Séminaire Lotharingien de Combinatoire **93B** (2025)
Article #34, 12 pp.

Proceedings of the 37th Conference on Formal Power Series and Algebraic Combinatorics (Sapporo)

Lusztig q-weight multiplicities and KR crystals

Hyeonjae Choi*[1], Donghyun Kim†[1], and Seung Jin Lee‡[1]

[1] *Department of Mathematical Sciences, Seoul National University, Seoul, Korea*

Abstract. Lusztig q-weight multiplicities extend the Kostka–Foulkes polynomials to a broader range of Lie types. In this work, we investigate these multiplicities through the framework of Kirillov–Reshetikhin crystals. Specifically, for type C and type B spin weights, we present a combinatorial formula for Lusztig q-weight multiplicities in terms of energy functions of Kirillov–Reshetikhin crystals, generalizing the charge statistic on semistandard Young tableaux for type A. Additionally, we introduce level-restricted q-weight multiplicities for nonexceptional types, and prove its positivity by providing their combinatorial formulas.

Keywords: Lusztig q-weight multiplicity, Kirillov–Reshetikhin crystals, Energy function, Level-restricted q-weight multiplicity, Kostka–Foulkes polynomial

1 Introduction

The *Kostka–Foulkes polynomial* $K_{\lambda,\mu}(q)$ is the coefficient of the modified Hall–Littlewood polynomials when expressed in the Schur basis. One of the famous formulas for $K_{\lambda,\mu}(q)$ is due to Lascoux–Schützenberger [3]:

$$K_{\lambda,\mu}(q) = \sum_{T \in \mathrm{SSYT}(\lambda,\mu)} q^{\mathrm{charge}(T)} \tag{1.1}$$

where the sum is taken over all semistandard Young tableaux of shape λ with weight μ and charge(T) is a certain nonnegative integer associated to a semistandard Young tableau T. The Kostka–Foulkes polynomial originates from geometry and representation theory, with connections to flag varieties, Springer fibers, and intersection homology [1]. Due to its ubiquity and rich structure, studying Kostka–Foulkes polynomial has been an active area of research. There exist several ways to generalize the Kostka–Foulkes polynomials, including Macdonald polynomials and Lusztig q-weight multiplicity. Lusztig q-weight multiplcity is a generalization of Kostka–Foulkes polynomials for any Lie type. In this paper, we mainly consider Lusztig q-weight multiplicities beyond type A.

---

*2pun4ani@snu.ac.kr. H. J. Choi is supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (No.0450-20240021)

†hyun920310@snu.ac.kr. D.H. Kim was supported by NRF grants 2022R1I1A1A01070620.

‡lsjin@snu.ac.kr. S. J. Lee was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIT) (No.0450-20240021)

Let $\mathfrak{g}_n$ be a classical simple Lie algebra and let $L : R^+ \to \mathbb{Z}$ be a function from the set of positive roots. For dominant weights λ, μ, we define

$$\mathrm{KL}^{\mathfrak{g}_n,L}_{\lambda,\mu}(q) = \sum_{w\in W}(-1)^w[e^{w(\lambda+\rho)-(\mu+\rho)}]\prod_{\alpha\in R^+}\frac{1}{1-q^{L(\alpha)}e^{\alpha}} \tag{1.2}$$

where $[e^\beta]f$ denotes the coefficient of e^β in f, $\rho = \frac{1}{2}\sum_{\alpha\in R^+}\alpha$, and W is the Weyl group of $\mathfrak{g}_n$. Notably, $\mathrm{KL}^{\mathfrak{g}_n,L}_{\lambda,\mu}(1)$ equals the weight multiplicity. While, in general, $\mathrm{KL}^{\mathfrak{g}_n,L}_{\lambda,\mu}(q)$ is not a polynomial with nonnegative coefficients, we examine two cases where it is.

First, we take $L \equiv 1$, i.e., $L(\alpha) = 1$ for all $\alpha \in R^+$. Then $\mathrm{KL}^{\mathfrak{g}_n,L\equiv 1}_{\lambda,\mu}(q)$, which we simply denote by $\mathrm{KL}^{\mathfrak{g}_n}_{\lambda,\mu}(q)$, is a famous *Lusztig q-weight multiplicity* [8]. Its nonnegativity follows from the theory of the affine Kazhdan–Lusztig polynomial. Since the positivity of (affine) Kazhdan–Lusztig polynomials relies heavily on geometric machinery, specifically related to intersection homology theory, finding a combinatorial formula for these polynomials remains an important open problem.

When $\mathfrak{g}_n$ is of type A, i.e., $\mathfrak{g}_n = A_{n-1}$, the Lusztig q-weight multiplicity $\mathrm{KL}^{A_{n-1}}_{\lambda,\mu}(q)$ recovers the Kostka–Foulkes polynomial, whose combinatorics are well understood via the charge formula by Lascoux and Schützenberger (1.1). The natural question to ask is whether there is a generalization of the charge statistic beyond type A that provides a positive combinatorial formula for the Lusztig q-weight multiplicities. This question has been a long-standing open problem for decades, and even when restricting to specific types, only partial results have been obtained. We provide a combinatorial formula for $\mathrm{KL}^{C_n}_{\lambda,\mu}(q)$ and $\mathrm{KL}^{B_n}_{\lambda,\mu}(q)$ for spin weights λ and μ, using the energy function in the column KR crystals (Theorem 3.1 and 3.4). Note that the energy function in the column KR crystals for affine type A is essentially the same as the charge statistic [9].

