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On the Stanley–Stembridge conjecture

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Abstract. The Stanley–Stembridge conjecture was a long-standing problem in algebraic combinatorics which states that the chromatic symmetric function for any $(3+1)$-free graph expands positively in terms of elementary symmetric functions. We explain how to find an inductive and positive formula for the elementary symmetric function expansion of chromatic quasisymmetric function for any unit interval graph, which in particular implies the Stanley–Stembridge conjecture.

Keywords: Stanley–Stembridge conjecture, chromatic quasisymmetric function, unit interval graph, affine Hecke algebra

1 Introduction

For any graph, Stanley [29] introduced a symmetric function called the chromatic symmetric function which refines the chromatic polynomial of the graph. In [29, 30], Stanley–Stembridge conjectured that it expands positively in terms of elementary symmetric functions for any $(3+1)$-free graph. This conjecture was a main open problem in this area and there are many researches including [2, 4, 8, 9, 10, 11, 13, 16, 19, 22, 24, 26, 31, 33] which prove the conjecture for special cases.

The aim of this extended abstract is to give a more detailed introduction to our paper [17] which proves the Stanley–Stembridge conjecture in general. Since the details of the proof already appeared in [17], we mainly focus on explaining how to find the proof. We also explain the contents of [18] on a (q,t)-analogue of chromatic symmetric functions which motivate our proof.

2 Background

2.1 Unit interval graphs

By a work of Guay-Paquet [14], the Stanley–Stembridge conjecture reduces to the case of both $(3+1)$-free and $(2+2)$-free graphs, which are known to be obtained by forgetting the orientations of unit interval graphs. The set of unit interval graphs corresponds

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bijectively to the set of Dyck paths, which is also an important object of study in algebraic combinatorics. In particular, the number of unit interval graphs with n vertices is equal to the n-th Catalan number. See for example [7] for an appearance of chromatic quasisymmetric function as a plethystic substitution of unicellular LLT polynomial.

Instead of defining the notions of $(a+b)$-free graph and unit interval graph, let us give a parametrization of unit interval graphs in terms of area sequences as in [3]. We set $[n] := \{1, 2, \ldots, n\}$ and

$$\mathbb{A}_n := \{\mathsf{a} : [n] \to \mathbb{Z} | 0 \le \mathsf{a}(i) < i,\ \mathsf{a}(i+1) \le \mathsf{a}(i) + 1)\}$$

which is the set of area sequences for unit interval graphs. For each $\mathsf{a} \in \mathbb{A}_n$, we associate an oriented graph Γ_{a} with vertex set $[n]$ and edges

$$i - \mathsf{a}(i) \to i,\ i - \mathsf{a}(i) + 1 \to i,\ \ldots,\ i - 1 \to i$$

for all $i \in [n]$. The oriented graph obtained in this way is called *unit interval graph*. We note that in [3], the notion of *circular unit arc digraph* is defined by weakening the condition $0 \le \mathsf{a}(i) < i$ by $0 \le \mathsf{a}(i) < n$ and considering the indices modulo n.

For our later purpose, it is convenient to introduce another set $\mathbb{E}_n$ for parametrizing unit interval graphs. We define

$$\mathbb{E}_n := \{\mathsf{e} : [n] \to \mathbb{Z} | 0 \le \mathsf{e}(i) < i,\ \mathsf{e}(i) \le \mathsf{e}(i+1)\}.$$

We have a bijection $\mathbb{A}_n \cong \mathbb{E}_n$ given by $\mathsf{a}(i) = i - 1 - \mathsf{e}(i)$ and we denote by Γ_{e} the unit interval graph corresponding to $\mathsf{e} \in \mathbb{E}_n$ under this bijection. We also note that $\mathbb{E}_n$ corresponds bijectively to the set of Hessenberg functions

$$\mathbb{H}_n := \{\mathsf{h} : [n] \to [n] | \mathsf{h}(i) \ge i,\ \mathsf{h}(i) \le \mathsf{h}(i+1)\}$$

by $\mathsf{h}(i) = n - \mathsf{e}(n+1-i)$.

