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Interpreting the chromatic polynomial coefficients via hyperplane arrangements

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Abstract. A recent result of Lofano and Paolini expresses the characteristic polynomial of a real hyperplane arrangement in terms of a projection statistic on the regions of the arrangement. We use this result to give an alternative proof for Greene and Zaslavsky's interpretation for the coefficients of the chromatic polynomial of a graph. We also show that this projection statistic has a nice combinatorial interpretation in the case of the braid arrangement, which generalizes to graphical arrangements of natural unit interval graphs. We use this generalization to give a new proof of the formula for the chromatic polynomial of a natural unit interval graph.

1 Introduction

In [3], Greene and Zaslavsky gave an interpretation for the coefficients of the chromatic polynomial of a graph G in terms of source components of acyclic orientations of G (formally defined in Section 2). Recently, in [5], Lofano and Paolini gave an expression for the characteristic polynomial of a hyperplane arrangement $\mathcal{A}$ as a generating function of the regions of $\mathcal{A}$, counted according to a projection statistic with respect to a fixed point $v \in \mathbb{R}^n$. The goal of this paper is to see how these two results are related in the case of graphical arrangements.

Definition 1.1. Let $G = ([n], E)$ be a graph. The *graphical arrangement* $\mathcal{A}_G$ is defined as the collection of the following hyperplanes:

$$\mathcal{A}_G := \{H_{i,j} \mid \{i, j\} \in E, i < j\},$$

where $H_{i,j} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i - x_j = 0\}$.

It is known that for any graph G,

$$\chi_G(t) = \chi_{\mathcal{A}_G}(t), \tag{1.1}$$

that is, the chromatic polynomial of a graph is equal to the characteristic polynomial of the corresponding graphical arrangement. Further, we can label each region of the graphical arrangement with a unique acyclic orientation of G.

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This raises the question of whether there exists a point $v \in \mathbb{R}^n$ such that for each region of $\mathcal{A}_G$, the projection statistic given in [5] equals the number of source components of the acyclic orientation of G labeling the region as given in [3]. In Section 4, we find a set of points in $\mathbb{R}^n$ such that this holds (see Theorem 4.4). This shows that the Greene and Zaslavsky result can be obtained using the Lofano and Paolini statistic.

We get a particularly nice combinatorial interpretation of this projection statistic in the case of the braid arrangement $\mathcal{B}_n = \mathcal{A}_{K_n}$ (see Theorem 3.2), which generalizes to natural unit interval graphs (see Theorem 5.2). We use this interpretation to give an alternative proof of the form of the chromatic polynomial of a natural unit interval graph (see Theorem 5.3).

Remark 1.2. Zaslavsky's theorem states that the number of regions of a hyperplane arrangement $\mathcal{A}$ is $(-1)^n \chi_{\mathcal{A}}(-1)$. Hence, we can expect to find an expression for $\chi_{\mathcal{A}}$ as a statistic on the regions of $\mathcal{A}$. No such statistic was known until the recent results of Lofano and Paolini [5] and Kabluchko [4]. Our work shows that this projection statistic can be viewed as a natural combinatorial statistic in the case of graphical arrangements.

2 Background

We assume a standard background on graphs and hyperplane arrangements as given in [7]. We use *graph* to mean an undirected finite graph without loops or multiple edges. For the rest of this section, let $G = ([n], E)$ be a graph.

Definition 2.1. The *chromatic polynomial* of G, denoted by χ_G, is the polynomial which when evaluated at a non-negative integer q gives the number of proper q-colorings of G.

We now define *source components* as in [1].

Definition 2.2. Let γ be an acyclic orientation of G. For $i \in [n]$, let R_i be the set of vertices reachable from i by a directed path of γ (with $i \in R_i$). We define $S_1, S_2, \ldots$ recursively: for $k \geq 1$, if $\bigcup_{i<k} S_i = [n]$, then $S_k = \emptyset$. Otherwise, define $S_k = R_m \setminus \bigcup_{i<k} S_i$ where $m = \min\{[n] \setminus \bigcup_{i<k} S_i\}$. The non-empty subsets S_k thus defined are the *source components* of γ.

The following result was proved by Greene and Zaslavsky.

Theorem 2.3. *[3] Let $G = ([n], E)$ be a graph, and k be a non-negative integer. Then, the coefficient of $(-1)^{n-k} q^k$ in $\chi_G(q)$ is the number of acyclic orientations of G with exactly k source components.*

The characteristic polynomial of a hyperplane arrangement $\mathcal{A}$, denoted by $\chi_{\mathcal{A}}$, is classically defined using the Mobius function of the intersection lattice. For our part, we use a characterization of the characteristic polynomial given by Lofano and Paolini [5] and Kabluchko [4].