We introduce another natural q-version of the weight multiplicity for nonexceptional types, which we call the *level-restricted q-weight multiplicity*. Consider a function L_A, where $L_A(\alpha) = 1$ if α is positive root of type A, i.e., $\alpha = \varepsilon_i - \varepsilon_j$ for $i < j$, and $L_A(\alpha) = 0$ otherwise. We denote the level-restricted q-weight multiplicity by $\mathrm{KL}^{\mathfrak{g}_n,L_A}_{\lambda,\mu}(q)$. We provide a combinatorial formula for $\mathrm{KL}^{\mathfrak{g}_n,L_A}_{\lambda,\mu}(q)$ for nonexceptional types (Theorem 4.1). Notably, our findings refine the $X = K = {}^{\infty}\mathrm{KL}$ theorem for tensor products of row KR crystals [6, 11] and provide new combinatorial objects for weight multiplicities.

The following table summarizes the main results:

Type	Lusztig q-weight multiplicity	Level-restricted q-weight multiplicity
B	column KR crystal of type $D^{(2)}_{N+1}$(spin)	row KR crystal of type $D^{(2)}_{N+1}$
C	column KR crystal of type $B^{(1)}_N$	row KR crystal of type $C^{(1)}_N$
D	?	row KR crystal of type $B^{(1)}_N$

It is natural to try to complete the missing parts of the table. For Lusztig q-weight multiplicity in type D, we suspect that a formula related to the column KR crystal of affine type $C_N^{(1)}$ might exist. For non-spin weights in type B, we may need to step beyond the current framework of crystal theory, which is a big and fascinating challenge.

Another potential direction is to develop crystal structures on GSSOT or SSROT (see Section 2.2), extending the work of Lee [7]. Additionally, constructing a bijection between our new objects and existing combinatorial objects for weight multiplicities offers an interesting challenge for combinatorialists.

2 Preliminaries

2.1 KR crystal

Let $\mathfrak{g} \supset \mathfrak{g}' \supset \mathfrak{g}_0$ be an affine Kac–Moody algebra with index set $I = J \cup \{0\}$, where $\mathfrak{g}'$ is the derived subalgebra and $\mathfrak{g}_0$ is the simple Lie algebra. The corresponding quantized universal enveloping algebras are $U_q(\mathfrak{g}) \supset U_q'(\mathfrak{g}) \supset U_q(\mathfrak{g}_0)$. For each pair $(r, s) \in J \times \mathbb{Z}_{\geq 0}$, there exist a finite-dimensional irreducible $U_q'(\mathfrak{g})$-module $W_s^{(r)}$, called the Kirillov–Reshetikhin (KR) module. The KR modules $W_s^{(r)}$ have a crystal basis $B^{r,s}$, called the Kirillov–Reshetikhin (KR) crystals, for nonexceptional types [10].

Let $\mathcal{C}$ be the category of tensor products of KR crystals. The category $\mathcal{C}$ has the following remarkable properties [5, 6].

1. For any $B_1, B_2 \in \mathcal{C}$, there exists a unique affine crystal isomorphism $R = R_{B_1,B_2} : B_2 \otimes B_1 \to B_1 \otimes B_2$, called the combinatorial R-matrix.
2. Let $B = B_n \otimes \cdots \otimes B_1$ where $B_1, \ldots, B_n \in \mathcal{C}$. There is a map $\overline{D}_B : B \to \mathbb{Z}$, called the energy function for $B \in \mathcal{C}$, such that $\overline{D}_B$ is constant on each J-component. From now on, we will simply denote it as $\overline{D}$ instead of $\overline{D}_B$.

In general, computing $\overline{D}$ is challenging, but when $B = (B^{1,1})^{\otimes n}$, the energy function is easy to compute. Let

$$a_n \otimes a_{n-1} \otimes \cdots \otimes a_1$$

be an element in $(B^{1,1})^{\otimes n}$. For the KR crystals of type $B_N^{(1)}$ and $C_N^{(1)}$, the energy function $\overline{D}$ is defined as $\overline{D} = \sum_{i=1}^{n-1}(n-i)H(a_{i+1}, a_i)$. For the KR crystals of type $B_N^{(1)}$, the function $H(b, a)$ is defined as follows:

$$H(b,a) = \begin{cases} 2 & \text{if } a = 1 \text{ and } b = \overline{1} \\ 1 & \text{if } b \succ a \text{ and } (b,a) \neq (\overline{1}, 1) \\ 0 & \text{if } b \preceq a \end{cases}$$

under the order $1 \prec 2 \prec \cdots \prec \bar{2} \prec \bar{1}$. For the KR crystals of type $C_N^{(1)}$, the function $H(b,a)$ is defined as follows:

$$H(b,a) = \begin{cases} 1 & \text{if } b \succ a \\ 0 & \text{if } b \preceq a \end{cases}$$

under the order $1 \prec 2 \prec \cdots \prec \bar{2} \prec \bar{1}$.