For two unit interval graphs Γ of n-vertices and Γ' of n' vertices, we can consider their disjoint ordered union $\Gamma \cup \Gamma'$ whose vertices are labeled by $1, \ldots, n$ for Γ and $n+1, \ldots, n+n'$ for Γ'. This is again a unit interval graph. We define $\mathsf{e}_n \in \mathbb{E}_n$ by $\mathsf{e}_n(i) = 0$ for any $i = 1, \ldots, n$ and $\mathsf{e}_\mu = \mathsf{e}_{\mu_1} \cup \cdots \cup \mathsf{e}_{\mu_l}$ for a composition $\mu = (\mu_1, \ldots, \mu_l)$ of n. The graph Γ_{e_n} corresponding to $\mathsf{e}_n \in \mathbb{E}_n$ is the complete graph of n-vertices.

2.2 Chromatic quasisymmetric functions

Shareshian–Wachs [27, 28] and Ellzey [12] introduced a refinement of the chromatic symmetric function for any oriented graph called chromatic quasisymmetric function. Let us recall its definition briefly. Let Γ be an oriented graph. A map $\kappa : \Gamma \to \mathbb{Z}_{>0}$ from the set of vertices of Γ to the set of positive integers is called a proper coloring if

$\kappa(u) \neq \kappa(v)$ for any $u, v \in \Gamma$ which are connected by an edge in Γ. For a proper coloring $\kappa : \Gamma \to \mathbb{Z}_{>0}$, we denote by $\mathrm{asc}(\kappa)$ the number of edges $u \to v$ in Γ such that $\kappa(u) < \kappa(v)$. The *chromatic quasisymmetric function* $\mathcal{X}_\Gamma(t)$ of Γ is defined by

$$\mathcal{X}_\Gamma(t) := \sum_{\kappa} t^{\mathrm{asc}(\kappa)} \prod_{v \in \Gamma} X_{c(v)},$$

where the sum runs over the set of all proper colorings of Γ and $X_1, X_2, \ldots,$ are indeterminates. It is known that $\mathcal{X}_\Gamma(t)$ is a symmetric function in $X_1, X_2, \ldots,$ if Γ is a unit interval graph [28] or more generally a circular unit arc digraph [3, 12]. By definition, $\mathcal{X}_\Gamma(1)$ is the *chromatic symmetric function* defined by Stanley [29].

Our main object of study is the elementary symmetric function expansion

$$\mathcal{X}_{\Gamma_{\mathsf{e}}}(t) = \sum_{\lambda \vdash n} c_\lambda(\mathsf{e}; t) e_\lambda(X)$$

of $\mathcal{X}_{\Gamma_{\mathsf{e}}}(t)$ for $\mathsf{e} \in \mathbb{E}_n$, where λ runs over the set of partitions of n and $e_\lambda(X)$ is the elementary symmetric function in $X_1, X_2, \ldots,$ corresponding to λ. The Stanley–Stembridge conjecture [29, 30] states that $c_\lambda(\mathsf{e}; 1) \geq 0$ for any $\mathsf{e} \in \mathbb{E}_n$ and $\lambda \vdash n$. Shareshian–Wachs [27, 28] conjectured that more strongly, we have $c_\lambda(\mathsf{e}; t) \in \mathbb{Z}_{\geq 0}[t]$. The main result of [17] implies that $c_\lambda(\mathsf{e}; t) \geq 0$ for any $\mathsf{e} \in \mathbb{E}_n$, $\lambda \vdash n$, and $t \in \mathbb{R}_{>0}$.

2.3 Modular law

In the study of chromatic quasisymmetric functions for unit interval graphs, the work of Abreu–Nigro [1] is extremely useful. It is written in terms of the Hessenberg functions $\mathbb{H}_n$, but we state it here in terms of $\mathbb{E}_n$ for later purpose[1].

