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A combinatorial formula for rank 2 scattering diagrams

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Abstract. Cluster algebras are celebrated for their intriguing positivity properties. Combining two distinct approaches to positivity, we give a directly computable, manifestly positive, and elementary (yet highly nontrivial) formula describing generalized cluster scattering diagrams in rank 2. This formula enumerates new combinatorial objects called tight gradings on maximal Dyck paths, inspired by the greedy basis construction for cluster algebras. Using the positivity of rank 2 generalized cluster scattering diagrams, we prove the Laurent positivity of generalized cluster algebras of all ranks, resolving a conjecture of Chekhov–Shapiro from 2014.

Keywords: scattering diagrams, generalized cluster algebras, positivity

1 Introduction

Scattering diagrams (or *wall-crossing structures*) emerged from efforts to construct mirror manifolds [13, 12] growing out of the Strominger–Yau–Zaslow conjecture [21] in mirror symmetry. Since then, this structure has also been utilized to encode enumerative geometric invariants and categorical invariants that count stable objects. These two themes have notably overlapped in the *cluster algebras* discovered by Fomin and Zelevinsky [7] and subsequent studies, where the techniques of scattering diagrams are fundamental in solving problems in algebraic combinatorics [9].

Cluster algebras, originally devised as a combinatorial framework to address total positivity and (dual) canonical bases in Lie theory, have themselves given rise to a wide range of intriguing algebraic and combinatorial questions. Among these, one of the most notable is the *positivity phenomenon*, conjectured by Fomin and Zelevinsky [7, Section 3].

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After remaining unsolved for over a decade, this positivity was first proven by Lee and Schiffler [16] in the generality for all skew-symmetric cluster algebras using an explicit rank-2 formula that sums over compatible pairs on Dyck paths. This breakthrough led to the construction of the *greedy basis* by Lee, Li and Zelevinsky [14].

In their seminal work [9], Gross, Hacking, Keel, and Kontsevich introduced ideas and tools from log Calabi–Yau mirror symmetry, including scattering diagrams, broken lines, and theta functions, into the study of cluster algebras. Due to the positivity of the scattering diagram developed in [9], the theta functions, which contain all cluster monomials, satisfy Laurent positivity. For the same reason, their multiplicative structure constants are also positive, a property referred to as *strong positivity*. We combine and extend the methods of Lee–Schiffler [15, 16] and Gross–Hacking–Keel–Kontsevich [9] to derive new positivity results for generalized cluster algebras [4].

In this extended abstract, a scattering diagram in a real vector space is defined as a collection of codimension-one rational cones, referred to as *walls*, each associated with a formal power series, called a *wall-function*. We first focus on rank-2 generalized cluster scattering diagrams in $\mathbb{R}^2$, defined as the consistent completion of an initial scattering diagram with two coordinate axes having wall-functions respectively polynomials P_1 and P_2. The wall-function $f_{(a,b)}(P_1, P_2)$ on the ray $\mathbb{R}_{\leq 0}(a, b)$ for any positive coprime integers (a, b) is notoriously difficult to compute, even when P_1 and P_2 are binomials of low degrees. Although there are Coxeter-type symmetries and cluster-type discrete structures governing the appearance of some rays [10], little is known about the wall-functions in a 2-dimensional sector known as the "Badlands", when $\deg P_1 \cdot \deg P_2 > 4$.

In Section 4, we present a directly computable, manifestly positive, elementary, yet highly nontrivial formula describing all wall-functions $f_{(a,b)}(P_1, P_2)$. We show that each coefficient of the wall-functions enumerates a new class of combinatorial objects that we call *tight gradings* on a maximal Dyck path. The *maximal Dyck path* $\mathcal{P}(m, n)$ is the lattice path from $(0, 0)$ to (m, n) that is closest to the main diagonal, without crossing strictly above it. A *grading* on $\mathcal{P}(m, n)$ is an assignment of a nonnegative integer value to each edge of $\mathcal{P}(m, n)$. A grading is *tight* if it satisfies a certain combinatorial compatibility condition (see Section 2 for precise details).

Pictorially, tight gradings can be represented as "tilings" by rectangles on rotations of the maximal Dyck path, as in the image to the right. The size of the first rectangle extending from each edge corresponds to its value in the grading. The relatively small space between the (light) blue and (dark) red rectangles encodes the tightness condition, and the fact that the rectangles are disjoint encodes the compatibility condition.

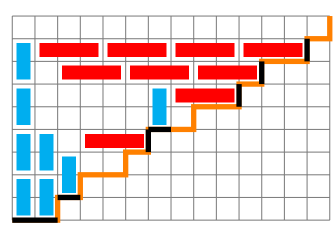


Figure 1: *A tight grading.*

Theorem 1.1. *In a generalized cluster scattering diagram of rank* 2*, each coefficient of the wall-function* $f_{(a,b)}(P_1, P_2)$ *is equal to the sum of weights of the corresponding tight gradings on some maximal Dyck path.*

In [11], the coefficients in $\log f_{(a,b)}$ are proven to be interpreted by relative Gromov–Witten invariants on toric surfaces. Therefore the above theorem yields a combinatorial formula for computing these Gromov–Witten invariants in terms of tight gradings.