On the other hand, for the KR crystals of type $D_{N+1}^{(2)}$, the energy function $\overline{D}$ is defined as $\overline{D} = \sum_{i=1}^{n-1} 2(n-i)H(a_{i+1}, a_i) + \mathrm{vac}$, where vac denotes the number of $\emptyset$ elements in $(B^{1,1})^{\otimes n}$. The function $H(b,a)$ is defined as follows:

$$H(b,a) = \begin{cases} 1 & \text{if } b \succ a \text{ or } a = b = \emptyset \\ 0 & \text{if } b \preceq a \end{cases}$$

under the order $\emptyset \prec 1 \prec 2 \prec 3 \prec \cdots \prec \bar{3} \prec \bar{2} \prec \bar{1}$. Note that for column KR crystals of affine type A, the energy function essentially coincides with the charge statistic [9].

In [2], Lascoux introduced the *standardization map*, which maps SSYT to SYT while preserving the cocharge. A generalized version of Lascoux's standardization map exists, mapping an element in $\otimes_i B^{a_i,b_i}$ to an element in $(B^{1,1})^{\otimes \sum_i a_i b_i}$ while preserving the coenergy function. More details can be found in [5, 6, 11].

In this paper, we only consider row KR crystals, i.e., $B^{1,r}$, and column KR crystals, i.e., $B^{c,1}$. For a nonnegative integer vector $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$, we employ the following notations:

$$B_\mu := B^{1,\mu_n} \otimes \cdots \otimes B^{1,\mu_1}, \qquad B_\mu^t := B^{\mu_n,1} \otimes \cdots \otimes B^{\mu_1,1}.$$

We say that $u \in B$ is *classical highest weight element* if $e_i(b) = 0$ for all $i > 0$. We define $\mathrm{HW}(B)$ to be the set of classical highest weight elements in B, and define $\mathrm{HW}(B,\lambda) = \{b \in \mathrm{HW}(B) : \mathrm{wt}(b) = \lambda\}$.

2.2 Combinatorial objects and embedding into KR crystals

We introduce the combinatorial objects and embed them into KR crystals for certain types.

Definition 2.1. Oscillating horizontal strip (ohs for short) (μ, ν, λ) of length r is a sequence of three partitions satisfying:

- ν/μ and ν/λ are horizontal strips,
- $|\nu/\mu| + |\nu/\lambda| = r$.

We say that ohs (μ, ν, λ) is g-bounded if $\nu_1 \leq g$.

Definition 2.2. Generalized oscillating horizontal strip (gohs for short) (μ, ν, λ) of length r is a sequence of three partitions satisfying:

- ν/μ and ν/λ are horizontal strips,
- $|\nu/\mu| + |\nu/\lambda| = r \quad \text{or} \quad r-1$.

We say that gohs (μ, ν, λ) is g-bounded if

- $\nu_1 \leq g$, if $|\nu/\mu| + |\nu/\lambda| = r$,
- $\nu_1 + \frac{1}{2} \leq g$, if $|\nu/\mu| + |\nu/\lambda| = r-1$.

Definition 2.3. Reverse oscillating horizontal strip (rohs for short) (μ, ν, λ) of length r is a sequence of three partitions satisfying:

- μ/ν and λ/ν are horizontal strips,
- $|\mu/\nu| + |\lambda/\nu| = r$.

We say that rohs (μ, ν, λ) is g-bounded if

$$\mu_1 + (\lambda_1 - \nu_1) + \max(\mu_2, \lambda_2) \leq 2g.$$

In any cases, we define the *initial shape* denoted by $I(\mu, \nu, \lambda)$ to be μ, and the *final shape* denoted by $F(\mu, \nu, \lambda)$ to be λ.

A *semistandard oscillating tableau* (SSOT for short) of shape λ with weight μ is a sequence of ohs's $T = (T_1, \ldots, T_n)$, where each T_i is an ohs of length μ_i, and the following conditions hold:

- $I(T_1) = \varnothing$ and $F(T_n) = \lambda$,
- $F(T_i) = I(T_{i+1})$ for $1 \leq i \leq n-1$.

We define $c(T)$ to be the minimal number such that every T_i is $c(T)$-bounded. The set of all SSOTs of shape λ with weight μ is denoted by $\mathrm{SSOT}(\lambda, \mu)$, and we write $\mathrm{SSOT}_g(\lambda, \mu)$ for the subset of $\mathrm{SSOT}(\lambda, \mu)$ consisting of SSOTs T such that $c(T) \leq g$. Similarly, we define a *generalized semistandard oscillating tableau* (GSSOT for short) and a *semistandard reverse oscillating tableau* (SSROT for short) using gohs and rohs, respectively.

For a length n vector $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and a positive number g, we define the *orthogonal complement* of λ with respect to g, denoted by $\mathrm{oc}(\lambda, g)$, to be $(g - \lambda_n, \ldots, g - \lambda_2, g - \lambda_1)$. We also define $\overline{\mathrm{oc}}(\lambda, g)$ to be $(g - \lambda_1, g - \lambda_2, \ldots, g - \lambda_n)$. There is a bijection between King tableaux of shape λ weight μ and $\mathrm{SSOT}(\mathrm{oc}(\lambda, g), \overline{\mathrm{oc}}(\mu, g))$ [7].