Definition 2.1. *We say that a function $\chi : \mathbb{E}_n \to \mathbb{Q}(t)$ satisfies the modular law if we have*

$$(1+t)\chi(\mathsf{e}) = t\chi(\mathsf{e}') + \chi(\mathsf{e}'')$$

for any triple $(\mathsf{e}, \mathsf{e}', \mathsf{e}'')$ in $\mathbb{E}_n$ satisfying one of the following conditions:

(i) There exists $1 < i \leq n$ such that $\mathsf{e}(i-1) < \mathsf{e}(i) < \mathsf{e}(i+1)$ and $\mathsf{e}(\mathsf{e}(i)) = \mathsf{e}(\mathsf{e}(i)+1)$. Moreover, we have $\mathsf{e}'(j) = \mathsf{e}''(j) = \mathsf{e}(j)$ for $j \neq i$ and $\mathsf{e}'(i) = \mathsf{e}(i)+1$ and $\mathsf{e}''(i) = \mathsf{e}(i)-1$.

(ii) There exists $i \in [n-1]$ such that $\mathsf{e}(i+1) = \mathsf{e}(i)+1$ and $\mathsf{e}^{-1}(i) = \emptyset$. Moreover, we have $\mathsf{e}'(j) = \mathsf{e}''(j) = \mathsf{e}(j)$ for $j \neq i, i+1$ and $\mathsf{e}'(i) = \mathsf{e}'(i+1) = \mathsf{e}(i+1)$ and $\mathsf{e}''(i) = \mathsf{e}''(i+1) = \mathsf{e}(i)$.

[1] We note that our combination of identifications $\mathbb{H}_n \cong \mathbb{E}_n \cong \mathbb{A}_n$ leads to another parametrization of the unit interval graphs that was used in [1], but since they are related by the transpose of the graph, their chromatic quasisymmetric functions are the same.

Here, we understand that $\mathsf{e}(n+1) = n-1$ *if* $i = n$ *in (i).*

One of the main results of [1] is that a function satisfying the modular law is uniquely characterized by its values at e_μ for any composition μ of n. Moreover, they give an algorithm for the computation of such a function from its values at e_μ. We actually used this algorithm to calculate chromatic quasisymmetric functions on our computers.

Theorem 2.2 (Abreu–Nigro [1])**.** *Let* $\lambda = (\lambda_1, \ldots, \lambda_l)$ *be a partition of* n. *If a function* $\chi : \mathbb{E}_n \to \mathbb{Q}(t)$ *satisfies the modular law and*

$$\chi(\mathsf{e}_\mu) = \begin{cases} \prod_{i=1}^{l} [\lambda_i]_t! & \text{if } \lambda \text{ is a rearrangement of } \mu, \\ 0 & \text{otherwise,} \end{cases}$$

then we have $\chi(\mathsf{e}) = c_\lambda(\mathsf{e}; t)$ *for any* $\mathsf{e} \in \mathbb{E}_n$.

Here, $[m]_t!$ means the t-factorial of $m \in \mathbb{Z}$ defined by

$$[m]_t := \frac{1-t^m}{1-t}, \quad [m]_t! := \prod_{i=1}^{m} [i]_t.$$

3 (q,t)-chromatic symmetric functions

Next we review the results of [18] mainly for motivational purpose. Unfortunately, the results of this section turn out not to be logically needed in the proof of the Stanley–Stembridge conjecture. However, they also offer a quantum multiplication on the ring of symmetric functions similar to the quantum multiplication in the theory of quantum cohomology, which we hope to be of independent interest.

3.1 Geometric interpretations

Shareshian–Wachs [27] conjectured that $\omega \mathcal{X}_\Gamma(t)$ is given by the Frobenius series of the cohomology of regular semisimple Hessenberg variety of type A associated with the Hessenberg function corresponding to unit interval graph Γ. Here, ω is the involution on the ring of symmetric functions as in [23] and the action of the n-th symmetric group on the cohomology of regular semisimple Hessenberg variety of type A is given by Tymoczko [32]. This conjecture was proved by Brosnan–Chow [6] and Guay-Paquet [15] independently.