Definition 2.4. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$ and let $v \in \mathbb{R}^n$ be an arbitrary point. Let R be a region of $\mathcal{A}$. We define the *projection from v to R* to be the unique point in R that has the minimum Euclidean distance from v, and denote it by $\mathrm{proj}_v(R)$. Further, we define the *projection dimension of v on R* to be the dimension of the unique face of R that contains $\mathrm{proj}_v(R)$ in its interior, and denote it by $\mathrm{pd}_v(R)$.

Theorem 2.5. *[4, 5] Let $\mathcal{A}$ be a real hyperplane arrangement in $\mathbb{R}^n$, and let $v \in \mathbb{R}^n$ be a generic point. Let $R(\mathcal{A})$ be the set of regions of $\mathcal{A}$. Then, the characteristic polynomial of $\mathcal{A}$ is given by*

$$\chi_{\mathcal{A}}(t) = \sum_{R \in R(\mathcal{A})} (-1)^{n-\mathrm{pd}_v(R)} t^{\mathrm{pd}_v(R)}.$$

Equivalently, for a non-negative integer k, the coefficient of $(-1)^{n-k}t^k$ in $\chi_{\mathcal{A}}(t)$ is the number of regions R of $\mathcal{A}$ such that the projection dimension of v on R is k.

3 The Braid Arrangement

In this section, we show that for the braid arrangement $\mathcal{B}_n$, the projection statistic defined in Theorem 2.5 can be expressed as a combinatorial statistic on the permutations labeling the regions of $\mathcal{B}_n$.

It is easy to see that each region of the braid arrangement is uniquely determined by a total ordering of the coordinates. As a consequence, there is a bijection between the regions of the braid arrangement $\mathcal{B}_n$ and permutations of $[n]$. When considered in one line notation, the permutation indicates the relative order of the coordinates. For example, the permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ is associated to the region consisting of all points with coordinates $x_{\sigma_1} > x_{\sigma_2} > \ldots > x_{\sigma_n}$.

Moreover, the flats of the arrangement can be represented by partitions of $[n]$, the blocks indicating which coordinates are equal. Finally, the faces of the arrangement can be represented by ordered partitions of $[n]$, the relative order of the blocks indicating the relative order of the corresponding coordinates. Note that the number of blocks of the partition is the dimension of the corresponding face. Given a face F of $\mathcal{A}$, we denote by $\Pi(F)$ the ordered partition labeling it, and given an ordered partition Π of $[n]$, we denote by F_Π the face it labels.

Definition 3.1. Let $v \in \mathbb{R}^n$ with coordinates $(v_1, \ldots, v_n)$ such that $v_1 > \ldots > v_n$ and $v_i - v_{i+1} > n(v_{i+1} - v_n)$ for all $i < n$. For instance, $v_i = (n+1)^{-i}$ for all $i \in [n]$ defines such a point.

We now state the main result of this section.

Theorem 3.2. *Let $v \in \mathbb{R}^n$ be as in Definition 3.1, let $\sigma \in \mathfrak{S}_n$ be a permutation and let R_σ be the region of the braid arrangement $\mathcal{B}_n$ labeled by σ. Then the projection dimension of v on R_σ equals the number of right-to-left minima of σ, that is,* $\mathrm{pd}_v(R_\sigma) = \mathrm{RLmin}(\sigma)$.

The rest of this section is dedicated to the proof of Theorem 3.2.

Let $v \in \mathbb{R}^n$ be as in Definition 3.1, and R be a region of $\mathcal{B}_n$. There is a unique face of R that contains $\text{proj}_v(R)$ in its interior. In order to prove the above theorem, we first prove some lemmas which allow us to characterize this face. We state the following lemma without proof.

Lemma 3.3. *Let $v \in \mathbb{R}^n$ and let $\Pi = \{B_1, \ldots, B_k\}$ be an unordered set partition of $[n]$. If $i \in B_j = \{j_1, \ldots, j_s\}$, then the projection of v on the flat labeled by Π has the i^{th} coordinate given by $\frac{v_{j_1}+\ldots+v_{j_s}}{s}$.*

Definition 3.4. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$, and let $v \in \mathbb{R}^n$ be a point. Let F be a face of $\mathcal{A}$. We say F is a *good face* if the projection of v on $\text{span}(F)$ lies in the interior of F.

The following lemma characterizes the good faces of $\mathcal{B}_n$.