Given an ohs (μ, ν, λ), we define $\mathrm{cind}(\mu, \nu, \lambda)$ to be

$$\mathrm{cind}(\mu, \nu, \lambda) := \{i : \mu_i^t + 1 = \nu_i^t\} \cup \{\bar{i} : \lambda_i^t + 1 = \nu_i^t\}.$$

We also define $\mathrm{rind}(\mu,\nu,\lambda)$ to be the multi-set consisting of $(\nu_i-\mu_i)$-many i's and $(\nu_i-\lambda_i)$-many $\bar{i}$'s. In other words, $\mathrm{cind}(\mu,\nu,\lambda)$ (respectively $\mathrm{rind}(\mu,\nu,\lambda)$) records column indices (respectively row indices) from the ohs. We abuse the notation by identifying a set $\{v_1,\dots,v_n\}$ with the word $v_1\dots v_n$ where $v_1\prec\cdots\prec v_n$. Here we employ the total order $\prec$ given by

$$1\prec 2\prec 3\prec\cdots\bar{3}\prec\bar{2}\prec\bar{1}.$$

For example, we identify the set $\{1,3,\bar{3}\}$ with the word $13\bar{3}$.

Now, we interpret the sets $\mathrm{SSOT}(\lambda,\mu)$, $\mathrm{GSSOT}(\lambda,\mu)$, and $\mathrm{SSROT}(\lambda,\mu)$ as sets of classical highest weights in specific KR crystals. We choose N to be sufficiently large so that $N,\bar{N}\notin\mathrm{cind}(Y)$ and $N,\bar{N}\notin\mathrm{rind}(Y)$ for any ohs, gohs, or rohs Y appearing in $\mathrm{SSOT}(\lambda,\mu)$, $\mathrm{GSSOT}(\lambda,\mu)$, or $\mathrm{SSROT}(\lambda,\mu)$.

Lemma 2.4. *1. There exists a bijection*

$$\phi_c:\mathrm{SSOT}(\lambda,\mu)\to\mathrm{HW}(B^t_\mu,\lambda^t)$$

where B^t_μ is of type $B^{(1)}_N$.

2. There exists a bijection

$$\phi_c:\mathrm{GSSOT}(\lambda,\mu)\to\mathrm{HW}(B^t_\mu,\lambda^t)$$

where B^t_μ is of type $D^{(2)}_{N+1}$.

Lemma 2.5. *1. There exists a bijection*

$$\phi_r:\mathrm{SSOT}(\lambda,\mu)\to\mathrm{HW}(B_\mu,\lambda)$$

such that $c(T)=\varepsilon_0(\phi_r(T))$, where B_μ is of type $C^{(1)}_N$.

2. There exists a bijection

$$\phi_r:\mathrm{GSSOT}(\lambda,\mu)\to\mathrm{HW}(B_\mu,\lambda)$$

such that $2c(T)=\varepsilon_0(\phi_r(T))$, where B_μ is of type $D^{(2)}_{N+1}$.

3. There exists a bijection

$$\phi_r:\mathrm{SSROT}(\lambda,\mu)\to\mathrm{HW}(B_\mu,\lambda)$$

such that $2c(T)=\varepsilon_0(\phi_r(T))$, where B_μ is of type $B^{(1)}_N$.

In the above, $\varepsilon_0(x)$ denotes the maximal number of times the e_0 operator can be applied to x.

In other words, we can identify our combinatorial objects, i.e., SSOT, GSSOT, and SSROT, with specific highest weights of KR crystals. The map ϕ_c in Lemma 2.4 can be directly described using cind for an ohs, as demonstrated in the example below. However, the map ϕ_r requires a modification after applying rind, which is not addressed here.

Example 2.6. Let $\lambda = (2), \mu = (3,3)$. Consider the following tableaux in $\mathrm{SSOT}(\lambda,\mu)$ and $\mathrm{GSSOT}(\lambda,\mu)$:

$$T_1 = ((\varnothing, \ydiagram{2}, \ydiagram{1}), (\ydiagram{1}, \ydiagram{2,1}, \ydiagram{2})) \in \mathrm{SSOT}(\lambda,\mu),$$
$$T_2 = ((\varnothing, \ydiagram{2}, \ydiagram{2}), (\ydiagram{2}, \ydiagram{2,1}, \ydiagram{2})) \in \mathrm{GSSOT}(\lambda,\mu).$$

For T_1, the cind of the first ohs is $\{1,2,\bar{2}\}$, and for the second ohs, it is $\{1,2,\bar{1}\}$. Due to the admissibility condition, T_1 maps to $[2] \otimes [1] \in B^{3,1} \otimes B^{3,1}$ via the map ϕ_c. Similarly, T_2 maps to $[\varnothing] \otimes [12] \in B^{3,1} \otimes B^{3,1}$ via the map ϕ_c.

3 Lusztig q-weight multiplicity

A partition $\mu = (\mu_1, \ldots, \mu_\ell)$ is a weakly decreasing sequence of positive integers. We say that μ is a partition of n, denoted by $\mu \vdash n$, if $|\mu| := \sum_i \mu_i = n$. We denote the length of μ, i.e., the number of its parts, by $\ell(\mu)$. We define Par_n to be the collection of partitions μ such that $\ell(\mu) \leq n$. Given $\mu \in \mathrm{Par}_n$, we naturally regard μ as a length n vector by appending zeros if necessary. The set of dominant weights for type C_n is Par_n, and for type B_n, dominant weights consist of μ and $\mu^\sharp := \mu + (\frac{1}{2})^n$ for $\mu \in \mathrm{Par}_n$. We refer to $\mu^\sharp$ as a *spin weight*.