Recently, Kato [21] found another geometric realization of the chromatic quasisymmetric function for any unit interval graph Γ. Namely, Kato [20] constructed a smooth proper variety $\mathscr{X}_\Gamma$ with a $GL_n[\![z]\!]$-action and $GL_n[\![z]\!]$-equivariant proper morphism m_Γ :

$\mathscr{X}_\Gamma \to \mathrm{Gr}$ to the affine Grassmannian of GL_n. By the decomposition theorem [5], one can decompose the pushforward of the constant sheaf on $\mathscr{X}_\Gamma$

$$\mathsf{m}_{\Gamma *}\underline{\mathbb{C}}_{\mathscr{X}_\Gamma} \cong \oplus_\lambda \mathsf{IC}_\lambda[-i]^{\oplus m^\Gamma_{\lambda,i}}$$

into a shifted direct sum of $GL_n[\![z]\!]$-equivariant simple perverse sheaves IC_λ on Gr, where λ runs over the set of partitions of length at most n. By the geometric Satake correspondence [25], $H^\bullet(\mathsf{IC}_\lambda)$ has a structure of $GL_n(\mathbb{C})$-module whose character is given by the Schur function s_λ. Kato [21] proves that

$$\mathcal{X}_\Gamma(t) = t^{-d_\Gamma} \sum_{\lambda,i} t^{i/2} m^\Gamma_{\lambda,i}\, s_\lambda(X)$$

for some $d_\Gamma \in \mathbb{Z}_{\geq 0}$ and obtained a new formula for $\mathcal{X}_\Gamma(1)$ using the affine Weyl group of type A_{n-1} and its action on the Laurent polynomial ring $\mathbb{C}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ as a corollary.

3.2 Affine Hecke algebras of type A

It is a natural question to ask whether one can refine Kato's formula for $\mathcal{X}_\Gamma(1)$ to obtain a formula for $\mathcal{X}_\Gamma(t)$ by using the affine Hecke algebras of type A. In fact, one can add an additional parameter and define a (q,t)-analogue $\mathcal{X}_\Gamma(q,t)$ of the chromatic symmetric function for any unit interval graph Γ in [18]. Let us briefly explain the construction.

Let $\mathscr{H}_m$ be the affine Hecke algebra of GL_m, i.e., the $\mathbb{Q}(t)$-algebra generated by T_i for $i \in \mathbb{Z}/m\mathbb{Z}$ and $\Pi^{\pm 1}$ satisfying the following relations:

- $(T_i - t)(T_i + 1) = 0$, for any $i \in \mathbb{Z}/m\mathbb{Z}$,
- $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$ for any $i \in \mathbb{Z}/m\mathbb{Z}$ if $m > 2$,
- $T_i T_j = T_j T_i$ for any $i, j \in \mathbb{Z}/m\mathbb{Z}$ such that $j \neq i, i \pm 1$,
- $\Pi T_i = T_{i+1}\Pi$ for any $i \in \mathbb{Z}/m\mathbb{Z}$.

We consider the polynomial representation of $\mathscr{H}_m$ on $\mathbb{Q}(q,t)[X_1^{\pm 1}, \ldots, X_m^{\pm 1}]$ defined by

$$T_i(F) = t\sigma_i(F) + (t-1)\frac{F - \sigma_i(F)}{1 - X_i X_{i+1}^{-1}},$$
$$\Pi(F) = X_1 F(X_2, \ldots, X_m, q^{-1}X_1),$$

for any $F = F(X_1, \ldots, X_m) \in \mathbb{Q}(q,t)[X_1^{\pm 1}, \ldots, X_m^{\pm 1}]$ and $i = 1, \ldots, m-1$, where we set $\sigma_i(F) = F(\ldots, X_{i+1}, X_i, \ldots)$. For each $a \in \mathbb{Z}_{\geq 0}$, we define $\mathbb{S}_a^{(m)} \in \mathscr{H}_m$ by

$$\mathbb{S}_a^{(m)} := \begin{cases} (1 + T_1^{-1} + T_2^{-1}T_1^{-1} + \cdots + T_{m-1-a}^{-1} \cdots T_1^{-1})\Pi & \text{if } 0 \leq a < m, \\ 0 & \text{if } a \geq m. \end{cases}$$

For example, we have $\mathbb{S}_{m-1}^{(m)} = \Pi$. It is easy to check that the action of $\mathbb{S}_a^{(m)}$ preserves the polynomial part $\mathbb{Q}(q,t)[X_1, \ldots, X_m]$ of $\mathbb{Q}(q,t)[X_1^{\pm 1}, \ldots, X_m^{\pm 1}]$.