Lemma 3.5. *Let $v \in \mathbb{R}^n$ be as in Definition 3.1, and let $\Pi = (B_1, \ldots, B_k)$ be an ordered partition of $[n]$. Then F_Π is a good face of $\mathcal{B}_n$ if and only if $\min(B_i) < \min(B_{i+1})$ for all $i < k$.*

Proof. Let $B = \{i_1, \ldots, i_k\}$ and $B' = \{i'_1, \ldots, i'_\ell\}$ be two blocks of our partition, and write their values in increasing order. Then $\min(B) = i_1$ and $\min(B') = i'_1$. Without loss of generality, let $i_1 < i'_1$. Let $X = \text{span}(F_\Pi)$, that is, X is the flat labeled by the unordered partition $\{B_1, \ldots, B_k\}$. Let $\text{proj}_v(X) = (p_1, \ldots, p_n)$.

Then, from Lemma 3.3 and the fact that v satisfies Definition 3.1, we have:

$$p_{i_1} - p_{i'_1} \geq \frac{(k-1)v_n + v_{i'_1-1}}{k} - v_{i'_1} \geq \frac{v_{i'_1-1} - v_{i'_1} - n(v_{i'_1} - v_n)}{k} > 0$$

Hence B must appear before B' in the ordered partition. $\square$

Lemma 3.6. *Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$, let $v \in \mathbb{R}^n$ be a point and let R be a region of $\mathcal{A}$. Then,*

$$\mathit{proj}_v(R) = \mathit{proj}_v(F)$$

where F is a good face of $\mathcal{A}$ incident to R.

Proof. Clearly, there is a unique face F of R that contains $\text{proj}_v(R)$ in its interior. Hence, $\text{proj}_v(R) = \text{proj}_v(F)$, and we need to show that F is a good face.

Let $X = \text{span}(F)$. Suppose F is not a good face. Then, $\text{proj}_v(X) \neq \text{proj}_v(F)$. Let L be the line joining $\text{proj}_v(X)$ and $\text{proj}_v(F)$. As $\text{proj}_v(X)$ is the orthogonal projection of v onto X and L lies in X, the distance between v and a point $p \in L$ increases as p moves from $\text{proj}_v(X)$ to $\text{proj}_v(F)$. But then $\text{proj}_v(F)$ must be on the boundary of F as otherwise we would have points on L in F closer to v, contradicting the fact that $\text{proj}_v(F)$ minimizes distance. This contradicts the fact that $\text{proj}_v(F)$ is in the interior of F, so F must be a good face. $\square$

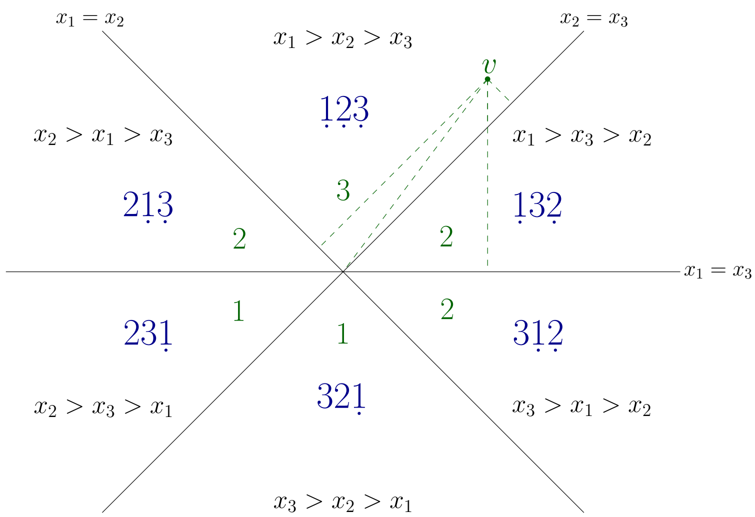

Figure 1: $\mathcal{B}_3$ with projection from a point v as in Definition 3.1 onto the regions, with the regions labeled by permutations (in blue) with right-to-left minima marked by dots, and dimension of projection/number of right-to-left minima (in green).

Now, we finally have enough information to prove Theorem 3.2.

Proof of Theorem 3.2. Let $\sigma = \sigma_1 \ldots \sigma_n$, and let F be a face of R_σ. Then, the ordered partition labeling F is of the form $(\{\sigma_1, \ldots, \sigma_{i_1}\}, \{\sigma_{i_1+1}, \ldots, \sigma_{i_2}\}, \ldots, \{\sigma_{i_k+1}, \sigma_n\})$ where $k \geq 0$, and $0 < i_1 < \ldots < i_k < n$.