Theorem 3.1. *For $\lambda, \mu \in \mathrm{Par}_n$, we have*

$$\mathrm{KL}^{C_n}_{\lambda,\mu}(q) = \sum_{T \in \mathrm{SSOT}_g(\mathrm{oc}(\lambda,g), \overline{\mathrm{oc}}(\mu,g))} q^{\overline{D}(\phi_c(T))}$$

where g is a positive integer such that $g \geq \lambda_1$, and ϕ_c is the map in Lemma 2.4.

Remark 3.2. Let α be any rearrangement of $\overline{\mathrm{oc}}(\mu,g)$. Then,

$$\sum_{T \in \mathrm{SSOT}_g(\mathrm{oc}(\lambda,g), \overline{\mathrm{oc}}(\mu,g))} q^{\overline{D}(\phi_c(T))} = \sum_{T \in \mathrm{SSOT}_g(\mathrm{oc}(\lambda,g), \alpha)} q^{\overline{D}(\phi_c(T))}.$$

We choose $\alpha = \overline{\mathrm{oc}}(\mu,g)$ for the convenience of the proof. Moreover, it can be shown that the right-hand side of Theorem 3.1 is independent of the choice of $g \geq \lambda_1$.

Example 3.3. Let $\lambda = (1,1,0,0), \mu = (0,0,0,0)$, and $g = 1$. Then, we have $\mathrm{KL}^{C_4}_{\lambda,\mu}(q) = q^6 + q^4 + q^2$. The set $\mathrm{SSOT}_1(\mathrm{oc}(\lambda,g), \overline{\mathrm{oc}}(\mu,g))$ consists of:

$$T_1 = ((\varnothing, \ydiagram{1}, \ydiagram{1}), (\ydiagram{1}, \ydiagram{1}, \varnothing), (\varnothing, \ydiagram{1}, \ydiagram{1}), (\ydiagram{1}, \ydiagram{1,1}, \ydiagram{1,1})),$$
$$T_2 = ((\varnothing, \ydiagram{1}, \ydiagram{1}), (\ydiagram{1}, \ydiagram{1,1}, \ydiagram{1,1}), (\ydiagram{1,1}, \ydiagram{1}, \ydiagram{1}), (\ydiagram{1}, \ydiagram{1,1}, \ydiagram{1,1})),$$
$$T_3 = ((\varnothing, \ydiagram{1}, \ydiagram{1}), (\ydiagram{1}, \ydiagram{1,1}, \ydiagram{1,1}), (\ydiagram{1,1}, \ydiagram{1,1}, \ydiagram{1,1,1}), (\ydiagram{1,1,1}, \ydiagram{1,1}, \ydiagram{1,1})).$$

Via the map ϕ_c, we embed T_i's into $(B^{1,1})^{\otimes 4}$ of type $B_N^{(1)}$ as follows:

$$\phi_c(T_1) = 1 \otimes 1 \otimes \bar{1} \otimes 1, \quad \phi_c(T_2) = 1 \otimes \bar{1} \otimes 1 \otimes 1, \quad \phi_c(T_3) = \bar{1} \otimes 1 \otimes 1 \otimes 1.$$

The energy function values are:

$$\overline{D}(\phi_c(T_1)) = 6, \quad \overline{D}(\phi_c(T_2)) = 4, \quad \overline{D}(\phi_c(T_3)) = 2.$$

These values coincide with $\mathrm{KL}^{C_4}_{\lambda,\mu}(q) = q^6 + q^4 + q^2$.

For Lusztig q-weight multiplicities of type B, we introduce a q,t-generalization. The set of positive roots R^+ consists of $(\varepsilon_i \pm \varepsilon_j)$'s for $i < j$, referred to as long roots, and ε_i's, referred to as short roots. Given dominant weights λ, μ, we define q,t-weight multiplicities by

$$\mathrm{KL}^{B_n}_{\lambda,\mu}(q,t) := \sum_{w \in W} (-1)^w [e^{w(\lambda+\rho)-(\mu+\rho)}] \prod_{\substack{\alpha \in R^+ \\ \alpha \text{ is a long root}}} \frac{1}{1-qe^{\alpha}} \prod_{\substack{\alpha \in R^+ \\ \alpha \text{ is a short root}}} \frac{1}{1-te^{\alpha}}.$$

The usual Lusztig q-weight multiplicities can be recovered by setting $t = q$. Consider B^t_μ of type $D^{(2)}_{N+1}$ and $b = b_n \otimes \cdots \otimes b_2 \otimes b_1 \in B^t_\mu$. Recall that $B^{r,1}$ of type $D^{(2)}_{N+1}$ decomposes as $\cup_{0 \le s \le r} B(sw_1)$ as a classical crystal. For each $b_i \in B^{\mu_i,1}$, we define a *vacancy*, denoted by $\mathrm{vac}(b_i)$, to be $(\mu_i - s)$ if $b_i \in B(sw_1)$, i.e., b_i corresponds to a word of length s. Then, we define $\mathrm{vac}(b) := \sum_{i=1}^n \mathrm{vac}(b_i)$. Finally, the q,t *-energy* of b is defined by

$$\mathrm{energy}_{q,t}(b) := q^{\frac{\overline{D}(b)-\mathrm{vac}(b)}{2}} t^{\mathrm{vac}(b)}.$$

Theorem 3.4. *We have*

$$\mathrm{KL}^{B_n}_{\lambda^\sharp,\mu^\sharp}(q,t) = \sum_{T \in \mathrm{GSSOT}_{g+\frac{1}{2}}(\mathrm{oc}(\lambda,g),\overline{\mathrm{oc}}(\mu,g))} \mathrm{energy}_{q,t}(\phi_c(T))$$

where g is a positive integer such that $g \ge \lambda_1$ and ϕ_c is the map in Lemma 2.4.