In Section 2, we define tight gradings. Section 3 contains preliminaries on rank-2 scattering diagrams, focusing on the generalized cluster case. We present our explicit formula for wall-function coefficients in terms of tight gradings in Section 4. We then discuss applications of this formula in Section 5, including showing that the Badlands wall-functions are all non-trivial and proving the Laurent positivity of generalized cluster algebras of all ranks. Further details can be found in our preprints [3] and [2].

2 Tight gradings

In this section, we introduce combinatorial objects called *tight gradings* that are central to our main results. Tight gradings form a subset of the compatible gradings defined by Rupel [19] and inspired by the compatible pairs of Lee–Li–Zelevinsky [14] (see Section 2).

Fix $m, n \in \mathbb{N} = \mathbb{Z}_{\geq 0}$. Consider a rectangle with vertices $(0,0)$, $(0,n)$, $(m,0)$, and (m,n) with a main diagonal from $(0,0)$ to (m,n).

Definition 2.1. A *Dyck path* $\mathcal{P}$ is a lattice path in $(\mathbb{Z} \times \mathbb{R}) \cup (\mathbb{R} \times \mathbb{Z}) \subset \mathbb{R}^2$ starting at $(0,0)$ and ending at (m,n), proceeding by only unit north and east steps and never passing strictly above the main diagonal. We also view $\mathcal{P}$ as the set of its unit north and east steps, where we refer to each step as an *edge*.

Given a set C of edges in $\mathcal{P}$, we denote the set of horizontal edges (east steps) by $C_{\mathbf{E}}$ and the set of vertical edges (north steps) by $C_{\mathbf{N}}$. We let $|C|$ denote the number of edges in C. For edges e and f in $\mathcal{P}$, let $\overrightarrow{ef}$ denote the subpath proceeding east from e to f (including both e and f), continuing cyclically around $\mathcal{P}$ if e is to the north or east of f.

The Dyck paths from $(0,0)$ to (m,n) form a partially ordered set by comparing the heights at all vertices. The *maximal Dyck path* $\mathcal{P}(m,n)$ is the maximal element under this partial order. When m and n are relatively prime, the maximal Dyck path $\mathcal{P}(m,n)$ corresponds to the lower Christoffel word of slope n/m. We label the horizontal edges from left to right by $u_1, u_2, \ldots, u_m$ and the vertical edges from bottom to top by $v_1, v_2, \ldots, v_n$.

Example 2.2. In Figure 2, the maximal Dyck path $\mathcal{P}(6,4)$ is shown in the top left and $\mathcal{P}(7,4)$ is shown in the top right.

Motivated by Lee–Schiffler [15], Lee, Li, and Zelevinsky [14] introduced combinatorial objects called *compatible pairs* to construct the *greedy basis* for rank-2 cluster algebras, consisting of indecomposable positive elements including the cluster monomials. Rupel [19, 20] extended this construction to the setting of *generalized* rank-2 cluster algebras by defining *compatible gradings*.

A function from the set of edges on $\mathcal{P}(m,n)$ to $\mathbb{N}$ is called a *grading*. For any grading ω and for any set of edges $S \subset \mathcal{P}$, let $\omega(S) := \sum_{e \in S} \omega(e)$.

Definition 2.3. A grading $\omega : \mathcal{P} \longrightarrow \mathbb{N}$ is called *compatible* if for every pair of $u \in \mathcal{P}_{\mathbf{E}}$ and $v \in \mathcal{P}_{\mathbf{N}}$ with $\omega(u)\omega(v) > 0$, there exists an edge e along the subpath $\overrightarrow{uv}$ so that at least one of the following holds:

$$\begin{aligned} e \in \mathcal{P}_{\mathbf{N}} \setminus \{v\} \quad &\text{and} \quad |\overrightarrow{ue}_{\mathbf{N}}| = \omega\left(\overrightarrow{ue}_{\mathbf{E}}\right); \\ e \in \mathcal{P}_{\mathbf{E}} \setminus \{u\} \quad &\text{and} \quad |\overrightarrow{ev}_{\mathbf{E}}| = \omega\left(\overrightarrow{ev}_{\mathbf{N}}\right). \end{aligned} \tag{2.1}$$

Example 2.4. For each $i \in \{1,2,3\}$, let $\omega_i : \mathcal{P}(i+5,4) \longrightarrow \mathbb{N}$ be the grading given by $\omega_i(u_1) = \omega_i(u_2) = 2$, $\omega_i(v_3) = \omega_i(v_4) = 3$, and $\omega_i(e) = 0$ for every edge e in $\mathcal{P}(i+5,4) \setminus \{u_1,u_2,v_3,v_4\}$ (see Figure 2 for $i = 1,2$). Then ω_1 is not compatible, but ω_2 and ω_3 are compatible. The main difference between ω_1 and ω_2 is that the edge $e = u_2$ in $\mathcal{P}(7,4)$ satisfies the second condition in (2.1) for $u = u_1$ and $v = v_4$, as both sides of the equation equal 6.