Now, by Lemma 3.6, $\text{proj}_v(R_\sigma)$ must lie on a good face of R_σ. Lemma 3.5 further gives us that these good faces are precisely those labeled by ordered partitions of $[n]$ of the form $\Pi = (B_1, \ldots, B_k)$ such that $\min(B_i) < \min(B_{i+1})$ for all $i < k$.

This implies that if F_Π is a good face of R_σ, then each block B_i contains at least one right-to-left minimum. If not, suppose j is the greatest index such that B_j does not contain a right-to-left minimum. Then, $j < k$ and $\min(B_j) > \min(B_{j+1})$, contradicting that F_Π is a good face. Hence, the dimension of a good face cannot be greater than $\text{RLmin}(\sigma)$.

Now, suppose that F_σ is the face of R_σ that v projects into. If the dimension of F_σ is less than $\text{RLmin}(\sigma)$, there is a block of $\Pi(F_\sigma)$ that has more than one right-to-left minimum. We can refine the partition (by breaking this block into two blocks, each containing at least one right-to-left minimum) to get a partition corresponding to a larger face F' which contains F_σ. This gives us the projection from v onto F' must be of shorter length (strictly shorter as F' is a good face) than the projection onto F_σ, which is a contradiction.

Hence we have $\text{pd}_v(R_\sigma) = \text{RLmin}(\sigma)$. $\square$

4 Graphical Arrangements

Let $G = ([n], E)$ be a graph and $\mathcal{A}_G$ be the corresponding graphical arrangement.

Definition 4.1. Let $R \in R(\mathcal{A}_G)$. We associate to R an acyclic orientation of G, denoted by γ_R, by directing the edge $\{i, j\} \in E$ towards i if and only if $x_i \geq x_j$ in R.

Lemma 4.2. *[6] The mapping γ which associates to each region $R \in R(\mathcal{A}_G)$ the orientation γ_R is a bijection between $R(\mathcal{A}_G)$ and the set of acyclic orientations of G.*

Recall that the faces of the braid arrangement can be identified with ordered partitions of $[n]$. As $\mathcal{A}_G$ is a sub-arrangement of $\mathcal{B}_n$, each flat of $\mathcal{A}_G$ corresponds to a partition of $[n]$ and each face of $\mathcal{A}_G$ corresponds to a set of ordered partitions of $[n]$. For a face F, we consider the representative ordered set partition where the blocks that do not have a relative order are ordered by increasing minimum elements, and denote it by $\Pi(F)$.

Note that for an acyclic orientation γ of G, the source components $(S_1, \ldots, S_k)$ of γ form an ordered partiton of $[n]$. We denote this ordered partition by $\Pi(\gamma)$.

Definition 4.3. Let $v \in \mathbb{R}^n$ with coordinates $(v_1, \ldots, v_n)$ such that $v_i > (6n^2 + 1)v_{i+1}$ and $v_n > 0$. For instance, $v_i = (6n^2 + 2)^{-i}$ for all $i \in [n]$ defines such a point.

Note that such a point $v \in \mathbb{R}^n$ will also satisfy the condition $v_i - v_{i+1} > n(v_{i+1} - v_n)$ for all $i < n$. Hence, any results that hold for a point v as in Definition 3.1 will also hold for a point v as in Definition 4.3.

The main result of this section is the following:

Theorem 4.4. *Let $G = ([n], E)$ be a graph and let v be as in Definition 4.3. Let R be a region of the graphical arrangement $\mathcal{A}_G$, and let $\Pi(\gamma_R) = (B_1, \ldots, B_k)$.*

Then, $\mathrm{pd}_v(R) = k$, that is, the projection dimension of v on R equals the number of source components of γ_R. In fact, the face of R that $\mathrm{proj}_v(R)$ lies in the interior of is $F_{\Pi(\gamma_R)}$.

In the case of the braid arrangement, the above theorem along with the following lemma which we state without proof give us Theorem 3.2.

Lemma 4.5. *The number of right-to-left minima of a permutation $\sigma \in \mathfrak{S}_n$ is equal to the number of source components of γ_{R_σ}, where R_σ is the region of $\mathcal{B}_n$ labeled by σ.*

Further, Theorem 4.4 along with Theorem 2.5 and Equation (1.1) give us an alternative proof of Greene and Zaslavsky's result about the interpretation of the coefficients of the characteristic polynomial (Theorem 2.3).

The rest of this section is devoted to the proof of Theorem 4.4.