Example 3.5. Let $\lambda = (1,1,1), \mu = (0,0,0)$, and $g = 1$. Then, we have $\mathrm{KL}^{B_3}_{\lambda^\sharp,\mu^\sharp}(q,t) = q^3t^3 + q^3t + q^2t + qt$. The set $\mathrm{GSSOT}_{1+\frac{1}{2}}(\mathrm{oc}(\lambda,g),\overline{\mathrm{oc}}(\mu,g))$ consists of:

$$\begin{aligned}
T_1 &= ((\varnothing,\varnothing,\varnothing),(\varnothing,\varnothing,\varnothing),(\varnothing,\varnothing,\varnothing)),\\
T_2 &= ((\varnothing,\varnothing,\varnothing),(\varnothing,\square,\square),(\square,\varnothing,\varnothing)),\\
T_3 &= ((\varnothing,\square,\square),(\square,\square,\varnothing),(\varnothing,\varnothing,\varnothing)),\\
T_4 &= ((\varnothing,\square,\square),(\square,\square,\square),(\square,\square,\varnothing)).
\end{aligned}$$

Via the map ϕ_c, we embed $T_i's$ into $(B^{1,1})^{\otimes 3}$ of type $D^{(2)}_{N+1}$ as follows:

$$\phi_c(T_1) = \varnothing \otimes \varnothing \otimes \varnothing, \quad \phi_c(T_2) = \bar{1} \otimes 1 \otimes \varnothing,$$
$$\phi_c(T_3) = \varnothing \otimes \bar{1} \otimes 1, \quad \phi_c(T_4) = \bar{1} \otimes \varnothing \otimes 1.$$

The q,t-energy values are:

$$\mathrm{energy}_{q,t}(\phi_c(T_1)) = q^3t^3, \quad \mathrm{energy}_{q,t}(\phi_c(T_2)) = q^3t,$$
$$\mathrm{energy}_{q,t}(\phi_c(T_3)) = q^2t, \quad \mathrm{energy}_{q,t}(\phi_c(T_4)) = qt.$$

These values coincide with $\mathrm{KL}^{B_3}_{\lambda^\sharp,\mu^\sharp}(q,t) = q^3t^3 + q^3t + q^2t + qt$.

3.1 Sketch of the proof of Theorem 3.1

Given $\lambda, \mu \in \mathrm{Par}_n$ for $n \geq 2$, we define $\lambda^{(1)}, \lambda^{(2)}, \dots, \lambda^{(n)}$ and μ' as follows:

$$\begin{aligned}
\lambda^{(1)} &= (\lambda_2, \dots, \lambda_n), & \mu' &= (\mu_2, \dots, \mu_n),\\
\lambda^{(2)} &= (\lambda_1 + 1, \lambda_3, \dots, \lambda_n), \\
\lambda^{(3)} &= (\lambda_1 + 1, \lambda_2 + 1, \lambda_4, \dots, \lambda_n), \\
&\ \ \vdots \\
\lambda^{(n)} &= (\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_{n-1} + 1).
\end{aligned}$$

We define $\mathrm{ROHS}_{\leq m}(\lambda, r)$ to be the set of rohs (λ, τ, ν) of length r such that $\ell(\tau), \ell(\nu) \leq m$. As proved in [4], the (C_n, C_{n-1})-recurrence is given as

$$\mathrm{KL}^{C_n}_{\lambda,\mu}(q) = \sum_{i\geq 1} (-1)^{i-1} \sum_{\substack{r+2m=\lambda_i-\mu_1+1-i \\ r,m\geq 0}} q^{r+m} \sum_{(\lambda^{(i)},\tau,\nu)\in \mathrm{ROHS}_{\leq n-1}(\lambda^{(i)},r)} \mathrm{KL}^{C_{n-1}}_{\nu,\mu'}(q). \tag{3.1}$$

Assuming that Theorem 3.1 holds for $\mathrm{KL}^{C_{n-1}}_{\lambda,\mu}$, the right-hand side of (3.1) becomes

$$\sum_{i\geq 1} (-1)^{i-1} \sum_{\substack{r+2m=\lambda_i-\mu_1+1-i \\ r,m\geq 0}} q^{r+m} \sum_{(\lambda^{(i)},\tau,\nu)\in \mathrm{ROHS}_{\leq n-1}(\lambda^{(i)},r)} \sum_{T\in \mathrm{SSOT}_g(\mathrm{oc}(\nu,g),\overline{\mathrm{oc}}(\mu',g))} q^{\overline{D}(\phi_c(T))} \tag{3.2}$$

where we choose a large enough g, say $g \geq \lambda_1 + \lambda_2$, so that $g \geq \nu_1$ for any ν appearing in the summation in (3.2).