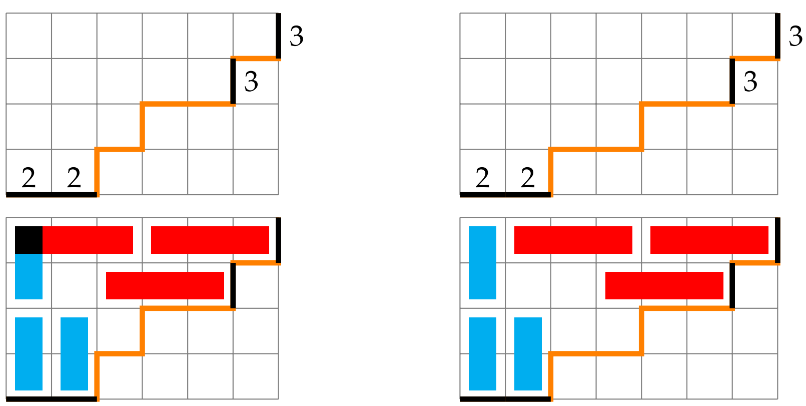


Figure 2: In the top images, we depict gradings ω_1 and ω_2 on the Dyck paths $\mathcal{P}(6,4)$ and $\mathcal{P}(7,4)$ from Example 2.4. Below, we depict the corresponding rectangular tilings. The grading ω_1 has overlapping rectangles and hence is not compatible, while the grading ω_2 is.

In their study of compatible pairs, Lee, Li, and Zelevinsky [14] introduced the notion of the "shadow" of a set of edges, which Rupel [20] extended to the setting of gradings.

Definition 2.5. Fix a maximal $\mathcal{P}$ and a grading $\omega \colon \mathcal{P} \to \mathbb{N}$. For each edge e in $\mathcal{P}$, we define its *shadow*, denoted by $\mathrm{sh}(e)$, as follows.

- If e is horizontal, then its shadow is $\overrightarrow{ev}_{\mathbf{N}}$, where $v \in \mathcal{P}_{\mathbf{N}}$ is chosen such that $\overrightarrow{ev}$ has minimal length with $|\overrightarrow{ev}_{\mathbf{N}}| = \omega(\overrightarrow{ev}_{\mathbf{E}})$. If no such v exists, let $\mathrm{sh}(e) = \mathcal{P}_{\mathbf{N}}$.

- If e is vertical, then its shadow is $\overrightarrow{ue}_{\mathbf{E}}$, where $u \in \mathcal{P}_{\mathbf{E}}$ is chosen such that $\overrightarrow{ue}$ has minimal length with $|\overrightarrow{ue}_{\mathbf{E}}| = \omega(\overrightarrow{ue}_{\mathbf{N}})$. If no such u exists, let $\mathrm{sh}(e) = \mathcal{P}_{\mathbf{E}}$.

For $S \subset \mathcal{P}$, let the *shadow* of S be $\mathrm{sh}(S) = \bigcup_{e \in S} \mathrm{sh}(e)$.

Example 2.6. Consider ω_2 as in Example 2.4. Then $\mathrm{sh}(v_3) = \{u_4, u_5, u_6\}$ and $\mathrm{sh}(v_4) = \{u_2, u_3, \ldots, u_7\} = \mathrm{sh}(\mathcal{P}_{\mathbf{N}})$.

Partially motivated by [1], we discovered the following definition.

Definition 2.7. Let ω be a grading on $\mathcal{P} = \mathcal{P}(m,n)$ with $p = \omega(\mathcal{P}_{\mathbf{N}})$ and $q = \omega(\mathcal{P}_{\mathbf{E}})$. The grading ω is a *tight grading* if ω is a compatible grading satisfying $p \leq m$, $q \leq n$,

$$|pn - qm| = \gcd(p,q)\,,$$

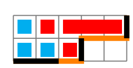
and at least one of

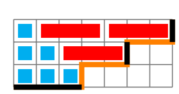
$$S_{\mathbf{E}} \subseteq \mathrm{sh}(\mathcal{P}_{\mathbf{N}}) \text{ or } S_{\mathbf{N}} \subseteq \mathrm{sh}(\mathcal{P}_{\mathbf{E}})\,,$$

where S is the set of edges e with $\omega(e) > 0$.