Lemma 4.6. *Let $G = ([n], E)$ be a graph, let R be a region of $\mathcal{A}_G$, and let γ_R be the acyclic orientation of G labeling R. Let $\Pi(\gamma_R) = (B_1, \ldots, B_k)$. Then $F_{\Pi(\gamma_R)}$ is a good face of R and R does not have a good face of dimension greater than k.*

Proof. Let Π be an ordered partition of $[n]$. Then $\text{span}(F_\Pi)$ is a flat of $\mathcal{A}_G$ if and only if for every block B of Π, the induced subgraph $G[B]$ is connected. This is because two coordinates x_i and x_j can be equated if and only if we have a set of hyperplanes $x_i = x_{k_1}$, $x_{k_1} = x_{k_2}, \dots, x_{k_t} = x_j$, which corresponds to a path between i and j in G.

Further, F_Π is a face of a region R if and only if any weak inequality in F_Π holds in R. Hence, it is clear that $F_{\Pi(\gamma_R)}$ is a face of R. Also, for $b_i = \min B_i$, we have $b_1 < b_2 < \dots < b_k$, and hence $F_{\Pi(\gamma_R)}$ is a good face of R by Lemma 3.5.

Now, let F' be a good face of R, and suppose $\Pi(F') = (D_1, \dots, D_\ell)$.

Claim: For all $j \in \{0\} \cup [k]$, $\bigcup\limits_{i=1}^{j} B_i \subseteq \bigcup\limits_{i=1}^{j} D_i$.

We prove this using induction on $j \in [k]$. The base case for $j = 0$ is trivial.

Now, suppose we have $\bigcup\limits_{i=1}^{j-1} B_i \subseteq \bigcup\limits_{i=1}^{j-1} D_i$. Then two cases arise:

Case 1: $b_j \in \bigcup\limits_{i=1}^{j-1} D_i$.

Let $m \in [n]$ be b_j-reachable. Then, $x_m > x_{b_j}$ for all $x \in R$. Now, if $m \in D_t$ for some $t \geq j$, we get $x_{b_j} > x_m$ for all $x \in R$, which is a contradiction.

Hence, $m \in \bigcup\limits_{i=1}^{j-1} D_i$, which gives us $B_j \subseteq \bigcup\limits_{i=1}^{j-1} D_i \subseteq \bigcup\limits_{i=1}^{j} D_i$.

Case 2: $b_j \notin \bigcup\limits_{i=1}^{j-1} D_i$.

Then $b_j \in D_j$ as if not, $b_j \in D_t$ for some $t > j$, which will give us $\min D_j > \min D_t$, contradicting the fact that F' is a good face by Lemma 3.5. Further, as $b_j \in D_j$, we have $B_j \subseteq \bigcup\limits_{i=1}^{j} D_i$ using the same argument as above.

Hence our claim is true by induction on j.

As $\bigcup\limits_{i=1}^{k} B_i = [n]$, we have $[n] \subseteq \bigcup\limits_{i=1}^{k} D_i$. So, $\ell \leq k$.

Thus R does not have a good face of dimension greater than k. $\square$

Lemma 4.7. *Let v be as in Definition 4.3. Let R be a region of $\mathcal{A}_G$, and let $\Pi(\gamma_R) = (B_1, \dots, B_k)$. Let p_B be the projection of v onto $F_{\Pi(\gamma_R)}$. Let $\Pi' = (D_1, \dots, D_\ell) \neq (B_1, \dots, B_k)$, be such that $F_{\Pi'}$ is a good face of R with $\ell \leq k$, and p_D be the projection of v onto $F_{\Pi'}$.*

Then, $||p - p_B|| < ||p - p_D||$.

Proof. From Lemma 3.3, $p_B = (p_1, \dots, p_n)$, where $\forall i \in B_j$, $p_i = \sum\limits_{m \in B_j} \dfrac{v_m}{|B_j|}$ and $p_D = (q_1, \dots, q_n)$, where $\forall i \in D_j$, $q_i = \sum\limits_{m \in D_j} \dfrac{v_m}{|D_j|}$.

Now, from the proof Lemma 4.6, $B_1 = D_1$ or $B_1 \subsetneq D_1$, and if $B_i = D_i$ for all $i < j$, $B_j = D_j$ or $B_j \subsetneq D_j$. As $(D_1, \dots, D_\ell) \neq (B_1, \dots, B_k)$ we have $D_j \supsetneq B_j$ for some j.

Let j be the first index where $D_j \supsetneq B_j$, let $|D_j| = d$, $|B_j| = b$, and let $\min(B_j) = b_j$. Then, as v satisfies Definition 4.3, $B_i = D_i$ for all $i < j$, and $d > b$, we get

$$||v - p_D||^2 - ||v - p_B||^2 \geq \left(\frac{1}{b} - \frac{1}{d} - \frac{1}{n^2}\right) v_{b_j}^2 > 0.$$

□

Theorem 4.4 is now a direct consequence of Lemmas 4.6 and 4.7.

