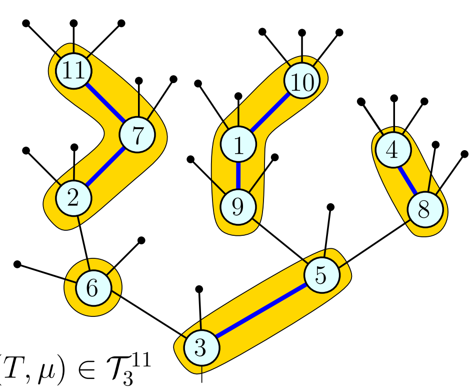
Definition 25. *Let $\mathcal{A} \subseteq \mathcal{A}^n_m$, let $B \subseteq [n]$ be a set, and let $\delta : [n] \to \mathbb{N}$ be a map. The pair (B, δ) is called* $\mathcal{A}$-connected *if the graph G with vertex set B and edge set $E := \{\{i, j\} \mid i, j \in B,\ \{x_i - x_j = \delta(i) - \delta(j)\} \in \mathcal{A}\}$ is connected. We define $\mathcal{P}(\mathcal{A})$ as the set of pairs $(\{B_1, \ldots, B_d\}, \delta)$, where $\{B_1, \ldots, B_d\}$ is a partition of $[n]$ and $\delta : [n] \to \mathbb{N}$ is a map such that, $\forall k \in [d]$, $\min(\delta(i) \mid i \in B_k) = 0$ and (B_k, δ) is $\mathcal{A}$-connected.*

Lemma 26. *Let $\mathcal{A} \subseteq \mathcal{A}^n_m$. The lattice $\mathcal{L}(\mathcal{A})$ is in bijection with $\mathcal{P}(\mathcal{A})$. The bijection $\Lambda : \mathcal{L}(\mathcal{A}) \to \mathcal{P}(\mathcal{A})$ associates to each affine subspace $L \in \mathcal{L}(A)$ the pair $\Lambda(L) = (\mathrm{blocks}(L), \delta_L)$.*

Given a marked tree $(T, \mu) \in \overline{\mathcal{T}}^{\mathrm{n}}_{\mathrm{m}}(\mathcal{A})$, we consider the set partition $\{B_1, \ldots, B_d\} = \mathrm{blocks}(\mu)$ with the convention $\min(B_1) < \min(B_2) < \cdots < \min(B_d)$. We define a map $\delta_\mu : [n] \to \mathbb{N}$ by setting $\delta_\mu(i) = \max(\mathrm{drift}_{\mathrm{T}}(i) - \mathrm{drift}_{\mathrm{T}}(j) \mid j \in B_k)$ for all $i \in B_k$. By definition, each block B_k is $\mathcal{A}$-connected (see Definition 14), which is equivalent to the fact that the pair (B_k, δ_μ) is $\mathcal{A}$-connected (see Definition 25). Thus, the pair $(\{B_1, \ldots, B_d\}, \delta_\mu)$ is in $\mathcal{P}(\mathcal{A})$, and corresponds to a subspace $L \in \mathcal{L}(\mathcal{A})$ via the bijection Λ of Lemma 26. Now consider the tree $\widetilde{T}$ obtained from T by

1. contracting all the marked edges: for all $k \in [d]$, the marked path of T corresponding to B_k is replaced by a node of $\widetilde{T}$ labeled k,
2. adding leaves as right children of each node of $\widetilde{T}$ so as to get a total of $mn + 1$ children for each node.

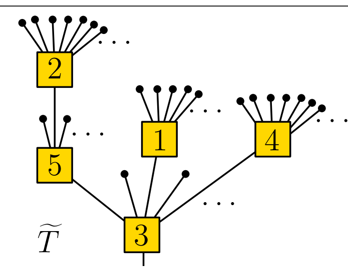
We define $\Gamma(T, \mu) := (L, \widetilde{T})$. This is illustrated in Figure 4.



$(T, \mu) \in \mathcal{T}^{11}_3$

block(μ)=$\{\{1, 9, 10\}, \{2, 7, 11\}, \{3, 5\}, \{4, 8\}, \{6\}\}$

i	1	2	3	4	5	6	7	8	9	10	11
$\delta_\mu(i)$	1	0	0	0	2	0	2	0	0	3	2



$\widetilde{T}$

Figure 4: The bijection Γ.

In order to conclude the proof of Theorem 16, one needs to check that $\widetilde{T}$ is in $\mathcal{T}^{\mathrm{d}}_{\mathrm{mn}}(\widetilde{\mathcal{A}}_L)$ (there is a technical subtlety there), and that any tree in $\mathcal{T}^{\mathrm{d}}_{\mathrm{mn}}(\widetilde{\mathcal{A}}_L)$ is obtained uniquely in this manner. Lastly, one must show that the diagram in Figure 3 is commutative.

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