We can represent compatible gradings as rectangular tilings as follows. First, rotate (i.e. cyclically shift) the maximal Dyck path so that the shadow of each horizontal (resp. vertical) edge does not extend beyond the left (resp. top) boundary of the rotated path. Such a rotation always exists for tight gradings. Then draw blue rectangles above each horizontal edge e with total height equal to the size of the shadow of e, partitioned into the vertical grading values contributing to the shadow. Similarly, we draw red rectangles to the left of each edge in $\mathcal{P}_{\mathbf{N}}$. The grading is compatible only if the resulting rectangles are non-overlapping. The condition on the shadows can be easily read off by seeing if every blue rectangle has a red rectangle above, or if every red rectangle has a blue rectangle to the left.

Example 2.8. (1) The grading ω_2 as in Example 2.4 is not tight despite $S_{\mathbf{N}} \subseteq \mathrm{sh}(\mathcal{P}_{\mathbf{E}})$, because $(m,n) = (7,4)$ does not satisfy $\beta_1 n - \beta_2 m = \pm\gcd(\beta_1,\beta_2)$ for $(\beta_1,\beta_2) = (6,4)$.
(2) Let $(\beta_1,\beta_2) = (2,1)$ and $(m,n) = (3,1)$. Consider $\mathcal{P}(3,1)$.  Suppose that $\omega(u_1) = 1$, $\omega(u_2) = \omega(u_3) = 0$, and $\omega(v_1) = 2$. Then ω is tight.

(3) Let $(\beta_1,\beta_2) = (4,2)$ and $(m,n) = (5,2)$. Consider $\mathcal{P}(5,2)$.  Suppose that $\omega(u_1) = \omega(u_2) = \omega(v_1) = 1$, $\omega(v_2) = 3$, and $\omega(u_3) = \omega(u_4) = \omega(u_5) = 0$. Then ω is tight.

(4) Let $(\beta_1,\beta_2) = (6,3)$ and $(m,n) = (7,3)$. Consider $\mathcal{P}(7,3)$.  Suppose that $\omega(v_2) = \omega(v_3) = 3$, $\omega(u_1) = \omega(u_2) = \omega(u_3) = 1$, and $\omega(v_1) = \omega(u_4) = \omega(u_5) = \omega(u_6) = \omega(u_7) = 0$. Then ω is tight.
(5) Let $(\beta_1,\beta_2) = (12,8)$ and $(m,n) = (14,9)$. There are total 14 tight gradings such that $\omega(h) = 2$ for exactly four horizontal edges h, $\omega(v) = 3$ for exactly four vertical edges v, and $\omega(e) = 0$ for all other edges on $\mathcal{P}(14,9)$.