5 Natural Unit Interval Graphs

In this section, we consider a special type of graphs known as natural unit interval graphs.

Definition 5.1. A graph $G = ([n], E)$ is a *natural unit interval graph* if for all $\{i, j\} \in E$, with $i < j$, we have for all $i < k < j$, $\{i, k\} \in E$ and $\{k, j\} \in E$.

We now consider the graphical arrangement $\mathcal{A}_G$ of a natural unit interval graph G. We know that every region of the braid arrangement is indexed by a permutation. As a region of a graphical arrangement is a union of adjacent regions of the braid arrangement, each region R of our graphical arrangement $\mathcal{A}_G$ is uniquely associated to a set of permutations. Let us denote this set by S_R.

We now state the main results of this section:

Theorem 5.2. *Let $G = ([n], E)$ be a natural unit interval graph, and let R be a region of $\mathcal{A}_G$. Let $\sigma \in \mathfrak{S}_n$ be the lexicographic minimum of S_R. Let v be as in Definition 4.3. Then,*

$$\mathrm{pd}_v(R) = \mathrm{RLmin}(\sigma).$$

Theorem 5.3. *Let $G = ([n], E)$ be a natural unit interval graph and $c_j = |\{i < j \mid \{i, j\} \in E\}|$ for all $j \in [n]$. Then,*

$$\chi_{\mathcal{A}_G}(q) = \prod_{j=1}^{n} (q - c_j).$$

The rest of this section is dedicated to the proof of Theorems 5.2 and 5.3. Note that Theorem 5.3 is known for the chromatic polynomial of a natural unit interval graph (see, for instance, [2]).

Definition 5.4. Let $G = ([n], E)$ be a natural unit interval graph. A permutation $\sigma = \sigma_1 \dots \sigma_n \in \mathfrak{S}_n$ is said to be a *G-local minimum* if for all $i \in [n-1]$ such that $\sigma_i > \sigma_{i+1}$, we have $\{\sigma_i, \sigma_{i+1}\} \in E$.

We now characterize G-local minima. To do this, we first define G-descents.

Definition 5.5. Let $G = ([n], E)$ be a natural unit interval graph and $\sigma = \sigma_1 \ldots \sigma_n \in \mathfrak{S}_n$. For $i \in [n-1]$ we say we have a *G-descent* at i (or that $\sigma_i\sigma_{i+i}$ is a *G-descent*) if $\sigma_i > \sigma_{i+1}$ and $\{\sigma_i, \sigma_{i+1}\} \in E$.

From the definitions of G-local minima and G-descents, it is clear that for a natural unit interval graph G, a permutation $\sigma \in \mathfrak{S}_n$ is a G-local minimum if and only if every descent of σ is a G-descent. We further have:

Lemma 5.6. *Let $G = ([n], E)$ be a natural unit interval graph. Let R be a region of $\mathcal{A}_G$ and let S_R be the set of permutations associated to R. Then the unique G-local minimum permutation in S_R is the lexicographic minimum of S_R.*

Proof. Let $\sigma = \sigma_1 \ldots \sigma_n$ be a G-local minimum in S_R and $\tau = \tau_1 \ldots \tau_n$ be the lexicographic minimum of S_R. It is clear that τ is a G-local minimum as if there was a descent $\tau_i > \tau_{i+1}$ of τ such that $\{\tau_i, \tau_{i+1}\} \notin E$, we could swap τ_i and τ_{i+1} and get a permutation in S_R that is less than τ in the lexicographic order.

We assume for contradiction that $\sigma \neq \tau$. Suppose they first differ at the i^{th} position, that is, $\sigma_1 \ldots \sigma_{i-1} = \tau_1 \ldots \tau_{i-1}$ and $\sigma_i \neq \tau_i$. As τ is the lexicographic minimum, $\tau_i < \sigma_i$. Further, as they are permutations, $\tau_i = \sigma_k$ for some $k > i$.

Now, as both σ and τ correspond to the same region of $\mathcal{A}_G$, we can obtain one from the other by a sequence of swapping adjacent elements in the one line notation, where the pairs of swapped elements are of the form $\{a, b\}$ with $\{a, b\} \notin E$.