3.3 Main results

For a unit interval graph $\Gamma = \Gamma_{\mathsf{a}}$ labeled by $\mathsf{a} \in \mathbb{A}_n$, we define

$$\mathcal{X}_\Gamma^{(m)}(q,t) := t^{n(m-1)} \mathbb{S}^{(m)}_{\mathsf{a}(1)} \cdots \mathbb{S}^{(m)}_{\mathsf{a}(n)}(1) \in \mathbb{Q}(q,t)[X_1, \ldots, X_m].$$

The main result of [18] is the following.

Theorem 3.1 ([18]). *$\mathcal{X}_\Gamma^{(m)}(q,t)$ is symmetric in $X_1, \ldots, X_m$ and we have*

$$\mathcal{X}_\Gamma^{(m')}(q,t)|_{X_{m+1}=\cdots=X_{m'}=0} = \mathcal{X}_\Gamma^{(m)}(q,t)$$

for any $0 < m < m'$. In particular, $\mathcal{X}_\Gamma(q,t) := \left(\mathcal{X}_\Gamma^{(m)}(q,t)\right)_{m \in \mathbb{Z}_{>0}}$ defines a symmetric function. Moreover, there exists a "quantum multiplication" $\star$ on the ring of symmetric functions over $\mathbb{Q}(q,t)$ which is commutative and associative, and reduces to the usual multiplication at $q = 1$ such that

$$\mathcal{X}_{\Gamma \cup \Gamma'}(q,t) = \mathcal{X}_\Gamma(q,t) \star \mathcal{X}_{\Gamma'}(q,t). \tag{3.1}$$

For any $\mathsf{e} \in \mathbb{E}_n$, we also have

$$\mathcal{X}_{\Gamma_{\mathsf{e}}}(q,t) = \sum_{\lambda \vdash n} t^{n(\lambda')} c_\lambda(\mathsf{e};t)\, e_{\lambda_1} \star e_{\lambda_2} \star \cdots \star e_{\lambda_l} \tag{3.2}$$

where $n(\lambda') = \sum_{i=1}^{l} \frac{\lambda_i(\lambda_i - 1)}{2}$ for a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ as in [23].

We call $\mathcal{X}_\Gamma(q,t)$ the *(q,t)-chromatic symmetric function* for a unit interval graph Γ. By taking $q = 1$ in (3.2), we find that $\mathcal{X}_\Gamma(1,t)$ and $\mathcal{X}_\Gamma(t)$ are related by sending e_λ to $t^{n(\lambda')} e_\lambda$. The quantum multiplication is uniquely determined by (3.1) since the chromatic quasisymmetric functions for unit interval graphs span the ring of symmetric functions.

We note that $\mathcal{X}_\Gamma(q,t)$ does not have any standard positivity properties such as e-positivity or Schur-positivity in general. However, one can extract some information about $c_\lambda(\mathsf{e};t)$ by taking another specialization for q.

3.4 Specialization at $q = \infty$

One can consider the limit $q \to \infty$ for $\mathcal{X}_\Gamma(q,t)$. For $\Gamma = \Gamma_{\mathsf{e}}$ with $\mathsf{e} \in \mathbb{E}_n$, it is given [18] by

$$\lim_{q \to \infty} \mathcal{X}_{\Gamma_{\mathsf{e}}}(q,t) = t^{\frac{n(n-1)}{2} - |\mathsf{e}|} [n]_t!\, e_n,$$

where we set $|\mathsf{e}| = \mathsf{e}(1) + \cdots + \mathsf{e}(n)$. On the other hand, a Pieri type formula for the quantum multiplication

$$e_1 \star e_r = (1 - q^{-1})[r+1]_t e_{r+1} + q^{-1} e_1 e_r$$

implies that for any $\lambda = (\lambda_1, \ldots, \lambda_l) \vdash n$, we have

$$\lim_{q \to \infty} e_{\lambda_1} \star \cdots \star e_{\lambda_l} = \frac{[n]_t!}{\prod_{i=1}^{l} [\lambda_i]_t!} \cdot e_n.$$