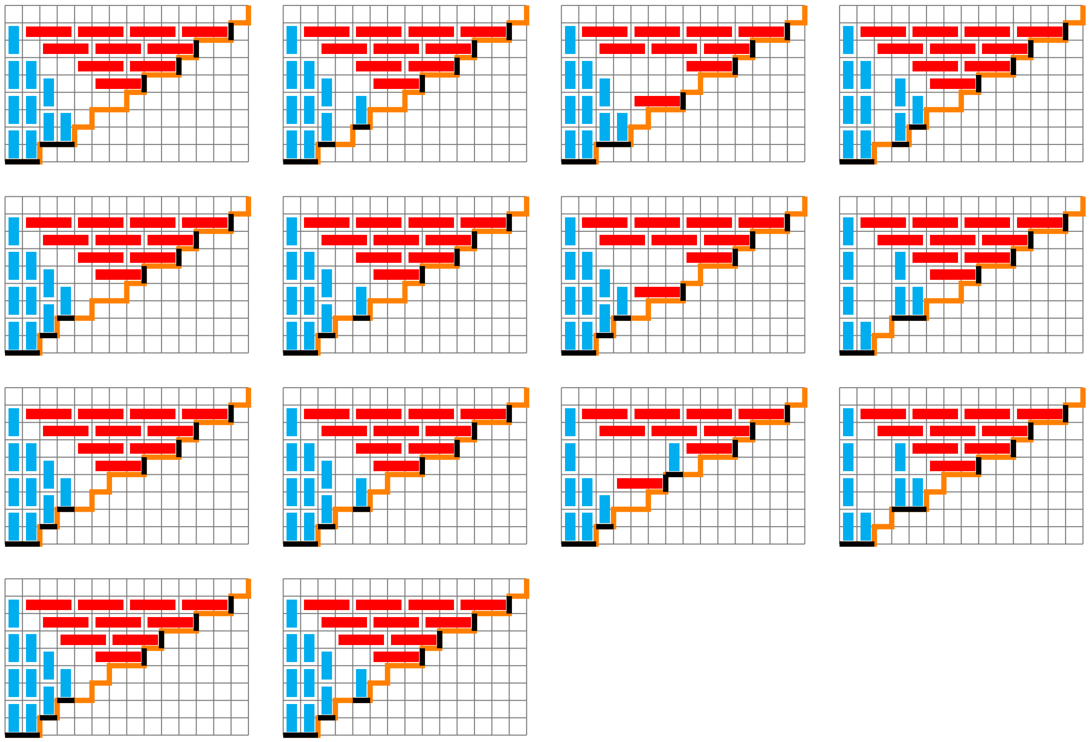


Figure 3: The 14 tight gradings on $\mathcal{P}(14,9)$ with $(\beta_1,\beta_2)=(12,8)$. These correspond to a coefficient of the central wall-function, whose sequence of coefficients is the Catalan numbers, in a certain cluster scattering diagram (see Example 3.6).

Remark 2.9. The word "tight" is inspired by the tight space between blue and red rectangles.

3 Rank 2 scattering diagrams

Fix a rank-2 lattice $M \cong \mathbb{Z}^2$ and choose a strictly convex rational cone σ in $M_\mathbb{R} := M \otimes \mathbb{R}$. We take the monoid $P = \sigma \cap M$ and denote $P^+ := P \setminus \{0\}$. Set $\widehat{\Bbbk[P]}$ to be the monoid algebra $\Bbbk[P]$ completed at the maximal monomial ideal $\mathfrak{m}$ generated by $\{x^m \mid m \in P^+\}$.

Definition 3.1. A *wall* is a pair $(\mathfrak{d}, f_\mathfrak{d})$ consisting of a *support* $\mathfrak{d} \subseteq M_\mathbb{R}$ and a *wall-function* $f_\mathfrak{d} \in \widehat{\Bbbk[P]}$, where

- $\mathfrak{d}$ is either a ray $\mathbb{R}_{\leq 0} w$ or a line $\mathbb{R} w$ for some $w \in P^+$;
- $f_\mathfrak{d} = f_\mathfrak{d}(x^w) = 1 + \sum_{k \geq 1} c_k x^{kw}$ for $c_k \in \Bbbk$.

Associated to a wall $(\mathfrak{d}, f_{\mathfrak{d}})$ and a direction $v \in M_{\mathbb{R}}$ transversal to $\mathfrak{d}$ is an algebra automorphism $\mathfrak{p}_{v,\mathfrak{d}} \in \mathrm{Aut}(\widehat{\Bbbk[P]})$ defined by $\mathfrak{p}_{v,\mathfrak{d}}(x^m) = x^m f_{\mathfrak{d}}^{n(m)}$ for $m \in P$, where $n \in \mathrm{Hom}(M, \mathbb{Z})$ is primitive and orthogonal to $\mathfrak{d}$ in the direction $n(v) < 0$.

Definition 3.2. A *scattering diagram* $\mathfrak{D}$ is a collection of walls such that the set

$$\mathfrak{D}_k := \{(\mathfrak{d}, f_{\mathfrak{d}}) \in \mathfrak{D} \mid f_{\mathfrak{d}} \not\equiv 1 \mod \mathfrak{m}^k\}$$

is finite for each $k \geq 0$.

A path $\gamma \colon [0,1] \to M_{\mathbb{R}}$ is called *regular* (with respect to $\mathfrak{D}$) if it is a smooth immersion with endpoints away from the support of any wall and only crosses walls transversally. For each $k \geq 1$, let $0 < t_1 < \cdots < t_s < 1$ be the longest sequence such that $\gamma(t_i) \in \mathfrak{d}_i$ for some wall $(\mathfrak{d}_i, f_{\mathfrak{d}_i}) \in \mathfrak{D}_k$. Consider the product $\mathfrak{p}^{(k)}_{\gamma,\mathfrak{D}} = \mathfrak{p}_{\dot\gamma(t_s),\mathfrak{d}_s} \circ \cdots \circ \mathfrak{p}_{\dot\gamma(t_1),\mathfrak{d}_1}$. We define the *path-ordered product* of γ to be $\mathfrak{p}_{\gamma,\mathfrak{D}} = \lim_{k\to\infty} \mathfrak{p}^{(k)}_{\gamma,\mathfrak{D}} \in \mathrm{Aut}(\widehat{\Bbbk[P]})$.

Definition 3.3. A scattering diagram $\mathfrak{D}$ is called *consistent* if the path-ordered product $\mathfrak{p}_{\gamma,\mathfrak{D}}$ equals the identity for any regular simple loop γ.