We have $\sigma = \tau_1 \ldots \tau_{i-1}\sigma_i \ldots \sigma_{k-1}\tau_i\sigma_{k+1} \ldots \sigma_n$. To obtain τ from σ, we would have to swap τ_i with σ_t for all $i \leq t \leq k-1$. Hence, for all $i \leq t \leq k-1$, $x_{\tau_i} = x_{\sigma_t}$ is not a hyperplane of $\mathcal{A}_G$, that is, $\{\tau_i, \sigma_t\} \notin E$.

Claim: For $i \leq t \leq k-1$, we have $\sigma_t < \tau_i$.

We prove this claim inductively.

Suppose $\sigma_{k-1} > \tau_i$. Then, $\sigma_{k-1}\tau_i$ is a descent which is not a G-descent, contradicting the fact that σ is a G-local minimum.

Now, suppose for $r < t \leq k-1$, we have $\sigma_t < \tau_i$.

Then, if $\sigma_r > \tau_i$, we have $\sigma_r > \sigma_{r+1}$. As σ is a G-local minimum, we have $\{\sigma_r, \sigma_{r+1}\} \in E$. Further, as G is a natural unit interval graph, we get that $\{\tau_i, \sigma_r\} \in E$, which is a contradiction. Hence, $\sigma_r < \tau_i$.

Hence our claim is proved.

But then we have $\sigma_i < \tau_i$ which is a contradiction. Hence, $\sigma = \tau$, that is, a G-local minimum in S_R is the lexicographic minimum of S_R and hence is unique. □

As the lexicographic minimum is unique for each region R of $\mathcal{A}_G$ where G is a natural unit interval graph, we can choose the representative of the region R to be the lexicographic minimum of S_R. In particular, these lexicographic minima will be precisely the permutations where all descents are G-descents.

Lemma 5.7. *Let v be as in Definition 4.3, and let $G = ([n], E)$ be a natural unit interval graph. Let R be a region of $\mathcal{A}_G$. Then $proj_v(R) \in R_\sigma$ where σ is the lexicographic minimum of S_R, and R_σ is the region of the braid arrangement $\mathcal{B}_n$ labeled by σ.*

Proof. Let $\sigma' \in S_R$ be such that $\mathrm{proj}_v(R) \in R_{\sigma'}$, where $R_{\sigma'}$ is the region of the braid arrangement $\mathcal{B}_n$ labeled by σ'. Suppose σ' is not the lexicographic minimum of S_R. Then, from Lemma 5.6, we have that σ' is not a G-local minimum. Hence for $\sigma' = \sigma'_1 \ldots \sigma'_n$, we have for some $i \in [n]$, $\sigma'_i > \sigma'_{i+1}$ and $\{\sigma'_i, \sigma'_{i+1}\} \notin E$. Then, $\sigma'' = \sigma'_1 \ldots \sigma'_{i+1}\sigma'_i \ldots \sigma'_n \in S_R$.

Now, as a consequence of Theorem 3.2 and Lemma 4.7, we have that the face F' of $R_{\sigma'}$ that v projects into is labeled by the ordered partition obtained by partitioning σ' at the right-to-left minima. Let us denote this partition by $\Pi(F')$.

In $\Pi(F')$, σ'_i and σ'_{i+1} will be in the same block (as σ'_i cannot be a right-to-left minimum). As swapping σ'_i and σ'_{i+1} does not affect the partition, F' is a common face of $R_{\sigma'}$ and $R_{\sigma''}$. Hence $\mathrm{proj}_v(R) \in R_{\sigma''}$, the region of the braid arrangement $\mathcal{B}_n$ labeled by σ''.

Continuing like this, we get that $\mathrm{proj}_v(R) \in R_\sigma$ where σ is the G-local minimum of S_R and hence the lexicographic minimum of S_R by Lemma 5.6, and R_σ is the region of the braid arrangement $\mathcal{B}_n$ labeled by σ. □

Lemma 5.8. *Let $G = ([n], E)$ be a natural unit interval graph, and let R be a region of $\mathcal{A}_G$. Let $\sigma \in \mathfrak{S}_n$ be the lexicographic minimum of S_R. Let Π be the ordered partition obtained by partitioning σ at the right-to-left minima. Then, F_Π is not contained in the interior of another face of $\mathcal{A}_G$.*

Proof. To show that F_Π is not contained in the interior of another face of $\mathcal{A}_G$, it is enough to show that $\mathrm{span}(F_\Pi)$ is a flat of $\mathcal{A}_G$. To show this, we show that for any block B of Π, the induced subgraph $G[B]$ is connected.

Suppose the induced subgraph $G[B]$ is not connected. Let i be the smallest vertex in B. As we get Π by partitioning at the right-to-left minima, i will appear after any other element of B in the one line notation of σ. Let j be the greatest vertex of the component with i, and k be a vertex in a different component of the induced subgraph.