By substituting them to (3.2), we obtain

$$\sum_{\lambda \vdash n} t^{|\mathbf{e}| - |\mathbf{e}_\lambda|} \frac{c_\lambda(\mathsf{e}; t)}{\prod_i [\lambda_i]_t!} = 1.$$

Actually, this formula itself can be easily proved by the modular law since the LHS satisfies the modular law with t replaced by t^{-1} and takes the value 1 at e_μ for any composition μ of n. This formula and the conjectural positivity of $c_\lambda(\mathsf{e}; t)$ suggest that

$$p_\lambda(\mathsf{e}; t) := t^{|\mathbf{e}| - |\mathbf{e}_\lambda|} \frac{c_\lambda(\mathsf{e}; t)}{\prod_i [\lambda_i]_t!}$$

for $\lambda \vdash n$ give a probability on the set of partitions of n for any $t \in \mathbb{R}_{>0}$. This is why we prefer to use $p_\lambda(\mathsf{e}; t)$ instead of $c_\lambda(\mathsf{e}; t)$. It turns out that this small change of normalization is crucial to reveal their inductive structure on n.

4 Idea of the proof of [17]

We now explain how to prove the Stanley–Stembridge conjecture. This is done by proving an inductive and positive formula for $p_\lambda(\mathsf{e}; t)$. The most nontrivial point is to find such formula, since it is rather straightforward to check the modular law whose details are explained in [17]. Hence we mainly focus on how to find the formula here.

4.1 First attempt

We consider the formal sum

$$\Phi_\mathsf{e} := \sum_{\lambda \vdash n} p_\lambda(\mathsf{e}; t) \cdot \lambda$$

in the $\mathbb{Q}(t)$-vector space $\mathsf{V}_n^{\mathrm{naive}}$ spanned by the partitions of n. As a first attempt, we try to find a linear map $\Omega_r^{\mathrm{naive}} : \mathsf{V}_n^{\mathrm{naive}} \to \mathsf{V}_{n+1}^{\mathrm{naive}}$ for $0 \le r \le n$ such that $\Omega_r^{\mathrm{naive}}(\Phi_\mathsf{e}) = \Phi_{\mathsf{e}'}$ for any $\mathsf{e} \in \mathbb{E}_n$, where $\mathsf{e}' \in \mathbb{E}_{n+1}$ is obtained by adding r to the end of e.

By experiments, we find that for example $\Omega_0^{\mathrm{naive}}(\lambda)$ should be obtained by adding a new box to the bottom row of λ, where we use French convention for drawing the Young diagram of λ. We also guess

$$\Omega_r^{\mathrm{naive}}((n)) = \frac{[r]_t}{[n]_t} \cdot (n, 1) + \frac{t^r [n-r]_t}{[n]_t} \cdot (n+1). \tag{4.1}$$

We observe that such a combination of coefficients appears many times in other examples. However, this naive approach does not work literally even at $n = 3$, where we should have $\Omega_2^{\mathrm{naive}}((2,1)) = (2,2)$ in order to obtain the correct $\Phi_{(0,0,1,2)}$, but we should have $\Omega_2^{\mathrm{naive}}((2,1)) = \frac{1}{[2]_t} \cdot (2,1,1) + \frac{t}{[2]_t} \cdot (3,1)$ in order to obtain the correct $\Phi_{(0,1,1,2)}$.

4.2 Refinement

The difference between $[2,1]$ in $\Phi_{(0,0,1)}$ and in $\Phi_{(0,1,1)}$ above is the order in which the boxes are added from the empty Young tableau. In order to remember such an order, we use standard Young tableaux instead of Young diagrams.