Theorem 3.4 ([13])**.** *Given any initial scattering diagram $\mathfrak{D}_{\mathrm{in}}$ of only lines, there is a unique minimal consistent scattering diagram* $\mathrm{Scat}(\mathfrak{D}_{\mathrm{in}})$ *containing $\mathfrak{D}_{\mathrm{in}}$ such that* $\mathrm{Scat}(\mathfrak{D}_{\mathrm{in}}) \setminus \mathfrak{D}_{\mathrm{in}}$ *consists of distinct rays with non-trivial wall-functions.*

While the use of scattering diagrams originated in the study of mirror symmetry, they have since found remarkable applications in cluster algebras by the celebrated work of Gross, Hacking, Keel, and Kontsevich [9]. We exhibit a collection of consistent scattering diagrams devised for (generalized) cluster algebras in rank 2. Let $M = \mathbb{Z}^2$, $e_1 = (1, 0)$, $e_2 = (0, 1)$. Choose σ to be the first quadrant of $M_{\mathbb{R}} = \mathbb{R}^2$. Denote $x = x^{e_1}$ and $y = x^{e_2}$. The initial scattering diagram will be two lines

$$\mathfrak{D}_{\mathrm{in}} = \{(\mathbb{R}e_1, P_1(x)), (\mathbb{R}e_2, P_2(y))\} \tag{3.1}$$

where $P_i(x^{e_i}) \in \Bbbk[x^{e_i}]$ with constant term 1. Denote $\mathfrak{D}(P_1, P_2) = \mathrm{Scat}(\mathfrak{D}_{\mathrm{in}})$. There are infinitely many rays in $\mathfrak{D}(P_1, P_2) \setminus \mathfrak{D}_{\mathrm{in}}$ of the form $(\mathbb{R}_{\leq 0}(a, b), f_{(a,b)})$ for coprime $(a, b) \in \mathbb{Z}^2_{>0}$ unless $\deg P_1 \cdot \deg P_2 < 4$, when there are finitely many.

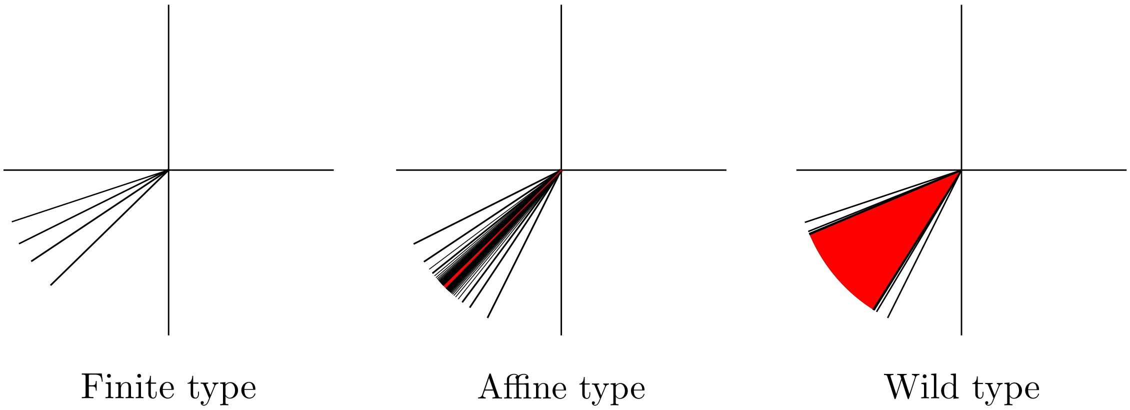


Figure 4: From left to right, we depict the scattering diagrams $\mathfrak{D}(1 + p_{1,1}x + p_{1,2}x^2 + p_{1,3}x^3, 1 + p_{2,1}y)$, $\mathfrak{D}_{(2,2)}$, and $\mathfrak{D}_{(3,2)}$.

Example 3.5. We depict in the left of Figure 4 the case $\deg P_1 = 3$ and $\deg P_2 = 1$. The remaining finite cases can be obtained by specializing certain coefficients to zero. In the scattering diagram on the left of Figure 4, the wall-functions on the added rays are

$$f_{(3,1)} = 1 + p_{1,3}p_{2,1}x^3y,$$
$$f_{(2,1)} = 1 + p_{1,2}p_{2,1}x^2y + p_{1,1}p_{1,3}p_{2,1}^2x^4y^2 + p_{1,3}^2p_{2,1}^3x^6y^3 \quad \text{(see Example 3.5 f(2,1))},$$
$$f_{(3,2)} = 1 + p_{1,3}p_{2,1}^2x^3y^2,$$
$$f_{(1,1)} = 1 + p_{1,1}p_{2,1}xy + p_{1,2}p_{2,1}^2x^2y^2 + p_{1,3}p_{2,1}^3x^3y^3.$$