Now, as i and j are in the same component, we have a path from i to j, say $ik_1k_2 \ldots k_tj$. Suppose we have $i < k < j$. Then, $k_s < k < k_{s+1}$ for some s (take $k_0 = i$ and $k_{t+1} = j$). But, as $\{k_s, k_{s+1}\} \in E$, and G is a natural unit interval graph, $\{k_s, k\} \in E$, that is, k is in the same component as k_s and hence the same component as i, a contradiction.

Hence $k > j$, that is, any other component of the induced subgraph has all vertices greater than j and hence greater than any vertex in the component containing i. We can order the components of $G[B]$ independently in the one line notation of σ and still have a permutation in S_R. The permutation obtained by ordering these components such that the component with i appears before any other component of $G[B]$ will be less than σ in the lexicographic order. This contradicts the fact that σ is the lexicographic minimum of S_R. Hence $G[B]$ is connected. □

Hence, by Lemmas 5.7 and 5.8, for a region $R \in R(\mathcal{A}_G)$ and v as in Definition 4.3, $\mathrm{pd}_v(R) = \dim(F_\Pi)$, where Π is the ordered partition of $[n]$ obtained by partitioning the lexicographic minimum σ of S_R at its right-to-left minima. As $\dim(F_\Pi)$ is clearly the number of right-to-left minima of σ, Theorem 5.2 follows.

We can now obtain the characteristic polynomial of the graphical arrangement of a natural unit interval graph.

Proof of Theorem 5.3. From Theorem 5.2, we have for a region R of the graphical arrangement $\mathcal{A}_G$, $\mathrm{pd}_v(R) = \mathrm{RLmin}(\sigma)$ where σ is the lexicographic minimum of S_R. We have also shown that these lexicographic minima are precisely the permutations where all descents are G-descents.

Hence, from Theorem 2.5 we get that

$$\chi_{\mathcal{A}_G}(-q) = (-1)^n \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{all descents of } \sigma \text{ are } G\text{-descents}}} q^{\mathrm{RLmin}(\sigma)}.$$

So, it is enough to show that

$$\sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{all descents of } \sigma \text{ are } G\text{-descents}}} q^{\mathrm{RLmin}(\sigma)} = \prod_{j=1}^{n} (q + c_j).$$

We prove this using induction on n. The base case is trivial.

Suppose that the statement holds for a natural unit interval graph on $n-1$ vertices.

Let $G = ([n], E)$ be a natural unit interval graph and let $G' = G[n-1]$. Then G' is also a natural unit interval graph.

Note that if we have a permutation of $[n]$ such that all descents are G-descents, removing n gives us a permutation of $[n-1]$ such that all descents are G-descents. This is because for $\sigma = \sigma_1 \ldots \sigma_n$, with $\sigma_i = n$ $(i < n)$, $n\sigma_{i+1}$ is a descent and hence a G-descent. If $\sigma_{i+1} < \sigma_{i-1}$, as $\{\sigma_{i+1}, n\} \in E$ and G is a natural unit interval graph, we have $\{\sigma_{i+1}, \sigma_{i-1}\} \in E$, and hence the only new descent formed on removing n is a G-descent. If $\sigma_{i+1} > \sigma_{i-1}$, no new descent is formed on removing n and hence all descents are G-descents.

Hence, we can uniquely obtain all permutations of $[n]$ such that all descents are G-descents by inserting n into a permutation of $[n-1]$ such that all descents are G-descents. There are two ways to do this:

We can insert n before any element it is adjacent to so that the descent thus formed is a G-descent. We have c_n ways of doing this. In this case, the number of right-to-left minima of the permutation will remain the same as n does not become a right-to-left minima and the ordering of the other elements is unchanged. The contribution of this to the characteristic polynomial will be $c_n \prod_{j=1}^{n-1} (q + c_j)$.

We can also insert n at the end of the permutation. This will not form any new descents, hence all descents will be G-descents. However, the number of right-to-left minima will increase by one as n is also a right-to-left minima. The contribution of this to the characteristic polynomial will be $q \prod_{j=1}^{n-1} (q + c_j)$.

Hence, we get that $$\sum_{\substack{\sigma \in \mathfrak{S}_n \\ \text{all descents of } \sigma \text{ are } G\text{-descents}}} q^{\mathrm{RLmin}(\sigma)} = \prod_{j=1}^{n} (q + c_j).$$

Hence, $\chi_{\mathcal{A}_G}(-q) = (-1)^n \prod_{j=1}^{n} (q + c_j)$. □

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