We consider the $\mathbb{Q}(t)$-vector space V_n spanned by the standard Young tableaux of size n and try to construct $\Psi_{\mathsf{e}} \in \mathsf{V}_n$ for any $\mathsf{e} \in \mathbb{E}_n$ such that $\pi(\Psi_{\mathsf{e}}) = \Phi_{\mathsf{e}}$, where $\pi : \mathsf{V}_n \to \mathsf{V}_n^{\mathrm{naive}}$ is the $\mathbb{Q}(t)$-linear map determined by $\pi(T) = \lambda$ if T is a standard Young tableau of shape λ. Moreover, we impose further restrictions that there exist $\mathbb{Q}(t)$-linear map $\Omega_r : \mathsf{V}_n \to \mathsf{V}_{n+1}$ for $0 \leq r \leq n$ such that we have

(i) $\Psi_{\mathsf{e}} = \Omega_{\mathsf{e}(n)} \Omega_{\mathsf{e}(n-1)} \cdots \Omega_{\mathsf{e}(1)}(\varnothing)$ for any $\mathsf{e} \in \mathbb{E}_n$, and

(ii) $\Omega_r(T)$ is a linear combination of standard Young tableaux T' obtained by adding new box labeled by $n+1$ to T on the top of some column.

Surprisingly, these two conditions consistently determine $\Omega_r(T)$ for many standard Young tableaux. Moreover, in most cases for small n, the linear combination appearing in (ii) is very simple and similar to (4.1). The first exception appears at $n = 6$, which takes the form

$$\Omega_4\left(\begin{array}{|c|c|cc}\hline 4 & 6 & & \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} \\ \hline\end{array}\right) = \frac{[3]_t}{[2]_t[4]_t}\begin{array}{|c|c|cc}\cline{1-1} 7 & \multicolumn{1}{c}{} & & \\ \hline 4 & 6 & & \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} \\ \hline\end{array}$$
$$+ \frac{t}{[2]_t^2}\begin{array}{|c|c|c|c}\cline{1-3} 4 & 6 & 7 & \\ \hline 1 & 2 & 3 & \multicolumn{1}{c|}{5} \\ \hline\end{array} + \frac{t^2[3]_t}{[2]_t[4]_t}\begin{array}{|c|c|ccc}\cline{1-2} 4 & 6 & & & \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{7} \\ \hline\end{array}$$

for example, where we paint the box by red if its label is greater than $r = 4$. We note that similar formulas also hold for the following standard Young tableaux

$$\begin{array}{|c|c|cc}\hline 4 & 5 & & \\ \hline 1 & 2 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{6} \\ \hline\end{array}\,, \quad \begin{array}{|c|c|cc}\hline 3 & 6 & & \\ \hline 1 & 2 & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{5} \\ \hline\end{array}\,, \quad \begin{array}{|c|c|cc}\hline 3 & 5 & & \\ \hline 1 & 2 & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{6} \\ \hline\end{array}\,.$$

The remaining task is to find out the rule for the operator Ω_r in general.

4.3 Formula

By inspection, we see that the places of boxes labeled by 5 or 6 (red boxes) in the standard Young tableaux above are the same. It turns out that this is a more general phenomenon.

Let us define a 0-1-sequence $\delta^{(r)}(T) = (\delta_i)_{i \in \mathbb{Z}}$ by setting $\delta_i = 1$ if $i \le 0$ or the top box of the i-th column of T has label larger than r, and $\delta_i = 0$ otherwise. Then the formula for $\Omega_r(T)$ depends only on the color pattern $\delta^{(r)}(T)$. This observation simplifies our codes for the calculation a lot and enables us to examine further examples, which lead to our general formula for $\Omega_r(T)$.

We set $W^{(r)}(T) := \{i \mid \delta_i = 0 \text{ and } \delta_{i-1} = 1\}$ and $R^{(r)}(T) := \{i \mid \delta_i = 1 \text{ and } \delta_{i-1} = 0\}$. For $c \in W^{(r)}(T)$, we denote by $f_c(T)$ the standard Young tableau obtained by adding a new box labeled by $n+1$ on the top of the c-th column of T. Finally, we define[2] $\Omega_r : \mathsf{V}_n \to \mathsf{V}_{n+1}$ by

$$\Omega_r(T) := \sum_{c \in W^{(r)}(T)} \frac{\prod_{i \in R^{(r)}(T)} [i-c]_t}{\prod_{i \in W^{(r)}(T) \setminus \{c\}} [i-c]_t} f_c(T) \tag{4.2}$$

for any standard Young tableau T of size n. Now we can state our main result in [17].