When $P_1(x) = 1 + x^{\ell_1}$ and $P_2(y) = 1 + y^{\ell_2}$, the resulting scattering diagram $\mathfrak{D}_{(\ell_1,\ell_2)} = \mathfrak{D}(P_1, P_2)$ [9] is famously responsible for the rank-2 cluster algebra $\mathcal{A}(\ell_1, \ell_2)$ [7]. When $\ell_1\ell_2 < 4$, its structure is directly derived from Example 3.5 by specializing coefficients. When $\ell_1\ell_2 \geq 4$, there is a discrete set of rays outside the closed cone spanned by

$$\left(-2\ell_1, -\ell_1\ell_2 - \sqrt{\ell_1^2\ell_2^2 - 4\ell_1\ell_2}\right) \quad \text{and} \quad \left(-\ell_1\ell_2 - \sqrt{\ell_1^2\ell_2^2 - 4\ell_1\ell_2}, -2\ell_2\right).$$

These rays are in bijection with the cluster variables $\{x_n \mid n \in \mathbb{Z},\ n \neq 0, 1, 2, 3\} \subset \mathcal{A}(\ell_1, \ell_2)$, where their directions are opposite to the d-vectors of cluster variables. The cone itself, known as the *Badlands*, has a much richer yet more elusive structure. It is known that $\mathfrak{D}_{(\ell_1,\ell_2)}$ has a ray at every rational slope within the Badlands; see [10, Section 4.7] and [8]. However, the wall-functions there were generally not understood.

Example 3.6. The scattering diagram $\mathfrak{D}_{(2,2)}$, depicted in the center of Figure 4, has only one non-cluster ray, depicted in red. The scattering diagram $\mathfrak{D}_{(3,2)}$ is depicted in the right of Figure 4, with the Badlands shown in red. The wall-function on the ray $\mathbb{R}_{\leq 0}(3, 2)$ is $1 + x^3y^2 + 2x^6y^4 + 5x^9y^6 + 14x^{12}y^8 + 42x^{15}y^{10} + \cdots$, where the coefficients are the Catalan numbers. The calculation of the coefficient 14 comes from Example 2.8(5).

We can even let the initial wall-functions in (3.1) be power series

$$P_i(x^{e_i}) = 1 + p_{i,1}x^{e_i} + \cdots + p_{i,j}x^{je_i} + \cdots \in \Bbbk[\![x]\!], \quad i = 1, 2.$$

In $\mathfrak{D}(P_1, P_2)$, the functions on the added rays $\mathbb{R}_{\leq 0}(a, b)$ with coprime $(a, b) \in \mathbb{N}^2$ are of the form

$$f_{(a,b)}(P_1, P_2) = 1 + \sum_{k \geq 1} \lambda(ka, kb)x^{ka}y^{kb}. \tag{3.2}$$

In Section 4, we will give a combinatorial formula for every $\lambda(ka, kb)$ in terms of tight gradings. Since the polynomial case can be obtain from the power series case by letting all but finitely many $p_{i,j}$ be zero, it will become clear that the formula given in Theorem 4.3 also applies to the polynomial case by considering certain subsets of tight gradings.

4 Combinatorial formula of wall-function coefficients

Definition 4.1. The *weight* of a grading $\omega : \mathcal{P}(m, n) \longrightarrow \mathbb{N}$ is defined as

$$\mathrm{wt}(\omega) = \prod_{i=1}^{m} p_{2,\omega(u_i)} \prod_{j=1}^{n} p_{1,\omega(v_j)}.$$

Example 4.2. If ω is as in Example 2.8(2), then $\mathrm{wt}(\omega) = p_{1,2}p_{2,1}$. If ω is as in Example 2.8(3), then $\mathrm{wt}(\omega) = p_{1,1}p_{1,3}p_{2,1}^2$. If ω is as in Example 2.8(4), then $\mathrm{wt}(\omega) = p_{1,3}^2 p_{2,1}^3$.

Our main theorem explicitly computes the elements $f_{(a,b)}$ in terms of tight gradings introduced in Section 2.

Theorem 4.3. *Fix coprime positive integers (a, b). For each $k \geq 1$, choose integers (m_k, n_k) such that $|an_k - bm_k| = 1$, $ka \leq m_k$, and $kb \leq n_k$. Then*

$$f_{(a,b)} = 1 + \sum_{k \geq 1} \sum_{\omega} \mathrm{wt}(\omega)x^{ka}y^{kb}, \tag{4.1}$$

where the second sum is over all tight gradings ω on the Dyck path $\mathcal{P}(m_k, n_k)$ of total horizontal weight kb and total vertical weight ka.