Lemma 3.6. *Let $A^{(i)}$ be a set of pairs (S, T) satisfying the following conditions:*

- *$S \in \mathrm{ROHS}_{\leq n-1}(\lambda^{(i)}, r)$ for some r such that there exists $m \geq 0$ with $r + 2m = \lambda_i - \mu_1 + 1 - i$,*

- $T \in \mathrm{SSOT}_g(\mathrm{oc}(\nu, g), \overline{\mathrm{oc}}(\mu', g))$ *where* $\nu = F(S)$.

Then, there exists a bijection

$$\Phi^{(i)} : A^{(i)} \to \mathrm{SSOT}_g(\mathrm{oc}(\lambda^{(i)}, g), \gamma^{(i)})$$

where $\gamma^{(i)} = (g - \mu_2, \ldots, g - \mu_n, \lambda_i - \mu_1 + 1 - i)$.

Lemma 3.7. *Keeping the notations in the previous lemma, let* $(S, T) \in A^{(i)}$ *such that* $S \in \mathrm{ROHS}_{\leq n-1}(\lambda^{(i)}, r)$. *Then, we have*

$$\overline{D}(\phi_c(\Phi^{(i)}(S, T))) = r + m + \overline{D}(\phi_c(T))$$

where m is an integer satisfying $r + 2m = \lambda_i - \mu_1 + 1 - i$.

Using Lemma 3.6 and 3.7, the equation (3.2) simplifies to:

$$\sum_{i \geq 1} (-1)^{i-1} \sum_{T \in \mathrm{SSOT}_g(\mathrm{oc}(\lambda^{(i)}, g), \gamma^{(i)})} q^{\overline{D}(\phi_c(T))}. \tag{3.3}$$

We now apply a sign-reversing involution to (3.3), generalizing the argument presented in Zabrocki's thesis, which proves the result for type A [12]. Theorem 3.4 can be proved using a similar argument.

4 Level-restricted q-weight multiplicity

We provide a combinatorial formula for the level-restricted q-weight multiplicity in terms of level-restricted paths in KR crystals. For a nonnegative integer vector $\alpha = (\alpha_1, \ldots, \alpha_n)$, we define $||\alpha|| = \sum_{i=1}^{n} (i-1)\beta_i$, where β is a rearrangement of α in weakly decreasing order.

Theorem 4.1. *We have the following formulas:*

$$\mathrm{KL}^{B_n, L_A}_{\lambda, \mu}(q) = \sum_{\substack{T \in \mathrm{GSSOT}(\hat{\lambda}, \bar{\mu}) \\ c(T) \leq 2g}} q^{||\bar{\mu}|| + |\bar{\mu}| - |\hat{\lambda}| - \lfloor \frac{\overline{D}(\phi_r(T))}{2} \rfloor}, \tag{4.1}$$

$$\mathrm{KL}^{C_n, L_A}_{\lambda, \mu}(q) = \sum_{\substack{T \in \mathrm{SSOT}(\hat{\lambda}, \bar{\mu}) \\ c(T) \leq g}} q^{||\bar{\mu}|| + |\bar{\mu}| - |\hat{\lambda}| - \overline{D}(\phi_r(T))}, \tag{4.2}$$

$$\mathrm{KL}^{D_n, L_A}_{\lambda, \mu}(q) = \sum_{\substack{T \in \mathrm{SSROT}(\hat{\lambda}, \bar{\mu}) \\ c(T) \leq 2g}} q^{||\bar{\mu}|| + \frac{|\bar{\mu}| - |\hat{\lambda}|}{2} - \overline{D}(\phi_r(T))}, \tag{4.3}$$

where $g \in \frac{\mathbb{Z}_{\geq 0}}{2}$ *such that* $g \geq \lambda_1$, $\hat{\lambda} = \mathrm{oc}(\lambda, g)$, *and* $\bar{\mu} = \overline{\mathrm{oc}}(\mu, g)$.

Note that the equations (4.1) and (4.3) also hold for spin weights. When λ and μ are spin weights, we choose g such that $g \geq \lambda_1$ and $2g$ is odd. When λ and μ are partitions, we choose g such that $g \geq \lambda_1$ and $2g$ is even.

Example 4.2. Let $\lambda = (1,1,1,1), \mu = (0,0,0,0)$, and $g = 1.5$. Then, we have $\mathrm{KL}^{D_4,L_A}_{\lambda^\sharp,\mu^\sharp}(q) = q^4 + q^2 + 1$. The set $\mathrm{SSROT}(\mathrm{oc}(\lambda^\sharp, g), \overline{\mathrm{oc}}(\mu^\sharp, g))$ with $c(T) \leq 3$ consists of:

$$
\begin{aligned}
T_1 &= ((\varnothing,\varnothing,\square),(\square,\square,\square\square),(\square\square,\square,\square),(\square,\varnothing,\varnothing)),\\
T_2 &= ((\varnothing,\varnothing,\square),(\square,\square,\begin{smallmatrix}\square\\\square\end{smallmatrix}),(\begin{smallmatrix}\square\\\square\end{smallmatrix},\square,\square),(\square,\varnothing,\varnothing)),\\
T_3 &= ((\varnothing,\varnothing,\square),(\square,\varnothing,\varnothing),(\varnothing,\varnothing,\square),(\square,\varnothing,\varnothing)).
\end{aligned}
$$

Via the map ϕ_r, we embed T_i's into $(B^{1,1})^{\otimes 4}$ of type $B_N^{(1)}$ as follows:

$$\phi_r(T_1) = \bar{1} \otimes \bar{1} \otimes 1 \otimes 1, \quad \phi_r(T_2) = \bar{1} \otimes \bar{2} \otimes 2 \otimes 1, \quad \phi_r(T_3) = \bar{1} \otimes 1 \otimes \bar{1} \otimes 1.$$

The energy function values are:

$$\overline{D}(\phi_r(T_1)) = 4, \quad \overline{D}(\phi_r(T_2)) = 6, \quad \overline{D}(\phi_r(T_3)) = 8.$$

Since $||\bar{\mu}|| + \frac{|\bar{\mu}|-|\hat{\lambda}|}{2} = 8$, we obtain $q^4 + q^2 + 1$, which coincides with $\mathrm{KL}^{D_4,L_A}_{\lambda^\sharp,\mu^\sharp}(q) = q^4 + q^2 + 1$.