Theorem 4.1 ([17])**.** *For any* $\mathsf{e} \in \mathbb{E}_n$*, we have*

$$\Phi_{\mathsf{e}} = \pi\, \Omega_{\mathsf{e}(n)} \Omega_{\mathsf{e}(n-1)} \cdots \Omega_{\mathsf{e}(1)}(\emptyset).$$

Since it is easy to check that all the coefficients of (4.2) are nonnegative, this in particular proves the Stanley–Stembridge conjecture.

4.4 Sketch of proof

The proof of our main theorem is a simple application of the modular law [1]. We only need to check the modular law in Definition 2.1 and also check the formula for $\mathsf{e} = \mathsf{e}_\mu$ for any composition μ of n. The case $\mathsf{e} = \mathsf{e}_\mu$ is easy since we have $R^{(r)}(T) = \emptyset$ and $|W^{(r)}(T)| = 1$ in all the steps for the calculation of the RHS.

It remains to check the modular law. Let us denote by $\tau_r(T)$ the standard Young tableau obtained by swapping r and $r+1$ in T if it is well-defined. For the first case in Definition 2.1, the modular law would follow if one has

$$\left([2]_t \Omega_r - \Omega_{r+1} - t\Omega_{r-1}\right)(R) \in \mathsf{K}_{r,m+1} \tag{4.3}$$

for any $R \in \mathsf{V}_m$, where $\mathsf{K}_{r,m+1} \subset \mathsf{V}_{m+1}$ is the subspace spanned by $T - \tau_r(T)$ for standard Young tableau T such that $\tau_r(T)$ is well-defined. Although this is not true in general, the

[2] The author thanks Mathieu Guay-Paquet for informing us of this simplified expression.

condition $\mathsf{e}(\mathsf{e}(i)) = \mathsf{e}(\mathsf{e}(i)+1)$ implies that $\Omega_{\mathsf{e}(i-1)} \cdots \Omega_{\mathsf{e}(1)}(\emptyset)$ can be written as a sum of certain linear combination of two standard Young tableau T and $\tau_{\mathsf{e}(i)}(T)$ for which (4.3) for $r = \mathsf{e}(i)$ holds. We need to be more careful whether (4.3) propagates to the n-th step or not, but in this case it is automatic.

For the second case in Definition 2.1, one can actually show that

$$\left([2]_t \Omega_{r+1}\Omega_r - \Omega_{r+1}\Omega_{r+1} - t\Omega_r\Omega_r\right)(R) \in \mathsf{K}_{m+1,m+2} \tag{4.4}$$

for any $R \in \mathsf{V}_m$. The condition $\mathsf{e}^{-1}(i) = \emptyset$ then guarantees that (4.4) propagates to the n-th step. This completes the sketch of proof and see [17] for more details of the calculations for (4.3) and (4.4).

4.5 Future directions

Finally, we briefly comment on possible future directions.

One obvious remaining problem is whether it is possible to prove Shareshian–Wachs conjecture using our formula or not. This is not straightforward since our modified formula for $c_\lambda(\mathsf{e};t)$ is still a priori a sum of rational functions on t. For example, the modified coefficient of $\Psi_{(0,0,0,1,1,2)}$ at

$$\begin{array}{|c|c|cc}\hline 4 & 6 & & \\ \hline 1 & 2 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{5} \\ \hline \end{array}\,,\quad \begin{array}{|c|c|cc}\hline 5 & 6 & & \\ \hline 1 & 2 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{4} \\ \hline \end{array}\,,\quad \begin{array}{|c|cccc}\hline 4 & & & & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{6} \\ \hline \end{array}\,,\quad \begin{array}{|c|cccc}\hline 5 & & & & \\ \hline 1 & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{6} \\ \hline \end{array}$$

are genuinely rational, but their sums become polynomials. If it is possible to understand why such cancelations of denominators occur, then one might be able to prove Shareshian–Wachs conjecture as well.

In [18, 17], we restrict our attention to the case of unit interval graphs since it is enough to prove the Stanley–Stembridge conjecture. It would be very interesting if one can generalize (q,t)-chromatic symmetric functions or Theorem 4.1 for more general graphs such as circular unit arc digraphs.

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