Example 4.4. Let $\deg P_1 = 3$, $\deg P_2 = 1$, and $(a, b) = (2, 1)$. Let $(m_k, n_k) = (ka + 1, kb)$ for all $k \geq 1$. Then the gradings as in Example 2.8(2)(3)(4) are the only tight gradings of the form $\omega : \mathcal{P} = \mathcal{P}(m_k, n_k) \longrightarrow \mathbb{N}$ with $\omega(\mathcal{P}_{\mathbf{E}}) = kb$ and $\omega(\mathcal{P}_{\mathbf{N}}) = ka$ with nonzero weight. Thus, Example 4.2 implies

$$f_{(2,1)}(P_1, P_2) = 1 + p_{1,2}p_{2,1}x^2y + p_{1,1}p_{1,3}p_{2,1}^2x^4y^2 + p_{1,3}^2p_{2,1}^3x^6y^3.$$

Remark 4.5. When $\deg P_i = \ell_i$ for $i = 1,2$, Reineke and Weist [18] showed that the coefficients of $f_{(a,b)}(P_1, P_2)$ can be expressed in terms of Euler characteristics of certain moduli spaces of framed stable representations of the complete bipartite quiver with ℓ_1 sources and ℓ_2 sinks. Therefore our tight grading formula Theorem 4.3 provides a manifestly-positive way to compute these Euler characteristics.

5 Applications of the tight grading formula

5.1 Every wall-function coefficient in the Badlands is non-trivial

As a direct application of our formula, we can prove that for $\mathfrak{D}(P_1, P_2)$ with positive coefficients and $\ell_1\ell_2 \geq 4$, every wall-function coefficient of every wall in the Badlands is strictly positive. Together with the known description of cluster rays, we obtain a full description of when $\lambda(ak, bk)$ is non-vanishing.

When $\ell_1 = \ell_2$, Gross and Pandharipande [10] showed that every rational ray in the Badlands has non-trivial wall-function using a result regarding the existence of stable quiver representations due to Reineke [10, Proposition 4.15]. This was extended to the skew-symmetrizable setting by Gränitz and Luo [8]. Davison and Mandel [6, Example 7.10] showed further that every coefficient is non-zero in the skew-symmetric setting.

Our approach requires only elementary combinatorics and extends naturally to the generalized cluster setting with positive initial wall-functions.

Theorem 5.1. *Suppose P_1 and P_2 have positive coefficients. In the scattering diagram $\mathfrak{D}(P_1, P_2)$, for any coprime $(a, b) \in \mathbb{Z}^2_{>0}$ with $\mathbb{R}_{\leq 0}(a, b)$ in the Badlands and any $k \geq 1$, there exists a maximal Dyck path $\mathcal{P}$ at least one tight grading $\omega \colon \mathcal{P} \longrightarrow \mathbb{N}$ with $\omega(\mathcal{P}_{\mathbf{E}}) = kb$, $\omega(\mathcal{P}_{\mathbf{N}}) = ka$, and non-zero weight with respect to P_1 and P_2.*

Therefore, every coefficient $\lambda(ka, kb)$ is strictly positive for every wall within the Badlands.

5.2 Positivity of generalized cluster algebras

Built on the rank-2 positivity demonstrated by our tight grading formula, we develop the positivity of higher-rank scattering diagrams towards applications in *generalized cluster algebras*. These algebras, axiomatized by Chekhov and Shapiro [4], accommodate polynomial mutation rules, in contrast to binomial exchange relations introduced by Fomin and Zelevinsky [7]. Following [9], the *generalized cluster scattering diagrams* [17, 5] are constructed to study these algebras.

Extending the positivity in rank 2 from Theorem 4.3 and using a *change of lattice trick* and a *perturbation trick* adapted from [11, 9], we obtain the following positivity result in all ranks. A coefficient is said to be *positive* if it is a polynomial in the coefficients of the initial exchange polynomials with positive integer coefficients.

Theorem 5.2 ([2, Theorem 9.9]). *There exists a representative for (the equivalence class of) a generalized cluster scattering diagram of any rank such that the coefficients of all wall-functions are positive.*

The strategy of the proof of Theorem 5.2 is to construct a sequence of finite scattering diagrams $\mathfrak{D}_k$ so that the limit of $(\mathfrak{D}_k)_{k\geq 1}$ is equivalent to $\mathfrak{D}$. The theorem is proven by showing inductively that each $\mathfrak{D}_k$ admits positivity where our rank-2 positivity Theorem 4.3 is crucial for the induction.

The positivity of theta function coefficients follows directly from that of the scattering diagram, as in the ordinary cluster case [9]. Since the cluster variables of a generalized cluster algebra are theta functions for the corresponding scattering diagrams [17, 5], we thus obtain the Laurent positivity of generalized cluster algebras.

Corollary 5.3 ([4, Conjecture 5.1]). *In a generalized cluster algebra of any rank, the Laurent expansion of any generalized cluster variable (in an initial cluster) has positive coefficients.*

The structure constants of the theta functions (see [9, Section 6] and [5, Section 5]) also directly inherit positivity from the positivity of the generalized cluster scattering diagrams.

Corollary 5.4. *The theta functions defined in a generalized cluster scattering diagram of any rank have strong positivity, that is, their multiplicative structure constants are positive.*

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