Remark 4.3. For any positive integer k, Theorem 4.1 implies the following:

$$
\begin{aligned}
\mathrm{KL}^{B_n,L_A}_{\lambda+k^n,\mu+k^n}(q) &= \sum_{\substack{T \in \mathrm{GSSOT}(\hat{\lambda},\bar{\mu}) \\ c(T) \leq 2g+2k}} q^{||\bar{\mu}||+|\bar{\mu}|-|\hat{\lambda}|-\lfloor \frac{\overline{D}(\phi_r(T))}{2} \rfloor},\\
\mathrm{KL}^{C_n,L_A}_{\lambda+k^n,\mu+k^n}(q) &= \sum_{\substack{T \in \mathrm{SSOT}(\hat{\lambda},\bar{\mu}) \\ c(T) \leq g+k}} q^{||\bar{\mu}||+|\bar{\mu}|-|\hat{\lambda}|-\overline{D}(\phi_r(T))},\\
\mathrm{KL}^{D_n,L_A}_{\lambda+k^n,\mu+k^n}(q) &= \sum_{\substack{T \in \mathrm{SSROT}(\hat{\lambda},\bar{\mu}) \\ c(T) \leq 2g+2k}} q^{||\bar{\mu}||+\frac{|\bar{\mu}|-|\hat{\lambda}|}{2}-\overline{D}(\phi_r(T))},
\end{aligned}
$$

where $\hat{\lambda} = \mathrm{oc}(\lambda, g)$ and $\bar{\mu} = \overline{\mathrm{oc}}(\mu, g)$. When k is sufficiently large, the right-hand sides becomes (up to a scalar)

$$\sum_{x \in \mathrm{HW}(B_{\bar{\mu}},\hat{\lambda})} q^{-\overline{D}(x)}$$

which is simply the 1-dim sum for certain tensor products of row KR crystals. Therefore, Theorem 4.1 can be regarded as a refinement of the $X = K = {}^\infty\mathrm{KL}$ theorem for tensor products of row KR crsytals [6, 11].

Acknowledgements

The authors are grateful to Mark Shimozono for Jae-Hoon Kwon for helpful comments.

References

[1] A. N. Kirillov. "Ubiquity of Kostka polynomials". *Physics and combinatorics 1999 (Nagoya)*. World Sci. Publ., River Edge, NJ, 2001, pp. 85–200. DOI.

[2] A. Lascoux. "Cyclic permutations on words, tableaux and harmonic polynomials". *Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989)*. Manoj Prakashan, Madras, 1991, pp. 323–347.

[3] A. Lascoux and M.-P. Schützenberger. "Sur une conjecture de H. O. Foulkes". *C. R. Acad. Sci. Paris Sér. A-B* **286**.7 (1978), A323–A324.

[4] C. Lecouvey. "Kostka-Foulkes polynomials cyclage graphs and charge statistic for the root system C_n". *J. Algebraic Combin.* **21**.2 (2005), pp. 203–240. DOI.

[5] C. Lecouvey, M. Okado, and M. Shimozono. "Affine crystals, one-dimensional sums and parabolic Lusztig q-analogues". *Math. Z.* **271**.3-4 (2012), pp. 819–865. DOI.

[6] C. Lecouvey and M. Shimozono. "Lusztig's q-analogue of weight multiplicity and one-dimensional sums for affine root systems". *Adv. Math.* **208**.1 (2007), pp. 438–466. DOI.

[7] S. J. Lee. "Crystal Structure on King Tableaux and Semistandard Oscillating Tableaux". *Transform. Groups* **30**.2 (2025), pp. 823–853. DOI.

[8] G. Lusztig. "Singularities, character formulas, and a q-analog of weight multiplicities". *Analysis and topology on singular spaces, II, III (Luminy, 1981)*. Vol. 101-102. Astérisque. Soc. Math. France, Paris, 1983, pp. 208–229.

[9] A. Nakayashiki and Y. Yamada. "Kostka polynomials and energy functions in solvable lattice models". *Selecta Math. (N.S.)* **3**.4 (1997), pp. 547–599. DOI.

[10] M. Okado and A. Schilling. "Existence of Kirillov-Reshetikhin crystals for nonexceptional types". *Represent. Theory* **12** (2008), pp. 186–207. DOI.

[11] M. Shimozono. "On the X=M=K Conjecture". 2005. arXiv:math/0501353.

[12] M. A. Zabrocki. *On the action of the Hall-Littlewood vertex operator*. Thesis (Ph.D.)–University of California, San Diego. ProQuest LLC, Ann Arbor, MI, 1998, 58 pp. Link.